Two-Path Convexity and Bipartite Tournaments of Small Rank

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Abstract

We study two-path convexity in bipartite tournaments. For a bipartite tournament, we obtain both a necessary condition and a sufficient condition on the adjacency matrix for its rank to be two. We then investigate 4-cycles in bipartite tournaments of small rank. We show that every vertex in a bipartite tournament of rank two lies on a four cycle, and bipartite tournaments with a maximum number of 4-cycles do not necessarily have minimum rank.

1 Introduction

Convexity has been studied in many contexts. For graphs and digraphs, the convex subsets of the vertex set are usually defined using some set of paths within the graph. More precisely, if $T = (V, E)$ is a (directed) graph and $\mathcal{P}$ a set of (directed) paths in T, a subset $A \subseteq V$ is $\mathcal{P}$-*convex* if, whenever $v, w \in A$, any (directed) path in $\mathcal{P}$ that originates at v and ends at w only involves vertices in A. Given a subset $S \subseteq V$, the *convex hull of* S, denoted $C(S)$, is defined to be the smallest convex subset containing S.

Several types of convexity have been studied in the literature. If $\mathcal{P}$ is the set of geodesics in T, the resulting convex sets are said to be *geodesically convex*. Geodesic convexity was introduced by F. Harary and J. Nieminen in [HN81] and also studied in [CFZ02] and [CCZ01]. When $\mathcal{P}$ is the set of all chordless paths, we have *induced path convexity* (see [Duc88]). Other types of convexity include *path convexity* (see [Pfa71] and [Nie81]) and *triangle path convexity* (see [CM99]).

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The most frequently studied parameters in convexity theory are the Helly number, the Radon number, and the Caratheodory number, which are based on notions of independence (see [vdV93]). These are all bounded above by the *rank*, which is defined as follows. A set is $F \subseteq V$ *convexly independent set* if $x \notin C(F - \{x\})$ for all $x \in F$. The *rank* $d(T)$ is the maximum size of a convexly independent set. Rank is also a measure of how computationally difficult it is to construct the convex subsets of a given convexity space. Specifically, it is an upper bound on the number of vertices required to generate a convex subset.

Convexity in tournaments is studied in [Var76], [EFHM72], [EHM72], and [Moo72]. Research in this area has focused on *two-path convexity*, where $\mathcal{P}$ is the set of all directed 2-paths. Other types of convexity lead to less interesting convex structures in most tournaments. For example, if all directed paths of length three or less are allowed, then the only convex subsets of a strong tournament are $\emptyset$ and the entire vertex set.

Recall that T is a *multipartite tournament* if one can partition V into n partite sets $P_1, P_2, \ldots, P_n$, $n \geq 2$ such that for all $i \neq j$ there is precisely one arc between each vertex in P_i and each vertex in P_j and no arcs between vertices in the same partite set. When $n = 2$, T is a *bipartite tournament.* As with tournaments, the study of two-path convexity in multipartite tournaments leads to rich convex structures, see [PWW08], [PWW06], and [PWW]. Since the results in this paper deal with two-path convexity in bipartite tournaments all references to convexity will mean two-path convexity.

The work in this paper is motivated by earlier work of Varlet in [Var76] where it is shown that all tournaments with at least two vertices have rank 2. Since numerous interesting results have been proven about cycles in tournaments (e.g., [GM72]), we consider whether those results are related to the fact that tournaments have rank 2 by studying bipartite tournaments with small rank. We first look at bipartite tournaments T of rank 2. Theorem 2.6 gives us a necessary and sufficient condition for T to have rank 2 and Theorem 2.8 gives a condition on the columns of the adjacency matrix that is sufficient for T to have rank 2. This result provides a mechanism to generate several examples of bipartite tournaments of rank 2.

We then investigate the connection between small rank and the number of 4-cycles present in T. In Theorem 3.4, we prove that if a bipartite tournament has rank 2, then every vertex of T lies on a 4-cycle. We then consider bipartite tournaments where one partite set has two vertices. Given partite sets with two and n vertices, we classify all bipartite tournaments of minimum rank (Theorem 3.6). We also show that if such bipartite tournaments have a maximum number of 4-cycles, then they do not have minimum rank unless $n \leq 4$ (Corollary 3.7).

Let $T = (V, E)$ be a digraph with vertex set V and arc set E. We denote an arc $(v, w) \in E$ by $v \to w$ and say that v dominates w. If $U, W \subseteq V$, then we write $U \to W$ to indicate that every vertex in U dominates every vertex in W. Two vertices are clones if they have identical insets and outsets, and T is *clone-free* if it has no clones. If $u, v, w \in V$ with $u \to v \to w$, we say that v *distinguishes* the vertices u and w. Note that in a clone-free multipartite tournament, for every pair of vertices u, w in the same partite set there is at least one vertex (not in that partite set) that distinguishes u and w. If A and B

are convex, we denote the convex hull of $A \cup B$ by $A \vee B$. If $v, w \in V$, we drop the set notation and write $\{v\} \vee \{w\}$ as $v \vee w$.

To facilitate our study of bipartite tournaments, it will be helpful to study their adjacency matrices. In the case of a bipartite tournament, however, the adjacency matrix can be represented more compactly. Let $\{v_1, \cdots, v_k\}$ and $\{w_1, \cdots, w_\ell\}$ be the partite sets of a bipartite tournament T. For each i and j with $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let $m_{i,j} = 1$ if $v_i \to w_j$ and let $m_{i,j} = 0$ otherwise. We will call $M = (m_{i,j})$ the *matrix of* T, and we say that T is the *bipartite tournament induced by* M. Notice that v_i distinguishes w_j and w_k if and only if $m_{i,j} \neq m_{i,k}$ and w_i distinguishes v_j and v_k if and only if $m_{j,i} \neq m_{k,i}$. In addition, identical rows or columns of the matrix of T correspond to clones.

2 Bipartite Tournaments of Rank 2

In this section we consider bipartite tournaments of rank 2 and give necessary and sufficient conditions under which a bipartite tournament has rank 2. Throughout this section, $T = (V, E)$ is a bipartite tournament with partite sets $P_1 = \{x_1, \cdots, x_m\}$ and $P_2 = \{y_1, \cdots, y_n\}$.

Lemma 2.1. Suppose T is a bipartite tournament of rank 2.

1. If $|P_2| \leq 2$, then each pair of vertices in P_1 is distinguished by each vertex in P_2.

2. If $|P_2| \geq 3$, then each pair of vertices in P_1 is distinguished by at least two vertices in P_2.

3. If $|P_2| \geq 2$, then there is no vertex $v \in P_1$ with either $v \to P_2$ or $P_2 \to v$.

Proof. Let $v, w \in P_1$. If v and w are clones, then it is clear that v, w, and any vertex in P_2 form a convexly independent set. Now suppose that v and w are distinguished by a unique vertex $x \in P_2$. For (1) and (2), it suffices to show that $|P_2| = 1$. If $|P_2| \geq 2$, then there is some $y \in P_2 - \{x\}$. We claim that $\{v, w, y\}$ is convexly independent. Clearly, $v \notin w \vee y = \{w, y\}$ and $w \notin v \vee y = \{v, y\}$. Also, $v \vee w = \{v, w, x\}$, since v and w are distinguished only by x. Thus, $y \notin v \vee w$, and so $\{v, w, y\}$ is convexly independent, a contradiction.

For (3), if such a v existed, then v and any two vertices in P_2 form a convexly independent set, so the result follows. □

Since every pair of vertices in the same partite set must be distinguished by at least one vertex, we have the following.

Corollary 2.2. Every bipartite tournament of rank 2 is clone-free.

It is not generally true that multipartite tournaments of rank 2 are clone-free. For instance, consider the tripartite tournament whose vertex set V consists of partite sets P_1, P_2, and P_3 having two vertices each, with arcs given by $P_1 \to P_2 \to P_3 \to P_1$. Any set of three vertices must then contain two vertices x and y in different partite sets. But then $x \vee y = V$, which contains the third vertex. Thus, $d(T) = 2$. But each vertex is a clone of the other vertex in its partite set, so T is not clone-free.

If we write Lemma 2.1 and Corollary 2.2 in terms of the matrix of T, we obtain the following.

Corollary 2.3. Suppose T is a bipartite tournament with rank 2 and matrix M, and assume that each partite set of T has at least *two* vertices. Then

1. The rows of M are distinct and the columns of M are distinct.
2. Each pair of rows of M differ in at least two positions. That is, for each $1 \leq i \neq j \leq m$, there exists k and l, $1 \leq k, l \leq n$ such that $m_{i,k} \neq m_{j,k}$ and $m_{i,l} \neq m_{j,l}$.
3. Each pair of columns of M differ in at least two positions. That is, for each $1 \leq k \neq l \leq n$, there exists i and j, $1 \leq i, j \leq m$ such that $m_{i,k} \neq m_{i,l}$ and $m_{j,k} \neq m_{j,l}$.
4. No row or column of M consists entirely of $0s$ or $1s$.

Now we are able to characterize bipartite tournaments of rank 2 with small partite sets.

Corollary 2.4. If T is a bipartite tournament of rank 2 such that one of its partite sets has at most two vertices, then T can be represented by one of the following matrices: $[1]$, $\begin{bmatrix} 1 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Proof. If one of the partite sets has at least three vertices, then it is impossible for each of these vertices to be distinguished by a single vertex in the partite set with at most two vertices. This violates Lemma 2.1(1), and so each partite set can have at most two vertices. The result follows easily from this. □

To determine which bipartite tournaments have rank 2, we can ask which binary matrices represent bipartite tournaments of rank 2. In the case of $|P_1|$ or $|P_2| = 3$, there are no such matrices. However, there are such matrices satisfying Corollary 2.3 (and thus Lemma 2.1). This indicates that Corollary 2.3, while being necessary for rank 2, is not sufficient.

Theorem 2.5. There are no bipartite tournaments of rank 2 with three vertices in one partite set. Up to isomorphism there is a unique bipartite tournament with three vertices in each partite set that satisfies the conclusions of Corollary 2.3.

Proof. Suppose that $|P_2| = 3$ and let M be the matrix of T. By Corollary 2.4, $|P_1| \geq 3$. By Corollary 2.3(4), the only possible columns for M are 100, 010, 001, 110, 101 and 011. It is then easy to check that the only matrices that satisfy the remaining conditions of Corollary 2.3 up to reordering of rows and columns are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

In each case, $\{x_1, x_2, x_3\}$ is convexly independent, so $d(T) = 3$. Note that these bipartite tournaments are isomorphic. □

The following theorem gives necessary and sufficient conditions on T to guarantee rank 2.

Theorem 2.6. Let $T = (V, E)$ be a bipartite tournament. Then $d(T) = 2$ if and only if for all vertices x and y in the same partite set, we have $x \vee y = V$.

Proof. In view of Corollary 2.4, we may assume each partite set of T has at least three vertices. Assume $d(T) = 2$ and x and y are in the same partite set of T. Without loss of generality we may assume that x and y are in partite set P_1. If $z \in P_2 - (x \vee y)$ then $\{x, y, z\}$ is convexly independent and $d(T) \geq 3$. Thus $P_2 \subseteq x \vee y$. If $z \in P_1$ then by Lemma 2.1(3), there exist vertices $u, v \in P_2$ with $u \to z \to v$. Since $P_2 \subseteq x \vee y$, we have $u, v \in x \vee y$, and so $z \in x \vee y$. The converse is clear. □

Given a bipartite tournament T, the columns of the matrix of T form a subset of $\mathbb{Z}_2^m$. Theorem 2.6 gives a method for finding vector subsets of $\mathbb{Z}_2^m$ that represent bipartite tournaments of rank 2. Let us introduce some notation. Given a vector $\mathbf{v} = v_1 v_2 \cdots v_m \in \mathbb{Z}_2^m$, we define the complement vector $\mathbf{v}_c = v_1' v_2' \cdots v_m' \in \mathbb{Z}_2^m$ to be the vector with $v_i' = v_i + 1$ for all i. With the notation introduced earlier, if y_i and y_j are in partite set P_2 of a bipartite tournament T and the corresponding columns c_i and c_j in the matrix of T are complementary then $P_1 \subseteq y_i \vee y_j$ as each vertex in P_1 distinguishes y_i and y_j. This observation motivates the following.

Definition 2.7. A subset $S \subset \mathbb{Z}_2^m$ is called rank 2 complementary (RTC) when the following hold.

1. $\mathbf{v} \in S \Rightarrow \mathbf{v}_c \in S$.
2. For each $1 \leq i \neq j \leq m$, there exists $\mathbf{v} \in S$ with $v_i \neq v_j$.
3. Each pair of vectors in S differs in at least two components.
4. $00 \cdots 0, 11 \cdots 1 \notin S$.

Except for (1), these conditions appear almost identical to those in Corollary 2.3. In fact, (3) and (4) are identical to Corollary 2.3(3) and (4). In addition, while (2) looks like a slightly weaker condition than Corollary 2.3(2), it is not. If $\mathbf{v}$ satisfies (2) in the definition, so does $\mathbf{v}_c$, so there are at least two vectors with $v_i \neq v_j$. We have the following.

Theorem 2.8. Let T be a bipartite tournament and let $S \subseteq \mathbb{Z}_2^m$ be the set of columns of the matrix of T. If S is RTC, then $d(T) = 2$.

Proof. If $x_i, x_j \in P_1$ then by Definition 2.7(2) there is a $y_k \in P_2$ such that $a_{i,k} \neq a_{j,k}$. Then $y_k \in x_i \vee x_j$. By Definition 2.7(1), there is a $y_\ell \in P_2$ such that c_k and c_l are complements. Then $a_{i,l} \neq a_{j,l}$ and $y_\ell \in x_i \vee x_j$. Since c_k and c_l are complements, $P_1 \subseteq y_k \vee y_\ell \subseteq x_i \vee x_j$. It then follows from Definition 2.7(4) that $x_i \vee x_j = V$.

Let $y_k, y_\ell \in P_2$. By Definition 2.7(2), $y_k \vee y_\ell$ contains at least two vertices $x_i, x_j \in P_1$. By the above argument, $V \subseteq x_i \vee x_j \subseteq y_k \vee y_\ell$ so $y_k \vee y_\ell = V$. Thus $d(T) = 2$ by Theorem 2.6. □

Example 2.9. This gives us a way to produce several examples of rank 2 bipartite tournaments. For instance, when $m = 4$, we can have $S = \{1000, 0111, 0100, 1011, 0010, 1101, 0001, 1110\}$, which gives us a bipartite tournament of rank 2 with partite sets of order 4 and 8.

We can get some examples with partite sets of odd cardinality as well. In the case $m = 5$, we can use $S = \{11000, 00111, 01100, 10011, 00010, 11101\}$ to get a bipartite tournament with partite sets of order 5 and 6.

Example 2.10. For some infinite classes of examples, let m be even, and let S be the set of all vectors with equal numbers of 0's and 1's. Since the degrees are balanced, S satisfies Definition 2.7(1). The other conditions follow similarly.

A set of examples of this is the following. Let T be the tournament with partite sets $P_1 = \{x_1, \cdots, x_{2r}\}$, $P_2 = \{y_1, \cdots, y_{2r}\}$. For $i \leq j$, we let $x_{2i-1} \to y_{2j-1} \to x_{2i} \to y_{2j} \to x_{2i-1}$. For $i > j$, we let $y_{2j-1} \to x_{2i-1} \to y_{2j} \to x_{2i} \to y_{2j-1}$. Since the $(2j-1)$st and $(2j)$th columns of the matrix of T are complements it is not difficult to verify that the set of columns of the matrix of T is RTC and thus $d(T) = 2$. Note that T is also regular.

3 Four-cycles and Small Rank

The examples of bipartite tournaments of rank 2 given in the previous section all contain relatively large numbers of 4-cycles. The reason is related to Definition 2.7(1). If $\mathbf{v}$ is a vector with $v_i = 0$, and $v_j = 1$, then $\mathbf{v}' = \mathbf{v_c}$ has entries $v'_i = 1$ and $v'_j = 0$. The vertices represented by $\mathbf{v}$, $\mathbf{v}'$, and the ith and jth rows of the vectors then form a 4-cycle.

In this section, we show that every vertex of a bipartite tournament of rank 2 with at least four vertices is a part of a 4-cycle. Thus all rank 2 bipartite tournaments have relatively large numbers of 4-cycles. We also carefully examine bipartite tournaments with two vertices in one partite set and a maximum number of 4-cycles.

Let u and v be vertices in the same partite set of T. We define

$$\begin{aligned} T^{u,v}_{10} &= \{x \in V : u \to x \to v\} \\ T^{u,v}_{01} &= \{x \in V : v \to x \to u\} \\ T^{u,v}_{11} &= \{x \in V : u \to x, v \to x\} \\ T^{u,v}_{00} &= \{x \in V : x \to u, x \to v\} \end{aligned}$$

The connection with 4-cycles is given by the following.

Lemma 3.1. Two vertices u and v in a bipartite tournament T lie on a common 4-cycle if and only if $T_{10}^{u,v}, T_{01}^{u,v} \neq \emptyset$. In fact, the number of 4-cycles containing u and v is $|T_{10}^{u,v}| \cdot |T_{01}^{u,v}|$.

The role of $T_{00}^{u,v}$ and $T_{11}^{u,v}$ is important in the proof of the main result.

Lemma 3.2. Let T be a bipartite tournament of rank 2 with at least 4 vertices and let $u, v \in V$ be in the same partite set. If u and v are not part of the same 4-cycle then $T_{11}^{u,v}, T_{00}^{u,v} \neq \emptyset$.

Proof. Let P_1 be the partite set containing u and v and let P_2 be the other partite set. Assume $T_{11}^{u,v} = \emptyset$. Since u and v are not in a common 4-cycle, then, without loss of generality, we may assume $T_{01}^{u,v} = \emptyset$. Thus, $P_2 = T_{10}^{u,v} \cup T_{00}^{u,v}$ so $P_2 \to v$. By Corollary 2.4, each partite set of T must have at least three vertices, contradicting Lemma 2.1(3). The proof for $T_{00}^{u,v}$ is similar. □

For the proof of our next theorem, we require the following notation from [HW96].

Definition 3.3. Let $U \subseteq V$, and define $C_k(U)$ inductively by

$$C_0(U) = U, \quad C_k(U) = C_{k-1}(U) \cup \{w \in V : x \to w \to y \text{ for some } x, y \in C_{k-1}(U)\}, k \geq 1$$

Note that $C_\infty(U)$ is the convex hull of U.

We can now prove the following.

Theorem 3.4. If T is a bipartite tournament with at least four vertices and $d(T) = 2$, then every vertex of T is part of a 4-cycle.

Proof. Since $|V| \geq 4$, Corollary 2.4 implies that each partite set of T has at least two vertices. Let $u \in V$ and assume u is not part of a 4-cycle. Let P be the partite set containing u and let $v \in P - \{u\}$. By Lemma 3.2, $T_{11}^{u,v}, T_{00}^{u,v} \neq \emptyset$. Without loss of generality, we may assume $T_{01}^{u,v} = \emptyset$. Let $U = \{u, v\}$. By Theorem 2.6, $u \vee v = V$. Let k be minimal so that $b \in C_k(U)$ for some $b \in T_{00}^{u,v}$. Then there exist $x, y \in P \cap C_{k-1}(U)$ such that $x \to b \to y$. Note that $x \neq u, v$. Thus, there exist $z, w \in C_{k-2}(U)$ such that $z \to x \to w$. By the minimality of k, we cannot have $z, w \in T_{00}^{u,v}$, which forces $z, w \in T_{11}^{u,v} \cup T_{10}^{u,v}$, and so $u \to z$. We then have $u \to z \to x \to b \to u$, and so u is part of a 4-cycle. □

Theorem 3.4 suggests that there may be a correlation between the number of 4-cycles and the rank of a bipartite tournament. Consequently, it makes sense to ask if maximizing the number of 4-cycles minimizes the rank. With this in mind, let T be a bipartite tournament with partite sets of order 2 and n. We have partite sets $P_1 = \{x_1, x_2\}$ and P_2 with disjoint subsets $T_{01}^{x_1,x_2}$, $T_{10}^{x_1,x_2}$, $T_{00}^{x_1,x_2}$, and $T_{11}^{x_1,x_2}$. To simplify notation we write T_{ij} instead of $T_{ij}^{x_1,x_2}$ and set $t_{ij} = |T_{ij}|$ for $i, j \in \{0, 1\}$. If T has a maximum number of 4-cycles, then $T_{00} = T_{11} = \emptyset$ and T_{10} and T_{01} are as close to the same order as possible. Then the union of the larger of T_{01} and T_{10} and either $\{x_1\}$ or $\{x_2\}$ is a maximum convexly independent set in V. Regardless of whether n is even or odd, we get the following.

Theorem 3.5. Let T be a bipartite tournament with a maximum number of 4-cycles. Then $d(T) = \lfloor \frac{n+1}{2} \rfloor + 1$.

To minimize $d(T)$, we must consider each possible convexly independent set. They are precisely the nonempty subsets of $T_{10} \cup T_{11}$, $T_{10} \cup T_{00}$, $T_{01} \cup T_{11}$, $T_{01} \cup T_{00}$, $T_{01} \cup \{x_1\}$, $T_{01} \cup \{x_2\}$, $T_{10} \cup \{x_1\}$, $T_{10} \cup \{x_2\}$, $T_{00} \cup T_{11}$, $T_{00} \cup \{x_1, x_2\}$, $T_{11} \cup \{x_1, x_2\}$, $T_{00} \cup \{x_1, y\}$, $T_{00} \cup \{x_2, z\}$, $T_{11} \cup \{x_2, y\}$, $T_{11} \cup \{x_1, z\}$, $T_{00} \cup \{y, z\}$ and $T_{11} \cup \{y, z\}$ where $y \in T_{01}$ and $z \in T_{10}$.

Note that since $|P_2| = n$, it follows that one of $T_{10} \cup T_{11}$ and $T_{01} \cup T_{00}$ must have at least $\frac{n}{2}$ vertices. Thus, $d(T) \geq \max(2, \lfloor \frac{n+1}{2} \rfloor)$. Assume $n \geq 3$ and let $r = \lfloor \frac{n+1}{2} \rfloor$. In order to be of minimum rank, we must have one of $t_{10} + t_{00}$ and $t_{01} + t_{11}$ equal to r (otherwise, one or the other is larger than r). If necessary, we can reverse the arcs of T (which does not change the convex subsets of T), so we can assume $t_{10} + t_{00} = r$, which means $t_{01} + t_{11} = n - r$.

By similar reasoning, one of $t_{10} + t_{11}$ and $t_{01} + t_{00}$ must be r. We take each of these cases in turn. If $t_{10} + t_{11} = r$, then we subtract $t_{10} + t_{00} = r$ from this to get $t_{11} = t_{00} = s$. We then get $t_{10} = r - s$ and $t_{01} = n - r - s$. Since $T_{00} \cup T_{11}$ is a convexly independent set, then $t_{00} + t_{11} \leq r$ so $2s \leq r$ and $s \leq \frac{r}{2}$. Since $T_{00} \cup \{x_1, y\}$ is a convexly independent set for $y \in T_{01}$ then $s \leq r - 2$. Finally, $T_{10} \cup \{x_1\}$ is a convexly independent set, so $s \geq 1$. Putting all of this together we have that $1 \leq s \leq \min(\frac{r}{2}, r - 2)$. This also implies that $r \geq 3$, so $n \geq 5$.

A similar argument in the case $t_{01} + t_{00} = r$ yields $t_{00} = s$, $t_{11} = n - 2r + s$, $t_{01} = t_{10} = r - s$, and $1 \leq s \leq \min(\frac{r}{2}, r - 2)$ when n is even and $1 \leq s \leq \min(\frac{r+1}{2}, r - 2)$ when n is odd. Again, $r \geq 3$, so $n \geq 5$. Putting this together and taking into account bipartite tournaments obtained by reversing all arcs, we obtain the following.

Theorem 3.6. Let T be a bipartite tournament with partite sets $P_1 = \{x_1, x_2\}$ and $P_2 = T_{01} \cup T_{10} \cup T_{00} \cup T_{01}$, with $|P_2| \geq 5$. Then $d(T) \geq \lfloor \frac{n+1}{2} \rfloor$ and if $d(T) = \lfloor \frac{n+1}{2} \rfloor$, then

1. if n is even, then T is isomorphic to the bipartite tournament with $t_{00} = t_{11} = s$ and $t_{01} = t_{10} = \frac{n}{2} - s$, where $1 \leq s \leq \min(\frac{n}{4}, \frac{n-4}{2})$.

2. if n is odd, then T is isomorphic to one of the following bipartite tournaments:
   (a) $t_{00} = t_{11} = s$, $t_{10} = \frac{n+1}{2} - s$, $t_{01} = \frac{n-1}{2} - s$ and $1 \leq s \leq \min(\frac{n+1}{4}, \frac{n-3}{2})$
   (b) $t_{00} = t_{11} = s$, $t_{01} = \frac{n+1}{2} - s$, $t_{10} = \frac{n-1}{2} - s$ and $1 \leq s \leq \min(\frac{n+1}{4}, \frac{n-3}{2})$
   (c) $t_{00} = s, t_{11} = s - 1$, $t_{10} = t_{01} = \frac{n+1}{2} - s$ and $1 \leq s \leq \min(\frac{n+3}{4}, \frac{n-3}{2})$
   (d) $t_{11} = s, t_{00} = s - 1$, $t_{10} = t_{01} = \frac{n+1}{2} - s$ and $1 \leq s \leq \min(\frac{n+3}{4}, \frac{n-3}{2})$

When $n = 1$ or $n = 2$ there are bipartite tournaments T with rank 2. They are $x_1 \to z \to x_2$ and $x_1 \to z \to x_2 \to y \to x_1$. When $n = 3$ or $n = 4$ there are no bipartite tournaments T with rank 2, but there are a number of tournaments with rank 3. These are precisely the bipartite tournaments T with $t_{00} \leq 1$, $t_{11} \leq 1$, $t_{01} \leq 2$ and $t_{10} \leq 2$. This and Theorem 3.5 give us the following.

Corollary 3.7. Let T be a bipartite tournament having partite sets with two and n vertices, $n \geq 1$. Then T can have a maximum number of 4-cycles and minimum rank subject to having partite sets with two and n vertices, respectively, if and only if $n \leq 4$.

Thus, while having a large number of 4-cycles does guarantee small rank, it does not correspond to minimum rank.

References

[CCZ01] G. Chartrand, A. Chichisan, and P. Zhang, *On convexity in graphs*, Cong. Numer. **148** (2001), 33–41.

[CFZ02] G. Chartrand, J.F. Fink, and P. Zhang, *Convexity in oriented graphs*, Discrete Applied Math. **116** (2002), 115–126.

[CM99] M. Changat and J. Mathew, *On triangle path convexity in graphs*, Discrete Math. **206** (1999), 91–95.

[Duc88] P. Duchet, *Convexity in graphs II. Minimal path convexity*, J. Combin. Theory, Ser. B **44** (1988), 307–316.

[EFHM72] P. Erdös, E. Fried, A. Hajnal, and E.C. Milner, *Some remarks on simple tournaments*, Algebra Universalis **2** (1972), 238–245.

[EHM72] P. Erdös, A. Hajnal, and E.C. Milner, *Simple one-point extensions of tournaments*, Mathematika **19** (1972), 57–62.

[GM72] M. Goldberg and J.W. Moon, *Cycles in k-strong tournaments*, Pacific J. Math. **40** (1972), 89–96.

[HN81] F. Harary and J. Nieminen, *Convexity in graphs*, J. Differential Geometry **16** (1981), 185–190.

[HW96] D.J. Haglin and M.J. Wolf, *On convex subsets in tournaments*, SIAM Journal on Discrete Mathematics **9** (1996), 63–70.

[Moo72] J.W. Moon, *Embedding tournaments in simple tournaments*, Discrete Math. **2** (1972), 389–395.

[Nie81] J. Nieminen, *On path- and geodesic-convexity in digraphs*, Glasnik Matematicki **16** (1981), 193–197.

[Pfa71] J.L. Pfaltz, *Convexity in directed graphs*, J. Combinatorial Theory **10** (1971), 143–152.

[PWW] D.B. Parker, R.F. Westhoff, and M.J. Wolf, *Convex independence and the structure of clone-free multipartite tournaments*, Submitted.

[PWW06] ———, *Two-path convexity in clone-free regular multipartite tournaments*, Australas. J. Combin. **36** (2006), 177–196.

[PWW08] ______, *On two-path convexity in multipartite tournaments*, European J. Combin. **29** (2008), 641–651.

[Var76] J.C. Varlet, *Convexity in tournaments*, Bull. Societe Royale des Sciences de Liege **45** (1976), 570–586.

[vdV93] M.L.J van de Vel, *Theory of Convex Structures*, North Holland, Amsterdam, 1993.