A variation on the game SET

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June 25, 2015

Abstract

SET is a very popular card game with strong mathematical structure. In this paper, we describe "Anti-SET", a variation on SET in which we reverse the objective of the game by trying to avoid drawing "*sets*". In Anti-SET, two players take turns selecting cards from the SET deck into their hands. The first player to hold a *set* loses the game.

By examining the geometric structure behind SET, we determine a winning strategy for the first player. We extend this winning strategy to all non-trivial affine geometries over $\mathbb{F}_3$, of which SET is only one example. Thus we find a winning strategy for an infinite class of games and prove this winning strategy in geometric terms. We also describe a strategy for the second player which allows her to lengthen the game. This strategy demonstrates a connection between strategies in Anti-SET and maximal caps in affine geometries.

1 Introduction

The card game SET is a very popular game among mathematics students. In addition to being an enjoyable pastime, it has a large amount of mathematical structure, including links to finite geometry, linear algebra, and combinatorics. This paper will focus on a particular variation of the game and the mathematics relevant to that variation. For much more information about the mathematics of SET as well as positional games that are similar to the game in this paper, see [3, 4] and the citations contained therein.

SET consists of a deck of cards. Each card is printed with several figures which have four attributes: number, color, filling, and shape. For example, the card in Figure 1 would be described as "two green striped ovals". The complete list of attributes is given in Table 1.

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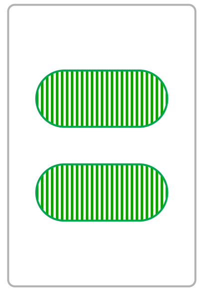


Figure 1: A Set card

Attribute	Values		
Number	1	2	3
Color	Red	Green	Purple
Filling	Open	Striped	Solid
Shape	Oval	Diamond	Squiggle

Table 1: Attributes of a Set card

There are four attributes with three values each, and every possible combination appears exactly once. Thus there are $3^4 = 81$ cards in a complete Set deck.

The game requires players to find a *set*: Three cards such that, for each attribute, all three cards are the same, or all three are different. Phrased differently, a *set* consists of three cards for which no attribute has two cards with one value, and another card with a different value. An example of a *set* is given in Figure 2: The number of cards is all the same (1), but the colors are all different. The shading is all the same (solid), and the shapes are all different. A non-example appears in Figure 3: two cards have solid shading, while the other is open. There are several other reasons why these cards are not a *set* as well. Note that, although the cards have three different colors, this *alone* is not enough to make a *set*.

Throughout this paper, we will use the notation Set to refer to the game, *set* to refer to a collection of cards as defined above, and "set" (without any special styling) to refer to the mathematical object consisting of an unordered collection of objects without repeated elements.

In the original game of Set, twelve cards are laid out at a time. Players compete to identify *sets* first, winning by collecting more *sets* than their opponents.

In this paper, we study a variation on Set which turns the usual goal upside down. Our game, *Anti*-Set, is a 2-player game played with a generalized Set deck, in which each card has d different attributes (traditional Set has $d = 4$). This situation corresponds to a d-dimensional affine geometry over $\mathbb{F}_3$, which will be described later. The players, who we will call Xavier and Olivia, begin

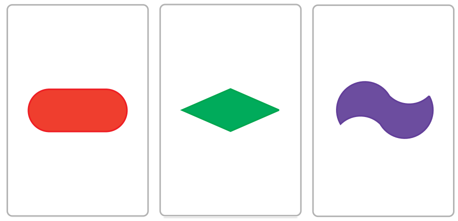


Figure 2: A *set*

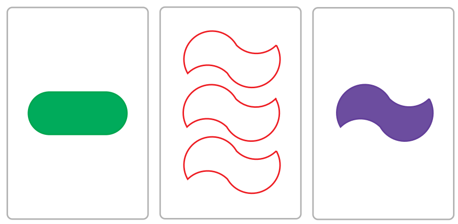


Figure 3: Not a *set*

with the entire SET deck laid out in front of them. Xavier and Olivia then take turns selecting cards from these cards and take them into their hands. The first player to have a *set* in his or her hand *loses* the game. Thus, players are faced both with the challenge of trying to avoid taking *sets* themselves but also trying to force the other player to take a *set*. As each player collects more cards in their hand, it becomes increasingly difficult to not take a *set*, as there are many more combinations of cards that can be made.

This game was inspired by a result of Pellegrino [5]. Translated into the language of SET (which did not exist at the time of Pellegrino's writing), we have the result[1]:

Proposition 1 (Pellegrino [5])**.** *Every set of 21* SET *cards contains a* set*.*

Thus, Anti-SET will always end once one player takes their 21st card, if not sooner. We initially created the game of Anti-SET to explore the consequences of Pellegrino's result in the context of a game.

In the following sections, we will analyze this game, provide a winning strategy for the first player that applies to *all* nontrivial generalized SET decks (that is, non-trivial affine geometries over $\mathbb{F}_3$), and examine the maximum and minimum number of turns required to win. Along the way, we will demonstrate some unexpected links between Pellegrino's result and the losing player's strategy.

[1] We acknowledge that the *three* different uses of the word "set" in this result may make the reader's head spin.

2 Example of Gameplay

Before we give precise mathematical background for SET, we present an extended example of gameplay for Anti-SET. For simplicity, we use a reduced version of Anti-SET as played with the 9 SET cards which are solid and have only one symbol per card. Later, we will justify this simplification geometrically and examine how it forms an important foundation for studying general Anti-SET.

Let Xavier be the first player. He may choose any of the cards shown in Figure 4. We will mark Xavier's hand of cards with an "X" and Olivia's with an "O".

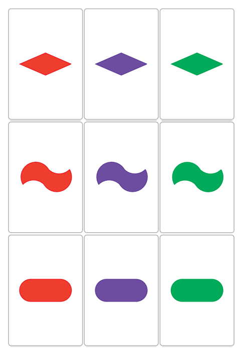


Figure 4: The 9-card reduced Anti-SET deck.

The moves are denoted as follows:

X_1**:** Xavier first arbitrarily chooses the red diamond.

O_1**:** Olivia, recognizing that every pair of cards determines a unique *set*, arbitrarily chooses the purple diamond.

The players hands at this point are represented in Figure 5a.

X_2**:** Xavier chooses the green diamond, knowing that it is part of a *set* (the three cards in the top row) from which Olivia already owns one card. Thus, he avoids at least one *set*.

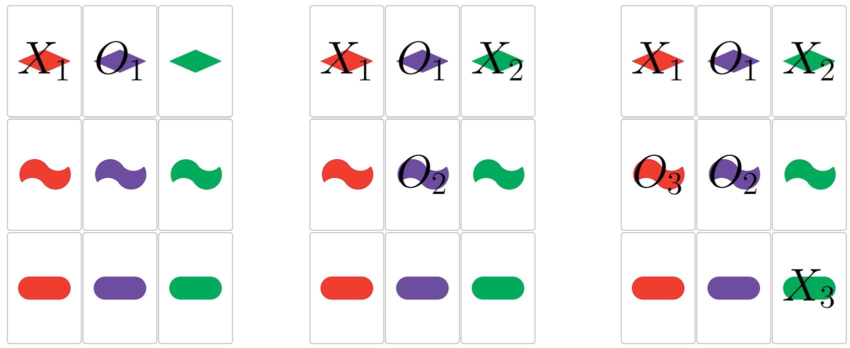
O_2**:** Olivia chooses the purple squiggle, again knowing that any pair of cards contains a *set* and thus all of her remaining choices are equally bad.

The players hands at this point are represented in Figure 5b.

X_3**:** Xavier again chooses a card, the green oval, which he knows is part of at least one *set* which he cannot obtain.

O_3**:** Olivia chooses the red squiggle, leaving Xavier with at least one possibility (the green squiggle) which could complete a *set*.

The players hands at this point are represented in Figure 5c.



(a) X_1 and O_1. (b) X_2 and O_2. (c) X_3 and O_3.

Figure 5: The first few steps of the Anti-Set game.

X_4**:** Finally, Xavier chooses the red oval. This leaves Olivia with two options, both of which complete a *set*. Thus, Xavier will win. (See Figure 6.)

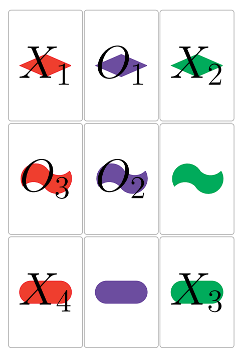


Figure 6: X_4 and Olivia's remaining options.

This nine-card example demonstrates the general flow of the game. In order to make valid conclusions about the game on a larger scale, we first need to describe the game mathematically, which we will do in the next section.

We also note that the board and style of play is similar to a backwards Tic-Tac-Toe game, with players trying to avoid getting three in a row. Indeed, Set as played with the 9 cards in this example can be thought of as playing Tic-Tac-Toe on a torus, a concept which is explored in depth in [4]. Our names "Xavier" and "Olivia", and the idea of marking their cards with X's and O's, were inspired by this interpretation.

3 Background

In this section, we will define the notation and concepts which will be used throughout the rest of the paper. Let X_n be the nth card Xavier picks, and let $\mathcal{X}(n) = \{X_1, X_2, \ldots, X_n\}$ denote the collection of Xavier's first n cards. Similarly, let O_n denote the nth card Olivia picks, and let $\mathcal{O}(n) = \{O_1, O_2, \ldots, O_n\}$ denote the collection of Olivia's first n cards. Note that $\mathcal{X}(n) \subset \mathcal{X}(n+1)$ and $\mathcal{O}(n) \subset \mathcal{O}(n+1)$.

Xavier is the first player. The game proceeds with all cards in a SET deck available to both players. The players alternately take cards into their hands in the order $X_1, O_1, X_2, O_2, \ldots$ until either $\mathcal{X}(n)$ or $\mathcal{O}(n)$ contains a *set*. The corresponding player loses on his or her nth turn. We call a pair of choices (X_n, O_n) a *round* of Anti-SET.

The mathematical structure of SET is an example of an *affine geometry*. For our purposes, we will define affine geometries from a coordinatized (vector-based) perspective, as described it [1, 2]. It is possible to do this from a purely axiomatic viewpoint as well (see [2]). For more details about affine geometry in the context of SET, see [4].

The affine geometry $AG(d, q)$ is an incidence structure whose points are d-dimensional vectors with entries in $\mathbb{F}_q$. That is, the points are the elements of $\mathbb{F}_q^d$. The k-dimensional subspaces of $AG(d, q)$, referred to as k-flats, are the k-dimensional linear subspaces of $\mathbb{F}_q^d$ together with their cosets. We note that for a given k-dimensional linear subspace L, the coset of L by the vector $\vec{h} \in \mathbb{F}_q^d$ is defined as $L + \vec{h} = \{\vec{x} + \vec{h} : \vec{x} \in L\}$.

The cards of SET correspond to the points of $AG(4, 3)$, and the *sets* are the 1-flats (usually called *lines*). More specifically, the points are all vectors of the form (x_1, x_2, x_3, x_4) where $x_i \in \{0, 1, 2\}$, with all arithmetic done modulo 3. The 1-flats correspond to the 1-dimensional subspaces of $\mathbb{F}_3^4$ and their cosets. Each such 1-flat contains 3 points, corresponding to the 3 cards in a *set*.

To give a more concrete interpretation of SET in this context, we note that each card corresponds to a unique vector, with each coordinate corresponding to a characteristic of the cards. We arbitrarily identify the coordinates with characteristics of the SET cards as shown in Table 2. There are many equivalent ways to map between attributes of SET cards and the entries of $\mathbb{F}_3$.

Example 1. *Consider the* set *in Figure 7. Using the correspondence from Table 2, these three cards, in order, form the vectors* $(0, 0, 0, 0)$*,* $(2, 1, 1, 1)$*, and* $(1, 2, 2, 2)$*.*

We will make extensive use of the following result about affine geometries:

Proposition 2 (Affine Collinearity Rule)**.** *Three points $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $AG(d, 3)$ form a line if and only if $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.*

Proof. A set ℓ in $AG(d, 3)$ is a line if and only if ℓ is a 1-dimensional subspace of

Vector (n, c, f, s):

Entry:	**0**	**1**	**2**
n**: Number**	3	1	2
c**: Color**	Red	Green	Purple
f**: Filling**	Open	Stripe	Solid
s**: Shape**	Squiggle	Oval	Diamond

Table 2: Correspondence between vectors in $\mathbb{F}_3^4$ and characteristics of SET cards.

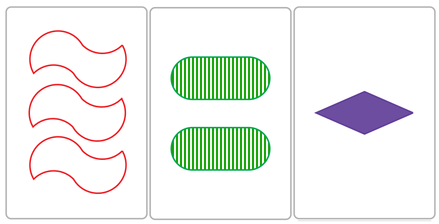


Figure 7: A *set*.

$\mathbb{F}_3^d$ or a coset thereof. Thus ℓ is a line if and only if there exist a nonzero vector $\vec{x}$ and a vector $\vec{h}$, both in $\mathbb{F}_3^d$, such that $\ell = \{\vec{h}, \vec{x}+\vec{h}, 2\vec{x}+\vec{h}\}$. (Note that $\vec{h} = \vec{0}$ is possible.) In particular, all lines in $AG(d, 3)$ contain 3 points. Then the sum of the elements in ℓ is $3\vec{h} + 3\vec{x} = \vec{0}$, since we are working in $\mathbb{F}_3$.

Conversely, suppose $\ell = \{\vec{a}, \vec{b}, \vec{c}\}$ such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$. Then $\vec{0}+(\vec{b}-\vec{a})+(\vec{c}-\vec{a}) = \vec{0} - 3\vec{a} = \vec{0}$ as well. Thus $\vec{c} - \vec{a} = 2(\vec{b} - \vec{a})$, and so $m = \{\vec{0}, \vec{b} - \vec{a}, 2(\vec{b} - \vec{a})\}$ is a linear subspace of $\mathbb{F}_3^d$. Thus $\ell = m + \vec{a}$ is a line. $\square$

In the context of SET, three cards $\{A, B, C\}$ form a *set* if and only if their corresponding vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (respectively) satisfy $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. To see this, consider three vectors whose associated cards form a *set*. The collection of three values in a given coordinate is limited to the following possibilities: {0,0,0}, {1,1,1}, {2,2,2}, or {0,1,2}. These collections of values constitute all possibilities for "all the same" or "all different." The sum of the numbers in each of these collections is 0 (mod 3). Furthermore, no other collection of three values sums to 0 (mod 3).

We will also use the following well-known proposition:

Proposition 3. *In $AG(d, q)$, every pair of points appears in exactly one line.*

This can be seen as follows: A line is a 1-dimensional subspace or a coset of such a subspace. Let x and y be distinct points in $AG(d, q)$. If $x = \alpha y$ for some $\alpha \in \mathbb{F}_q$, then $\{x, y\}$ appear together only in the 1-dimensional linear subspace defined by x. If x and y are not scalar multiples, then $\{0, y-x\}$ appear together only in the 1-dimensional linear subspace ℓ defined by $y-x$, and therefore $\{x, y\}$ appear together only in the coset $\ell + x$.

This corresponds to the well-known fact that every pair of SET cards is part of a unique *set*. Algebraically, given two points $\vec{a}$ and $\vec{b}$, there exists a unique vector $\vec{c} \in \mathbb{F}_q^d$ such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Because affine lines include both linear subspaces of $\mathbb{F}_q^d$ and their cosets, affine geometries naturally include parallel lines. All cosets of a given line ℓ are parallel to ℓ, and together this collection of cosets partitions the points of the geometry. Thus, for any *set* $\{A, B, C\}$ in the traditional 81 card SET game, there are $81/3 = 27$ *sets* (including $\{A, B, C\}$ itself) which are parallel to the original *set*. These 27 *sets* contain all 81 cards in the SET deck.

Example 2. *In Example 1, we saw a* set *consisting of the vectors*

$$S = \{(0, 0, 0, 0), (2, 1, 1, 1), (1, 2, 2, 2)\}.$$

The coset $S + (1, 0, 1, 2)$ *is*

$$S + (1, 0, 1, 2) = \{(1, 0, 1, 2), (0, 1, 2, 0), (2, 2, 0, 1)\},$$

which can be verified to be a set *sharing no points with* S*.*

In addition to points and lines, affine geometries contain other substructures with geometric interpretation. Of particular interest to us is the *affine plane* $AG(2, q)$, which can be viewed as a 2-dimensional subspace of a larger affine geometry. Affine planes are very well studied. In the case of SET, the set of vectors obtained by fixing any two coordinates of the vectors in $\mathbb{F}_3^4$ is isomorphic to an affine plane. With only two coordinates "free" to change, a plane contains $3^2 = 9$ points.

Example 3. *The 9 cards in Figure 4 form an affine plane. Here, the coordinate corresponding to "number" is fixed at 1, and the coordinate corresponding to "filling" is fixed at 2 (solid). This there are two free coordinates, giving a 2-dimensional plane.*

An affine plane is spanned by two non-parallel lines. In the affine plane $AG(2, 3)$, each line is part of a *parallel class* of three parallel lines.

Notice that there are twelve lines (that is, *sets*) in $AG(2, 3)$. As represented in Example 3, there are three horizontal lines, three vertical lines, and then three lines in each diagonal direction. (For example, the *set* containing the red squiggle, purple diamond, and green oval is one of these diagonal *sets*.)

The final substructure of particular interest to us is a *hyperplane*: a $(d - 1)$-dimensional subspace within $AG(d, q)$. Equivalently, a hyperplane is a subspace of maximal size, or of co-dimension 1. In SET, a hyperplane corresponds to a set of 27 cards with a single attribute fixed.

The remainder of this paper will primarily use geometric language when discussing SET. In particular, we will use "point" and "line" to refer to cards and *sets*, respectively, except when interpreting our results in terms of the original SET game.

We note that the results of this paper apply to $AG(d, 3)$ for all $d \geq 2$. That is, they apply not only to SET (which lives in $AG(d, 3)$) but also to "general" SET as played in $AG(d, 3)$. For example, a version of SET could be created in which each card has five attributes: the four usual ones, plus a scratch-and-sniff scent attribute with three different values. The game of Anti-SET could be played with these $3^5 = 243$ cards without change. When $d = 1$, the geometry $AG(1, 3)$ consists of a single line, in which no win nor loss of Anti-SET is possible.

4 Results

In this section we prove that Xavier has a winning strategy in Anti-SET, as played on any affine geometry $AG(d, 3)$, $d \geq 2$. We first reformulate Anti-SET in purely geometric terms:

*Anti-*SET is a 2-player game played on $AG(d, 3)$. The players, Xavier and Olivia, take turns (beginning with Xavier) selecting points from the geometry. The first player to have a line contained entirely in his or her hand *loses* the game.

Theorem 1 (Winning Strategy)**.** *Suppose Xavier and Olivia play Anti-*SET *using the points in $AG(d, 3)$, $d \geq 2$. Moves X_1 and O_1 may be chosen arbitrarily. After those moves, Xavier will always win by following this strategy: For each move $n \geq 2$, Xavier chooses X_n to be the unique third point on the unique line containing X_1 and O_{n-1}.*

Xavier's strategy depends on him following Olivia's moves. The first two moves are arbitrary, after which Xavier begins to follow Olivia by completing lines which are not completely contained in either player's hands. The worked example in Section 2 implements exactly this strategy on a 9-card affine plane.

We note that we require $d \geq 2$ only because $d = 1$ is a degenerate case: $AG(1, 3)$ consists of three points on a single line. Thus, every game ends in a tie, as neither player can fully collect the line. However, the condition that $q = 3$ is essential. Our strategy is highly dependent on the fact that each line contains exactly 3 points, a fact that is lost for $q \neq 3$.

The following lemmas are necessary to establish the correctness of this strategy.

Lemma 1 (Xavier can play)**.** *If Xavier consistently follows the strategy in Theorem 1, then Xavier can always choose the required point.*

Proof. Consider the nth round of the game. In the previous round, Olivia selected point O_{n-1}, and now Xavier wishes to choose as X_n the unique point C completing the line ℓ containing points $\{X_1, O_{n-1}\}$. Note that point C exists and is unique by Proposition 3. If C is unavailable, it must be in either $\mathcal{O}(n-2)$ (because Olivia's move O_{n-1} was not C) or $\mathcal{X}(n-1)$.

If Olivia previously chose C, then by following the strategy Xavier would have immediately chose the other point on ℓ. If Xavier previously chose C, then he must have done so immediately after Olivia chose the other point on ℓ.

In either case, *all* points on ℓ appear in $\mathcal{O}(n-2) \cup \mathcal{X}(n-1)$, and thus it was impossible for Olivia to choose any point on ℓ as O_{n-1}. Therefore we obtain a contradiction, and C must be available for Xavier to choose. $\square$

Lemma 2 (Xavier can't lose)**.** *If Xavier consistently follows the strategy in Theorem 1, then Xavier can't lose.*

Proof. Without loss of generality, assume that at least two rounds have occurred. In round $j \geq 1$, Olivia chooses O_j. In round $k > j$, Olivia chooses O_k. Following the Winning Strategy, Xavier chooses X_{j+1} and X_{k+1}, respectively. Thus $\{X_1, O_j, X_{j+1}\}$ and $\{X_1, O_k, X_{k+1}\}$ are lines. This is represented geometrically by solid lines connecting the points in Figure 8. Algebraically, $X_1 + O_j + X_{j+1} = \vec{0}$ and $X_1 + O_k + X_{k+1} = \vec{0}$.

Suppose that at some future round m, while following the Winning Strategy, Xavier chooses point X which completes a line $\{X_{j+1}, X_{k+1}, X\} \subseteq \mathcal{X}(m)$, causing him to lose. Thus $X_{j+1} + X_{k+1} + X = \vec{0}$.

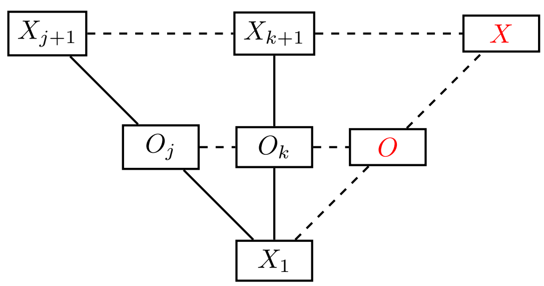


Figure 8: Diagram for proof of Lemma 2.

Xavier chose X in response to some move O by Olivia. Thus X is the unique third point on the line containing $\{X_1, O\}$. Therefore $X_1+O+X = \vec{0}$. Beginning with this fact and applying algebra, we have:

$$\begin{aligned}
0 &= X_1 + O + X \\
&= X_1 + O + (-X_{j+1} - X_{k+1}) && \text{because } \{X_{j+1}, X_{k+1}, X\} \text{ is a line} \\
&= X_1 + O + (X_1 + O_j) + (X_1 + O_k) && \text{because } \{X_1, O_j, X_{j+1}\} \text{ and } \{X_1, O_k, X_{k+1}\} \text{ are lines} \\
&= 3X_1 + O + O_j + O_k \\
&= O + O_j + O_k && \text{Because } 3X_1 \equiv 0 \pmod 3
\end{aligned}$$

Therefore $\{O, O_j, O_k\}$ is a line. Because Olivia chose O before Xavier was forced to chose X, Olivia would have immediately lost with a line in $\mathcal{O}(m-1)$.

Thus, it is impossible for Xavier to have a line in $\mathcal{X}(m)$, since Olivia would immediately lose before he could choose to complete such a line. Therefore, Xavier cannot lose when following the strategy. $\square$

Lemma 3 (There are no ties)**.** *If Xavier consistently follows the strategy in Theorem 1, then the game cannot end in a tie.*

Proof. We first note that there are no ties in any 9-card plane $AG(2,3)$. That is, it is impossible to partition the 9 points into two sets, neither of which contains a line. In particular, any set of at least 5 points from a 9-card plane must contain a line. This may be demonstrated by brute force, or with an elegant counting argument such as that in [4].

After Xavier's third turn, the set of points selected is $S = \{X_1, O_1, X_2, O_2, X_3\}$. Note that, by following the strategy, S contains two non-parallel lines: $\{X_1, O_1, X_2\}$ and $\{X_1, O_2, X_3\}$. These two lines span an affine plane P.

The game may proceed in two ways:

1. Olivia may choose to only select points in P. There are no ties in P and by Lemma 2, Xavier cannot lose. Thus Olivia must eventually lose.
2. Olivia may choose to select some point outside of P. If Olivia does not lose, she will eventually run out of points outside of P, and therefore must choose a point from within P. As argued above, Olivia must then lose. Note that the points in P remain available for Olivia to choose, because Xavier will only choose a point in P if Olivia also chooses a point in P. This is because no line of $AG(d,3)$ contains two points in a plane and one point outside of a plane.

Either way, Olivia loses. □

Together, these lemmas provide a proof of Theorem 1:

Proof. (Of Theorem 1)

By Lemma 1, Xavier can follow the strategy. By Lemma 2, Xavier can never lose when following the strategy. Finally, by Lemma 3, the game cannot end in a tie. Therefore, Xavier (the first player) wins. □

5 Length of the game

Now that we know that Xavier will always win, a reasonable question is "how many moves are required for Xavier to win?" Without assuming rational play, a game could be as short as three rounds: Olivia could choose three cards which form a *set* and lose after move O_3. But assuming rational play, Olivia can survive much longer.

In this section, we seek to answer the question: "How long can Olivia force the game to continue?" Because there is some room for ambiguity, we provide the following precise definition:

Definition 1. *The* length *of a game of Anti-*Set *is the lowest index n such that $\mathcal{O}(n)$ contains a line.*

Thus, for example, the game played in Section 2 has length 4. Note that, because Olivia plays second, length can be interpreted as the number of complete rounds played before the game ends.

We also require the concept of a *cap*:

Definition 2. *A* cap *in $AG(d, q)$ is a set of points which contain no line. A* maximal cap *is a cap with the largest possible size for a given set of parameters d, q, and its size is denoted $m_2(AG(d, q))$.*

Example 4. *Every set of five points in $AG(2, 3)$ contains a line. Consider the result of the sample game from Section 2, shown in Figure 6. The four points marked X contains no line and therefore form a maximal cap in $AG(2, 3)$. The three marked O forms a cap which is not maximal. Thus $m_2(AG(2, 3)) = 4$.*

A long-standing question in finite geometry is to determine the size of a maximal cap. While a variety of bounds are known, no exact formula is known in general. For $q = 3$, some currently known values for $m_2(AG(d, 3))$ are summarized in Table 3. For more information see [6] and references therein.

d	1	2	3	4	5	6
$m_2(AG(d, 3))$	2	4	9	20	45	112

Table 3: Sizes of maximal caps for some small affine geometries.

In the language of affine geometry, Proposition 1 can be stated:

Proposition 1 (Pellegrino [5])**.** *In $AG(4, 3)$, $m_2(AG(4, 3)) = 20$.*

In other words, every set of 21 Set cards must contain a *set*.

Theorem 2. *The maximum possible length of a game of Anti-*Set *played on $AG(d, 3)$ is $m_2(AG(d, 3))$.*

Proof. Let $m = m_2(AG(d, 3))$. Xavier is always the first to have k points in hand for any k, and thus $\mathcal{X}(m + 1)$ (if the game lasts so long) must contain a *set*. However, by Theorem 1, Xavier cannot lose. Thus, Olivia's previous move, O_m, must have ended with $\mathcal{O}(m)$ containing a *set*. Thus the length of the game is at most m. □

As a corollary, the length of Anti-Set played on $AG(4, 3)$ is at most 20. Computational simulations for small d suggest that Olivia can always achieve this bound, but we are unable to prove this.

Next, we determine a lower bound on the length of the game. We do this by demonstrating a strategy for Olivia which guarantees the game to last for a certain number of moves.

Lemma 4. *Let $\{S_1, S_2, S_3\}$ be three parallel hyperplanes in $AG(d, 3)$. Then any line which intersects S_1 and S_2 also intersects S_3.*

Proof. This is a direct consequence of the structure of the underlying vector space. Note that $\{S_1, S_2, S_3\}$ partition the points of $AG(d, 3)$, and also note that a line $\ell = \{x, y, z\}$ contains exactly three points. Because ℓ and each S_i are cosets of a linear subspace, $S_i \cap \ell$ must be a linear subspace (or coset) as well. In $\mathbb{F}_3^d$, each such subspace contains 3^k points for some $k \geq 0$. Thus ℓ must intersect each hyperplane in 0, 1, or 3 points. If ℓ intersects both S_1 and S_2 in at least one point, then ℓ cannot intersect either in all 3 points. Thus ℓ intersects each of S_1 and S_2 in exactly 1 point, and so its 3rd point must be in the remaining point set, S_3. $\square$

Theorem 3. *Suppose Xavier and Olivia play Anti-*Set *on $AG(d, 3)$, $d \geq 1$. Then Olivia can force the game to have length at least* $2 + \sum_{i=1}^{d-1} m_2(AG(i, 3))$.

Proof. We proceed by induction. As a basis, consider Anti-Set played on $AG(2, 3)$. This is a 9-point plane. We saw in Section 2 that Olivia may extend the game to 4 rounds simply by not choosing her 3rd point to be on the line defined by the first two. Furthermore, $2 + m_2(AG(1, 3)) = 4$ since a cap in $AG(1, 3)$ consists of any two points on the only line. (Recall that the 4th round ends with Olivia choosing her 4th card, which must complete a line in $\mathcal{O}(4)$.)

Assume, for Anti-Set played in $AG(d-1, 3)$, that Olivia has a strategy which makes the length of the game $2 + \sum_{i=1}^{d-2} m_2(AG(i, 3))$. Then she can play on $AG(d-1, 3)$ for $1 + \sum_{i=1}^{d-2} m_2(AG(i, 3))$ rounds *without* losing. Let S_1 be a copy of $AG(d-1, 3)$ embedded as a hyperplane in $AG(d, 3)$, and let $\{S_1, S_2, S_3\}$ be the three hyperplanes parallel to S_1 in $AG(d, 3)$. Olivia proceeds as follows:

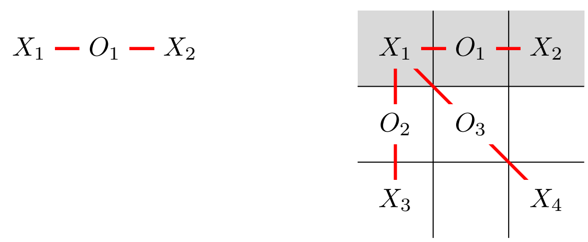
1. Inductively, Olivia plays for $1 + \sum_{i=1}^{d-2} m_2(AG(i, 3))$ moves entirely in S_1 without losing. Note that Xavier's moves also fall entirely in S_1.
2. Olivia then chooses the $m_2(AG(d-1, 3))$ points of a maximal cap entirely in S_2. Note that Olivia is free to choose these points, because Xavier's moves must now fall entirely in S_3 by Lemma 4.

This strategy describes Olivia's moves for $n = 1 + \sum_{i=1}^{d-1} m_2(AG(i, 3))$ rounds. Olivia never completes a line in $\mathcal{O}(n)$ by following this strategy. By our inductive assumption, no line exists within the subset of her moves falling in S_1. Because Olivia chooses the points of a cap in S_2, no line exists within her points in S_2. Finally, no line in $\mathcal{O}(n)$ can exist with one point in S_1 and another in S_2: By Lemma 4, the third point of such a line would be in S_3, but Olivia chooses no points in S_3.

Thus, Olivia does not lose by following the above strategy, and therefore Olivia can play for at least $2 + \sum_{i=1}^{n-1} m_2(AG(i,3))$ rounds. $\square$

Intuitively, Olivia's strategy works as follows: Olivia "fills up" a line with a cap, jumping up to a plane which she also fills with a cap. She continues jumping up to the next structure until she eventually runs out of room.

Example 5. *Theorem 3 is demonstrated in Figures 9 and 10. In Figure 9(a), play begins on a line (that is, $AG(1,3)$). In Figure 9(b), the line expands to a full plane (that is, $AG(2,3)$). Note that Olivia's play occurs only in the second row, which is one coset of the original line. Her two plays form a cap on this line. Similarly, Xavier's plays all occur in the third row, another coset of the original line.*



(a) Play in $AG(1,3)$. (b) Play expanded to $AG(2,3)$.

Figure 9: Visualization of Olivia's strategy from Theorem 3.

In Figure 10, play expands to cover the cosets of the plane. Figure 10 shows the original plane from Figure 9, now considered to be a subspace S_1. The other two planes in this figure are the cosets S_2 and S_3 of S_1. Note that Olivia plays only in coset S_2, and that her plays form a cap in $AG(2,3)$. Xavier's plays are forced into coset S_3.

6 Open problems

Naturally, a variation on Set such as Anti-Set leaves many open questions. The game Set has been widely studied, and several of the open problems below are based on generalizations and extensions already proposed for Set.

A more general category of games to which Anti-Set belongs could be named "Anti tic-tac-toe on a design." A t-(v, k, λ) *design* (or t-design) is a set of v points $\mathcal{P}$ together with a collection $\mathcal{B}$ of k-subsets of the points, called blocks, such that every t-subset of $\mathcal{P}$ appears in exactly λ blocks. The points and lines of $AG(d, 3)$ form an example of an *affine geometry design*. See [1] for further details. To play Anti tic-tac-toe on a given t-(v, k, λ) design, two players alternate selecting points of the design. The first player to select all points in

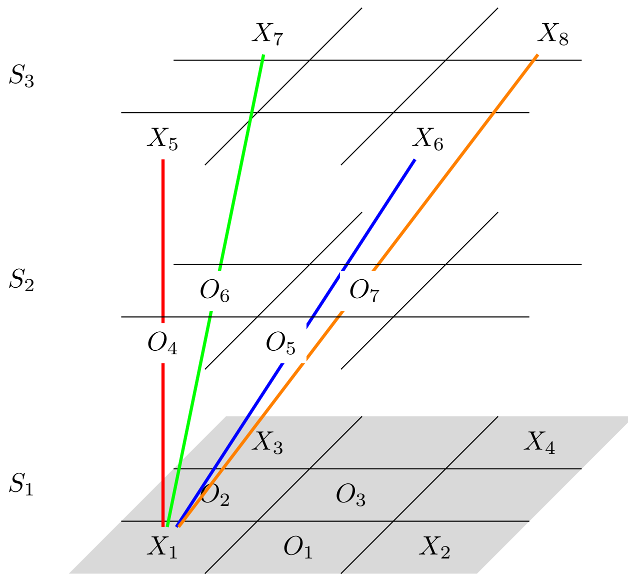


Figure 10: Continued visualization of Olivia's strategy from Theorem 3.

any block of the design loses. Thus the general question is: Is there a winning strategy for Anti tic-tac-toe on a design?

Specific instances of this general game will likely prove to be more tractable. For example:

- Play on non-ternary affine geometries, which are also examples of affine geometry designs. The Winning Strategy described in this paper depends heavily on working in $AG(d,3)$. Is it possible to have $q > 3$? The largest difference here is that lines now have more than 3 points, opening the possibility that Olivia plays on a point which completes a line, leaving Xavier unable to "follow" Olivia.

- Play on a projective geometry. It is possible to play Anti-Set on a projective geometry $PG(d,q)$? The set of points and k-dimensional subspaces of $PG(d,q)$ form a *projective geometry design.* (For information about "Projective Set", see [3].)

- Play on Steiner Triple Systems. This is a name given to the category of 2-$(v,3,1)$ designs. In this category, every pair of points determines a unique line, and every line has 3 points. This includes two key geometric features that figures in the strategy for Anti-Set.

Other open problems involve changing the parameters of play for Anti-Set:

- Play with 3 or more players. This must considerably change the strategy. Under the Winning Strategy described here, it would be possible for one player to “block” another player’s necessary move.

- Recovering from an error. If Xavier does not follow the winning strategy, when is it possible for Olivia to win? Is it possible for Xavier to recover from this error, and if so, under what conditions?

Finally, we believe that Theorem 3 can be improved:

- Determine a strategy for Olivia which always forces a game length of $m_2(AG(d,3))$ rounds, thus improving on Theorem 3.

7 Acknowledgments

The authors would like to thank the Mathematics Center for Educational Programs (MathCEP) at the University of Minnesota for supporting this research. This work was completed as part of the University of Minnesota Talented Youth in Mathematics (UMTYMP) undergraduate research program in summer 2013.

We would also like to thank the anonymous referee for their very helpful comments and suggestions.

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