What Mathematicians Do

When I first started teaching, I thought I knew exactly what students needed to do well in mathematics. I would give them carefully planned, clear lectures, and assign them homework problems to help them put the concepts to work. The underlying assumption was that if they were able to understand everything I lectured on, and if they practiced using these concepts, they would be successful.

As it turned out, I was wrong. It was common for students to come to me for help, extremely frustrated, saying something like "It was so clear when you wrote it on the board, but when I do my homework it's like I've never seen this before in my life". It was difficult for my students to transform my "lecture reasoning" into their own reasoning, *despite the fact that they thought my lectures were clear*. Even worse, it often happened that the homework that was supposed to solidify their understanding of mathematics was merely an opportunity to practice their mistakes.

My approach is to create an environment where students participate in some of the same activities that mathematicians do. This is not an original idea; in fact, I came upon it while I was studying a philosophy of mathematics called Social Constructivism (see [Ern91] or [Ern98]). Basically, it says that mathematics exists in and is created by the culture of mathematicians. A subversive way of saying this is that math is just a bunch of stuff people make up. But it cannot be just anything; it must be accepted and appreciated by the mathematical community.

There are two concepts embedded in the phrase "accepted and appreciated by the culture of mathematicians". One is the correctness of a given piece of mathematics. The other is how interesting the piece of mathematics is. Both of these aspects are important in mathematics. More importantly, both are subject to the whims of the community of mathematicians (even the issue of correctness). This makes the values and activities of mathematicians very important. It also makes the interactions between mathematicians important.

So what do mathematicians do? First and foremost, we make up and solve problems. They must be problems that have not been solved before, and they must be "interesting" problems (as determined by the community). And often we work in groups as collaborators. So in all of my classes, I put an emphasis on solving problems, and working collaboratively. Especially in calculus level and below, I spend less time talking at the board, and more time having students doing problems in groups. While they are toiling, I will wander among the groups to help them out. What this usually means is that I walk around and ask annoying questions. I will make them explain their reasoning, help them discover their mistakes, and help get them started in the right direction. The underlying philosophy is that they work as part of a community, and that they can create some of the mathematics within that community. Working together, students will correct each other, challenge each other, and bounce ideas off each other, just as mathematicians do. A side benefit is that they get some experience working problems correctly, as opposed to my old homework where they may or may not do the problems correctly and often have no way of knowing the difference. In class, correctness is negotiated between the students, often through me. Furthermore, I get to see what kind of reasoning (good and bad) that students are bringing to class. This helps me tailor my teaching to their strengths and weaknesses.

Once a mathematician solves the problem, he or she must write a paper that explains the work to the mathematical community. This is no small matter to mathematicians. Even if a mathematician's work is good, it may be misunderstood or ignored if it is not intelligible to the rest of the mathematical community. To this end, I give my students tough problems to turn in. I try to make them "interesting" (which usually means hard). I encourage them to work with others, and I demand that they write it up clearly and in complete sentences. If they do a lousy job of explaining a problem, I will take off points, even if their solution is otherwise correct. My justification to them is that the ability to explain technical information is important (which is true), but that is not the primary reason I require the writing. When they have to explain to me how they get their solution, they are forced to explain to themselves how they get their solution. If they make mistakes, they are more likely to pick them up. They are more likely to take care in solving the problem, and less likely to rush through it on the way to the next problem. And they make considerably fewer of those awful algebra mistakes that tempt the best of us to give up our profession. They simply take more care. The result is some solid writing and beautiful mathematics.

I do not want to understate the importance of lecturing, but I want to make two things clear. First, not all the mathematics students learn comes from lectures. Much (perhaps most?) of it comes from the hours of study, reflection, consultation with other students, and focused struggles with hard problems. Nothing we say in class can do this work for them, but some of this work can be done in class *by the students*, in an environment where they can ask questions precisely when they are having trouble. Second, as I mentioned before, "clear" lectures alone are not sufficient. They must not only transmit knowledge and understanding; they must also give insight into how mathematicians think. I believe that this is where we miss our greatest opportunity to inspire our students. The solution of one problem leads to more questions, more problems. Each discovery leads to more facts to discover. It is this evolution that makes some theorems more important than others and that generates the creation of the different areas of mathematics. This is where the definitions come from; they are not arbitrary.

In my linear algebra class, I have spent an entire day on the definition of isomorphisms of vector spaces. That may seem a bit extreme. I probably could have just written the definition and told them that isomorphisms tell us when two vector spaces are "the same" because they "preserve" addition and scalar multiplication (10 minutes, tops). As mathematicians, we have internalized that understanding. But our students have a tougher time connecting these vague explanations with the precise but mysterious definition. More importantly, they have no idea how anyone could have come up with that definition. Even worse, they probably never even thought to care about how the definition was derived. That is not the students' failure. That is our failure to teach them one of the most fundamentally important values of our academic culture. Our lectures should reflect that culture. For instance, when I do research, I often study examples, find patterns and counterexamples, and then make definitions to exclude the counterexamples. What results (sometimes) is a nice theorem. When I teach, I start a topic with an example (maybe more) that leads to a question. That is our "research problem". We work on this problem until we have a solution. Oftentimes, we find that the solution requires proving some obscure fact that, by itself, seems rather weird. This, of course, is a lemma. Then we reorganize our work and write a proof of the theorem. This gives students a genuine (if secondhand) mathematical experience. In addition, they see that the definitions and lemmas and theorems serve the *solution of a problem*. They are not ends in themselves. Incidentally, after a while, my students will spontaneously include lemmas in their homework solutions (I don't explicitly suggest that they do so). I did not start using lemmas until graduate school (how sad!). Of course, I do not keep all the fun to myself. I will occasionally put a problem on their homework where they work examples, guess the general pattern, and then prove a result based on their observations. It is not exactly research, but it is the type of thing a research mathematician would do.

So far, I have made a big deal out of the importance of solving problems. What I have not yet addressed is the creation of problems. Who makes them up? Who decides what problems are important and which problems are not? Once again, the community decides. In any case, I address this issue in my upper level classes. In addition to the four weekly problems that they have to solve, students must also pose a problem they are curious about. If I see a problem that is interesting and challenging, I put it on a list of "Open Problems" that students can work on for modest extra credit. I tried this in a topology class and an abstract algebra class, and I received some very interesting problems. No one has solved any yet (except me), but I think they are an interesting and important part of the class.

There is one last activity that I believe mathematicians participate in, one that tends to hide beneath the surface. That is the creation of mental models to help us understand the problems we are working on. Most of the time, especially on really hard problems, there is a space between the way mathematicians think about the problem and the rhetorical constructions we make to write a formal proof. The two are, of course, related. We may even convince ourselves that they are the same thing. But I do not believe they are. Mathematics educators recognize this reality, and use these models to help train teachers. I have recently begun wondering why we do not continue to emphasize mental model-building in all mathematics courses. For example, when I introduce sets in my introductory proof course, I use two models. One I call the bag model, which is thinking of a set as a bag with objects inside. This is a nice model, because the bag and the objects are both crucial to defining a set. It is especially helpful to distinguish between the sets $\{1, 2, 3\}$ and $\{\{1, 2\}, 3\}$. It can also be thought of as the model underlying the roster method of expressing a set. The other model I like is what I call the Gatekeeper model. This is a rule or collection of rules that must be satisfied to be in a given set. This model is particularly useful in helping students understand how to prove set identities, and it can be thought of as the model underlying the set-builder method of expressing a set. I continue to look for new ways to introduce models into my courses.

There is a lot of the mathematical culture I am missing, and a lot of "community building" that I have not fully addressed. For instance, could I use office hours to get students to interact and build collaborative relationships? Could I put students in study groups according to where they live? Obviously, most of the mathematical activity occurs outside the classroom. How can I create fertile ground for this activity to grow? Dang, I thought I had it all figured out.

References

[Ern91] P. Ernest, *The Philosophy of Mathematics Education*, Falmer Press, London, 1991.

[Ern98] ______, *Social Constructivism as a Philosophy of Mathematics (Electronic Book)*, State University of New York Press, Albany, New York, 1998.