Determining Properties of a Multipartite Tournament from its Lattice of Convex subsets

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Abstract

The collection of convex subsets of a multipartite tournament T forms a lattice $\mathcal{C}(T)$. Given a lattice structure for $\mathcal{C}(T)$, we deduce properties of T. In particular, we find conditions under which we can detect clones in T (i.e. vertices with identical arc orientations). We also determine conditions on the lattice which will imply that T is bipartite, except for a few cases. We classify the ambiguous cases. Finally, we study a property of $\mathcal{C}(T)$ we call the *anti-bipartite condition*. We prove a result on directed cycles in multipartite tournaments satisfying the anti-bipartite condition, and determine the minimum number of partitions in such multipartite tournaments.

1 Introduction

Let $T = (V, E)$ be a digraph. In [Pfa71], J. Pfaltz defines a convex subset to be $C \subseteq V$ which satisfies the condition that whenever $a, b \in C$, then all vertices in any directed path from a to b are contained in C. The collection $\mathcal{C}(T)$ of the convex subsets of T forms a lattice by inclusion, and he investigated some of the properties of this lattice. In particular, he proved that $\mathcal{C}(T)$ is complete, semimodular, and $\mathcal{A}$-regular. He also made inferences about T from the lattice theoretic properties of $\mathcal{C}(T)$.

In [Var76], J. Varlet looked at similar problems in the case where T is a tournament. He used a less restrictive definition of convexity; if $a, b \in C$, then only directed 2-paths

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had to be contained in C. With this definition, he showed that $\mathcal{C}(T)$ is weakly $\mathcal{A}$-normal, which implies that $\mathcal{C}(T)$ is weakly distributive. He also showed that $\mathcal{C}(T)$ has breadth 2, a result that has been important in creating algorithms to construct convex subsets of a given tournament. In [HW96], D. Haglin and M. Wolf used the fact that tournaments have breadth 2 to construct an algorithm that runs in $O(n^4)$ serial time. They improved this to $O(n^3)$ in [HW99].

In [PWW], the notion of the breadth of $\mathcal{C}(T)$ was studied in the case of T a *multipartite tournament*. Using Varlet's definition of convexity, they found upper bounds for the breadth of multipartite tournaments and upper bounds for the breadth of clone-free multipartite tournaments. They also classified all clone-free multipartite tournaments of maximum breadth.

Many of the results above were motivated by similar problems: how do the graph-theoretic properties of T reflect upon the lattice theoretic properties of $\mathcal{C}(T)$? In this paper, we consider the inverse problem. Given knowledge about the lattice structure of $\mathcal{C}(T)$, we seek to deduce properties of T. We approach this problem in the case where T is a multipartite tournament, using Varlet's definition of convexity.

As pointed out by Varlet, while each T gives rise to a unique $\mathcal{C}(T)$, it is often true that a given lattice can be the convex subset lattice of more than one multipartite tournament. For instance, if T^* is the multipartite tournament obtained from T by reversing the orientation of all of its arcs, then $\mathcal{C}(T) = \mathcal{C}(T^*)$, but only rarely is it true that $T \cong T^*$ (see [Var76, p. 581]). As we shall see in Theorem 3.3, it is possible for two multipartite tournaments to have identical convex subset lattices, yet not even have the same number of partitions. Thus, these inverse problems offer a great deal of mathematical richness.

In section 2, we consider the question of determining clones (i.e. vertices with identical arc orientations) from the lattice. We are able to derive conditions on $\mathcal{C}(T)$ which enable us to determine whether two vertices are clones, or at the very least detect the presence of clones.

In section 3, we study the problem of determining, given $\mathcal{C}(T)$, whether or not T is bipartite. We find a condition which is satisfied by all bipartite tournaments. In addition, this condition implies a multipartite tournament is bipartite in all but a very few cases. We classify the ambiguous cases.

In section 4, we study multipartite tournaments satisfying what we call the *anti-bipartite condition*. We prove a result concerning directed cycles in these multipartite tournaments. We also determine the minimum number of partitions in clone-free multipartite tournaments satisfying the anti-bipartite condition.

Recall that a *multipartite tournament* is a complete multipartite graph whose edges have been directed. As an abuse of notation, we use T to denote both the tournament and its vertex set. We denote an arc (v, w) as $v \to w$ and say that v *dominates* w. If $A, B \subseteq T$, then $A \to B$ means that for each $a \in A$, $b \in B$ we have $a \to b$. Two vertices $v, w \in T$ are called *clones* if, for each $u \in T$, we have $v \to u$ if and only if $w \to u$ and we have $u \to v$ if and only if $u \to w$. Note that this implies that v and w are in the same partition. A multipartite tournament without clones is said to be *clone-free*. A vertex w is said to *distinguish u and v* if either $u \to w \to v$ or $v \to w \to u$. Thus, vertices u and v

are clones if and only if they are in the same partition and no vertices distinguish them. As noted before, T^* is the multipartite tournament which is identical to T except that $v \to w$ in T^* if and only if $w \to v$ in T. We call it the *dual* of T.

Let $C_1, C_2 \in \mathcal{C}(T)$. It is easy to show that $C_1 \cap C_2 \in \mathcal{C}(T)$, and so we may express the infimum and supremum of C_1 and C_2 as $C_1 \wedge C_2 = C_1 \cap C_2$ and $C_1 \vee C_2 = \cap\{C \in \mathcal{C}(T) : C_1, C_2 \subseteq C\}$. Also note that all singleton sets of vertices are convex, so we may identify the vertices of T from the *atoms* of $\mathcal{C}(T)$ (i.e. the convex subsets C such that if A is convex, and $C' \leq C$, then either $C' = C$ or $C' = \emptyset$).

There is a natural algorithm for forming the convex hull of a given subset $U \subseteq T$. Begin by looking at all 2-paths between vertices in U. Then we repeat this process recursively until no new vertices can be obtained with 2-paths. This process is formalized in the following.

Definition 1.1. [HW96] Let $U \subseteq T$, and define $C_U(k)$ inductively by

$$C_U(0) = U, \quad C_U(k) = C_U(k-1) \cup M(C_U(k-1)), k \geq 1$$

where $M(X) = \{w \in T : x \to w \to y \text{ for some } x, y \in X\}$, for any $X \subseteq T$.

Then $C_U(k) \subseteq C_U(k+1)$ for all k, and the convex hull of U is $C_U(\infty) = \cup_{k=0}^{\infty} C_U(k)$.

2 Identifying Clones from $\mathcal{C}(T)$

Given the lattice of convex subsets for a multipartite tournament T, we would like to find a way of identifying the clones of T. This is not possible in general, as illustrated by the multipartite tournaments in Figure 1.

Figure 1:

The convex subsets are identical in each one, but only the second one has a pair of clones (namely, a and b). In some cases however, something can be said.

If $S \subseteq T$ is a set of clones, then it is clear that any subset of S is a convex subset. The following gives us a partial converse of this.

Lemma 2.1. Let $S \subseteq T$. If every subset of S is convex, then

1. S intersects at most two partitions of T nontrivially.

2. No two vertices in S can be distinguished by another vertex.

Proof. For (1), let $u, v, w \in S$ be in different partitions. Then there is a 2-path among u, v, and w, say $u \to v \to w$. Then $v \in u \vee w$, and so $\{u, w\} \notin \mathcal{C}(T)$, a contradiction.

For (2), any vertex distinguishing two vertices $u, v \in S$ will form a 2-path, and will thus be in any convex subset containing u and v. This is a contradiction. $\square$

In particular, Lemma 2.1(2) implies any two vertices which form a doubleton convex subset are either clones, or they have an arc between them and identical arc orientations otherwise.

Lemma 2.2. Suppose that $u, v \in T$ are clones. Then, for all $a \in T - \{u, v\}$, we have $(u \vee a) - \{u, v\} = (v \vee a) - \{u, v\}$. The converse is not true.

Proof. Let $U = \{u, a\}$, $V = \{v, a\}$. It suffices to show that $C_U(k) - \{u, v\} = C_V(k) - \{u, v\}$ for all k. We induct on k. The case $k = 0$ is obvious. We assume $C_U(k) - \{u, v\} = C_V(k) - \{u, v\}$, and prove the result for $k + 1$. Let $x \in C_U(k + 1) - \{u, v\}$. Then there exists $r, s \in C_U(k)$ with $r \to x \to s$. We can not have both of r, s equal to u, v, since u and v are clones. If either r or s is v, then $x \in C_V(k + 1)$ by induction and the fact that $v \in C_V(k)$. If $r = u$, then we have $u \to x \to s$. Since u and v are clones, we then have $v \to x \to s$, and again $x \in C_V(k + 1)$. The result is similar if $u = s$. Finally, if neither r nor s is equal to u or v, then the result follows from induction.

The converse not being true follows from the example $u \to v \to a \leftarrow u$. $\square$

The above lemma suggests that we might want to consider when $u \in v \vee a$ and $v \in u \vee a$ for $a \in T - \{u, v\}$.

Lemma 2.3. If u and v are clones and $a \in T - \{u, v\}$, then $u \in v \vee a$ if and only if $v \in u \vee a$.

Proof. Suppose that $u \in v \vee a$. We have, by Lemma 2.2, that $(v \vee a) - \{v\} \subseteq u \vee a$. Let k be minimal with $u \in C_V(k)$, where $V = \{v, a\}$. Then there exist $x, y \in C_V(k - 1)$ with $x \to u \to y$. Since u and v are clones, we have $v \notin \{x, y\}$. Thus, $x, y \in (v \vee a) - \{v\} \subseteq u \vee a$. Again using the fact that u and v are clones, we have $x \to v \to y$, and so $v \in u \vee a$. The other part of the if and only if follows by symmetry. $\square$

Thus, given a vertex $a \notin \{u, v\}$, we will want to consider whether or not u is in $v \vee a$. The following theorem details what happens in the case $u \notin v \vee a$.

Theorem 2.4. If $\{v, w\} \in \mathcal{C}(T)$ and if there exists $a \in T - \{u, v\}$ such that $u \notin v \vee a$, then the following hold:

1. If $v \in u \vee a$, then u and v are not clones.
2. If $v \notin u \vee a$, and if either $\{u, a\} \notin \mathcal{C}(T)$ and $\{v, a\} \in \mathcal{C}(T)$ or $\{u, a\} \in \mathcal{C}(T)$ and $\{v, a\} \notin \mathcal{C}(T)$, then u and v are not clones.
3. If $v \notin u \vee a$, $\{u, a\} \notin \mathcal{C}(T)$ and $\{v, a\} \notin \mathcal{C}(T)$, then u and v are clones.

4. If $v \notin u \vee a$, $\{u, a\} \in \mathcal{C}(T)$ and $\{v, a\} \in \mathcal{C}(T)$, then there are clones among u, v and a.

Proof. (1) follows directly from Lemma 2.3.

For (2) suppose without loss of generality that $\{u, a\} \notin \mathcal{C}(T)$ and $\{v, a\} \in \mathcal{C}(T)$. Then there exists $w \in T$ such that, without loss of generality, $u \to w \to a$ (or $w \in u \vee a$). If u and v are clones, then it must be the case that $v \to w$ because $u \to w$. But, then we would have $v \to w \to a$ which contradicts the assumption that $\{v, a\} \in \mathcal{C}(T)$. Thus, u and v are not in the same partition and therefore are not clones.

For (3) in order to show that u and v are clones, suppose, without loss of generality, that $u \to v$. Then u and v are in distinct partitions, say P_1 and P_2 respectively. Without loss of generality $a \in P_1$ or in a partition P_3 distinct from P_1 and P_2. First, suppose $a \in P_1$. Since $v \notin u \vee a$ we have that $a \to v$ and since $\{v, a\} \notin \mathcal{C}(T)$, there exists $w \in P_3$, a third partition, such that $a \to w \to v$ or $v \to w \to a$. In the first case, $a \to w \to v$, so $w \in v \vee a$. But $w \to u \to v$, since $w \to v$ and $\{u, v\} \in \mathcal{C}(T)$, and w is in a different partition than u. Therefore $u \in v \vee a$, which is a contradiction. In the second case we have $u \to w \to a$, since $v \to w$, $\{u, v\}$ is convex, and w is in a different partition than u. Thus $w \in u \vee a$. But $a \to v \to w$, so that $v \in u \vee a$, which is again a contradiction. Thus $a \notin P_1$. In the case that $a \in P_3$, we see that again, $a \to v$. Now, if $u \to a$ then $u \to a \to v$ contradicts $\{u, v\} \in \mathcal{C}(T)$, and if $a \to u$, then $a \to u \to v$ contradicts $u \notin v \vee a$. Thus u, v are in the same partition and $\{u, v\} \in \mathcal{C}(T)$, so u and v are clones.

For (4) suppose that u, v, and a are in distinct partitions. Then there must be a two-path between them, say $u \to v \to a$. Then $\{u, a\} \notin \mathcal{C}(T)$ which contradicts the assumption. Thus, at least two are in the same partition. Say, u and v are in the same partition. Since $\{u, v\} \in \mathcal{C}(T)$ and u and v are in the same partition, u and v are clones. $\square$

As an immediate consequence of Theorem 2.4 (4), we have

Corollary 2.5. If there exists $S \subseteq T$ with $|S| \geq 3$ and every subset of S is contained in $\mathcal{C}(T)$, then T is not clone-free.

3 The Bipartite Condition

In this section, we consider the problem of determining the number of partitions in a multipartite tournament from its lattice of convex subsets. This is not always possible. Consider the tripartite tournament $x \to y \to z \leftarrow x$ and the bipartite tournament $x \to y \to z$. Each of these has the same lattice of convex subsets, and thus there is no lattice theoretic condition on the convex subsets that can uniquely determine the number of partitions of a multipartite tournament.

More specifically, we explore the problem of determining whether a multipartite tournament is bipartite. As demonstrated above, we will not be able to determine this in every case. However, we will solve this problem in most cases, and show, in detail, what happens in the ambiguous cases.

We begin by considering necessary and sufficient conditions for a tournament to be bipartite. The following is clear.

Lemma 3.1. If T is a bipartite tournament with partitions P_1 and P_2, then $\{x, y\} \in \mathcal{C}(T)$ for all $x \in P_1$ and $y \in P_2$.

We use this lemma as the basis for defining a lattice theoretic condition which is almost sufficient to imply that a tournament is bipartite. One can think of the partitions as nonempty disjoint sets whose union is the entire vertex set for the tournament.

Definition 3.2. Let T be a multipartite tournament. We say T satisfies the *bipartite condition* if there exist $A, B \subseteq T$ such that $A \cup B = T$ and $\{x, y\} \in \mathcal{C}(T)$ for any choice of $x \in A$, $y \in B$.

Note that the two examples at the beginning of this section satisfy the bipartite condition, with $A = \{x, z\}$ and $B = \{y\}$. We can also have $A = \{x\}$ and $B = \{y, z\}$. Thus, A and B are not necessarily unique, and are not necessarily the partitions of the multipartite tournament, even when it is bipartite.

While the bipartite condition cannot always tell us whether a multipartite tournament is bipartite, the following theorem says that it always implies that a multipartite tournament is bipartite unless it is impossible to determine the number of partitions from the lattice of convex subsets.

Theorem 3.3. Let T be a multipartite tournament satisfying the bipartite condition. Then

1. T has at most three partitions.
2. If T is tripartite, then T must be of the form $P_1 \to P_2 \to P_3 \leftarrow P_1$, where the P_i are the partitions of T.
3. Suppose there exist nonempty sets $X, Y, Z \in T$ such that $A = X \cup Z$, $B = Y$ are as in the definition of the bipartite condition. If, in addition, we have $x \vee z = \{x, z\} \cup Y$ for all $x \in X$, $z \in Z$, then T must be one of $X \to Y \to Z$, $X \to Y \to Z \leftarrow X$ and their duals. If there exist no such sets, then T is bipartite.

Before proving this theorem, we will need four lemmas. For each of these lemmas, we will assume that the multipartite tournament T has at least three partitions. Let the partitions be given by P_1, P_2, P_3, $\ldots$, and suppose that the sets A and B are as in the definition of the bipartite condition. Note that this implies either that two vertices in A are in distinct partitions or that two vertices in B are in distinct partitions. Without loss of generality, suppose that there exist $x_1, x_2 \in A$ with $x_1 \to x_2$. For convenience, let $x_1 \in P_1$ and $x_2 \in P_2$. We begin with the following lemma.

Lemma 3.4. All vertices of B are in the same partition of T.

Proof. Suppose not. Then there exist $y_1, y_2 \in B$ with $y_1 \to y_2$. If $x_1 \to y_1$, then $y_1 \in x_1 \vee y_2$, a contradiction. Similarly, we cannot have $y_1 \to x_1$, so $y_1 \in P_1$. An almost identical argument gives us that $y_2 \in P_2$. Similar reasoning gives us $x_1 \to y_2$ and $y_1 \to x_2$.

Since T is at least tripartite, let z be a vertex in a third partition. If $z \in A$, then $\{z, y_2\} \in \mathcal{C}(T)$, so we must have $\{x_1, y_1\} \to z$. Since $\{y_1, z\} \in \mathcal{C}(T)$, we must have $z \to y_2$ and $z \to x_2$. But now $y_1 \to z \to x_2$, so $z \in y_1 \vee x_2$, a contradiction. A similar contradiction occurs if $z \in B$. This proves the lemma. □

The next lemma helps make it easier to determine the arcs involving the vertices in the partition containing B.

Lemma 3.5. Let P be a partition of T with $B \subseteq P$, $B \neq P$, and let P' be any other partition. If $z \in P'$, then $P \to z$ or $z \to P$.

Proof. Suppose that $x \in P$ with $x \to z$. Since $B \subseteq P$, we have $z \in A$. In the case of $x \in A$, let $y \in B$. If $z \to y$, then $z \in x \vee y$, which violates the bipartite condition, and so $B \to z$. If $u \in A \cap P$ with $z \to u$, then $B \to z \to u$, which also violates the bipartite condition. Thus, $P \to z$. A similar argument gives us $z \to P$ when $z \to x$. The argument is almost identical in the case $x \in B$; merely reverse the roles of A and B. □

We have three possibilities. Either $B \subseteq P_1$, $B \subseteq P_2$, or B is in a third partition. We look at the first two cases with the following lemma.

Lemma 3.6.

1. If $B \subseteq P_1$, and if P_i is another partition of T ($i \geq 3$), then $P_i \to P_1 \to P_2 \leftarrow P_i$.
2. If $B \subseteq P_2$, and if P_i is another partition of T ($i \geq 3$), then $P_1 \to P_2 \to P_i \leftarrow P_1$.

Proof. Since $x_1 \to x_2$, we have $P_1 \to x_2$ by Lemma 3.5. If $x_2 \to x \in P_i$, then $B \to x_2 \to x$, violating the bipartite condition. Thus, $P_i \to x_2$. A similar argument gives $P_i \to B$, and so $P_i \to P_1$ by Lemma 3.5. It follows that $P_i \to P_2$, for otherwise $x' \to x \to B$ for some $x' \in P_2$, $x \in P_i$, a contradiction. Similarly, $B \to P_2$, and so $P_1 \to P_2$ by Lemma 3.5. This completes (1).

Part (2) follows directly from applying (1) to T^*, and so the lemma is proven. □

We now take a look at the case of B in a partition other than P_1 and P_2.

Lemma 3.7. Suppose that $B \subseteq P_i$, where P_i is a partition of T which is not P_1 or P_2. Then $P_1 \to P_i \to P_2 \leftarrow P_1$.

Proof. Recall our global assumption that $x_1, x_2 \in A$ with $x_1 \to x_2$, and $x_1 \in P_1$ and $x_2 \in P_2$.

In the case $B \neq P_i$, let $x \in P_i \cap A$. If $x \to x_2$, then the lemma follows from Lemma 3.6(1) by letting x play the role of x_1 and interchanging the roles of P_i and P_1. Similarly, if $x_2 \to x$, then the lemma follows from Lemma 3.6(2).

Now we consider the case of $B = P_i$. If $y \in B$ with $y \to x_1$, then we have $y \to x_1 \to x_2$, which violates the bipartite condition. Thus, $x_1 \to B$. Similarly, $B \to x_2$. If there is some

$x \in P_2$ with $x \to x_1$, then $x \to x_1 \to B$, which violates the bipartite condition. Thus, $x_1 \to P_2$, and, similarly, $P_1 \to x_2$. It is then easy to show that $P_1 \to B \to P_2$. It then quickly follows that $P_1 \to P_2$, and the lemma is proven. □

Now we are ready to prove the main theorem.

Proof. (of Theorem 3.3)

By Lemmas 3.4, 3.6, and 3.7, to prove (1) and (2), we need only show that T is at most tripartite. Suppose that T has partitions P_i for $i = 1, \ldots, 4$, where B is contained in either P_1, P_2, or P_3, $x_1 \in A \cap P_1$, $x_2 \in A \cap P_2$ with $x_1 \to x_2$. Then we must have $P_4 \subseteq A$.

In the case of $B \subseteq P_1$, we must have, by Lemma 3.6(1), $P_3 \to P_1 \cup P_2 \leftarrow P_4$. Let $x \in P_3$, $y \in P_4$. If $x \to y$, then we have $x \to y \to B$, a contradiction of the bipartite condition. If $y \to x$, then $y \to x \to B$, another contradiction. The case of $B \subseteq P_2$ follows similarly from Lemma 3.6(2). Thus, T is at most tripartite in these cases.

In the case of $B \subseteq P_3$, let $x \in P_1$, $y \in P_2$, and $z \in P_4$. By Lemma 3.7, we must have $P_1 \to P_3 \to P_2 \leftarrow P_1$. We cannot have $B \to y \to z$. Thus, $z \to y$. Also, for $u \in B$ we can not have $u \to z \to y$ so $z \to B$. Now we get $x \to z \to B$ or $z \to x \to B$, which are both contradictions of the bipartite condition. Thus T is at most tripartite, and (2) is proven.

For (3), let $x \in X$ and $z \in Z$. Since $Y \subseteq x \vee z$, there exists, without loss of generality, $y \in Y$ with $x \to y \to z$.

We first show that the vertices of X are in the same partition. Suppose there is some $x' \in X$ in a partition different from x. If $x' \to x$, then $x' \to x \to y$, which violates the bipartite condition. Suppose that $x \to x'$. Since $Y \subseteq x' \vee z$, there exists $y' \in Y$ such that either $x' \to y' \to z$ or $z \to y' \to x'$. If $x' \to y'$, then $x \to x' \to y'$, which violates the bipartite condition. Thus, we must have $z \to y' \to x'$. Now observe that we cannot have $z \to x$, for then $z \to x \to y$. If $x \to z$, then $x \to z \to y'$, which also violates the bipartite condition. Thus, x and z are in the same partition. Similarly, x' and z are in the same partition, which implies that x and x' are in the same partition, a contradiction. Thus, all the vertices of X are in the same partition. Similarly, all the vertices of Z are in the same partition.

We now show that all vertices in Y are in the same partition. Suppose that there is some $y' \in Y$ in a partition different from y. If $y \to y'$, then $x \to y \to y'$, which violates the bipartite condition. If $y' \to y$, then $y' \to y \to z$, which also violates the bipartite condition. Thus, all vertices in Y are in the same partition. Note that since $x \to y \to z$, all vertices in Y must be in a partition different from those of X and Z. Therefore Y is one of the partitions of T.

Now we show that $X \to Y$. Suppose that there exist $x' \in X$, $y' \in Y$ such that $y' \to x'$. Note that we must have $y \to x'$. Otherwise, $y' \to x' \to y$, and since $Y \subseteq x \vee z$, this would imply that $x' \in x \vee z$, a contradiction. But we have $y \in x' \vee z$. Since the only 2-path through y must come from vertices in $X \cup Z$, and since $(x' \vee z) \cap (X \cup Z) = \{x', z\}$, it must be true that y distinguishes x' and z. But we have just shown that $x' \leftarrow y \to z$, which contradicts this. Thus, $X \to Y$. Similarly, $Y \to Z$.

If T is tripartite, we must have that each of X and Z are partitions of T. To avoid violating the bipartite condition, we must have $X \to Z$. Thus, T is either $X \to Y \to Z$

or $X \to Y \to Z \leftarrow X$. If we started with $z \to y \to x$, then we would end up with the duals of these.

If there do not exist sets $X, Y, Z \in T$ such that $A = X \cup Z$, $B = Y$ satisfy the hypotheses of the theorem and such that $x \vee z = \{x, z\} \cup Y$ for all $x \in X$, $z \in Z$, then we specifically avoid the conditions which lead us to the tripartite tournament in (2). Thus T must be bipartite. $\square$

4 The Anti-Bipartite Condition

We now look at the opposite situation to the above.

Definition 4.1. Let T be a multipartite tournament. We say T satisfies the *anti-bipartite condition* if there exist disjoint subsets A and B of T with $A \cup B = T$ such that $\{x, y\} \notin \mathcal{C}(T)$ for any choice of $x \in A$, $y \in B$. We call the collection of such multipartite tournaments $A(m, n)$, where $m = |A|$ and $n = |B|$.

Note that $A(m, n) = A(n, m)$, as we can always relabel the sets A and B. Also, a tournament satisfying the anti-bipartite condition must have $|T| \geq 3$.

Example 4.2. Let $T \in A(1, 2)$ with $A = \{x\}$ and $B = \{y_1, y_2\}$. We know that $\{x, y_1\} \notin \mathcal{C}(T)$, so $y_2 \in x \vee y_1$. Without loss of generality, $x \to y_2 \to y_1$. Similarly, $y_1 \in x \vee y_2$. Since $y_2 \to y_1$, this forces $y_1 \to x$, and so $A(1, 2)$ consists of the 3-cycle.

Example 4.3. Consider $T \in A(2, 2)$ with $A = \{x_1, x_2\}$ and $B = \{y_1, y_2\}$. We claim that there is a 3-cycle in T. Since $\{x_1, y_1\} \notin \mathcal{C}(T)$, then, without loss of generality, $x_1 \to x_2 \to y_1$. In order to avoid a 3-cycle among x_1, x_2, and y_1, and to make sure $\{x_2, y_1\} \notin \mathcal{C}(T)$, we must have $x_2 \to y_2 \to y_1$ or $y_1 \to y_2 \to x_2$. The latter would create a 3-cycle among x_2, y_1, and y_2, so $x_2 \to y_2 \to y_1$. But now $\{x_2, y_2\} \notin \mathcal{C}(T)$ forces $y_2 \to x_1$, giving us the 3-cycle $x_1 \to x_2 \to y_2 \to x_1$.

Let us then suppose that $x_1 \to x_2 \to y_1 \to x_1$. In order for $\{x_2, y_2\} \notin \mathcal{C}(T)$, we must either have $y_2 \to x_1$ or $y_1 \to y_2$. In order for $\{x_1, y_2\} \notin \mathcal{C}(T)$, we have two cases. In the case $y_2 \to x_1$, we must have either $y_2 \to y_1$ or $x_2 \to y_2$. In either case, we get tripartite tournaments (call them T_1 and T_2, respectively). In the case $y_1 \to y_2$, we are forced to have $x_2 \in x_1 \vee y_2$, and so $x_2 \to y_2$. This tripartite tournament is isomorphic to T_2^*. Note that T_1 is self-dual. It follows that $A(2, 2) = \{T : T$ a multipartite tournament, $|T| = 4$, and T has T_1, T_2 or T_2^* as a subdigraph$\}$.

One might ask whether a multipartite tournament T satisfying the anti-bipartite condition can be bipartite. The answer is no, as the following lemma shows.

Lemma 4.4. Suppose T is a multipartite tournament satisfying the anti-bipartite condition. Then T is not bipartite.

Proof. Let $x_1 \in A, y_1 \in B$. Since $\{x_1, y_1\} \notin \mathcal{C}(T)$, there exists a 2-path between x_1 and y_1 through a vertex x_2. Without loss of generality, $x_2 \in A$. But $\{x_2, y_1\} \notin \mathcal{C}(T)$, so there must be a 2-path between x_2 and y_1. If the 2-path is through x_1, then there exists a 3-cycle, and hence T has at least 3 partitions. If the 2-path between x_2 and y_1 goes through a fourth vertex z, then T is at least tripartite (as x_2, y and z are in different partitions). □

Next, we consider the partitions of the multipartite tournaments in $A(m,n)$. One might expect these to have many partitions. It turns out, however, that for any choice of m and n, there always exists a $T \in A(m,n)$ that is tripartite. Consider the following generalization of Example 4.2:

Let $T = A \cup B$ be a multipartite tournament, where $A = A_1 \cup A_2$ and $|A| = m, |B| = n$. Let $A_1 \to A_2 \to B \to A_1$. Then T is tripartite and satisfies the anti-bipartite condition. Thus $A(m,n)$ contains a tripartite tournament regardless of the values of m and n. However, note that in this example, all elements of A_1 are clones, as are the elements of A_2 and of B. That means that a tournament T constructed in this fashion will have clones whenever $|T| \geq 4$.

One may then ask: What is the minimum number of partitions a *clone-free* multipartite tournament $T \in A(m,n)$ must have?

Theorem 4.5. For all m, n with $m + n > 2$, there exists a tournament $T \in A(m,n)$ that is clone-free and tripartite.

Proof. From Examples 4.2 and 4.3, we already know that the theorem holds for $A(1,2)$ and $A(2,2)$, so we will assume that $\max(m,n) > 2$.

First, suppose $m \geq 3, n = 1$. Let $A = \{x_1, \ldots, x_m\}, B = \{y_1\}$. Define T as follows:

If m is even, let $A \to y_1, x_m \to x_1$. In addition, let $x_i \to x_j$ whenever $i < j$, $i + j$ is odd and $(i,j) \neq (1,m)$.

If m is odd, let $A - \{x_1\} \to y_1, y_1 \to x_1, x_m \to x_2, x_i \to x_j$ for all $i < j$ for which $i + j$ is odd and $(i,j) \neq (2,m)$.

It is easily verified that T is tripartite with the following partitions:

$$P_1 = \{x_{2k+1} \mid 0 \leq k \leq \lceil \frac{m}{2} \rceil - 1\}, \quad P_2 = \{x_{2k} \mid 1 \leq k \leq \lfloor \frac{m}{2} \rfloor\}, \quad P_3 = \{y_1\}$$

Next, we will show $T \in A(m,1)$.

If m is even, we have $x_i \to x_{i+1} \to y_1$ for $1 \leq i \leq m-1$. In addition, $x_m \to x_1 \to y_1$, so $x_i \vee y_1$ is nontrivial for all i.

If m is odd, then $x_i \to x_{i+1} \to y_1$ for $1 \leq i \leq m-1$. In addition, $x_m \to x_2 \to y_1$, so $x_i \vee y_1$ is nontrivial for all i. Therefore T satisfies the anti-bipartite condition and is thus an element of $A(m,1)$.

Finally, we need to show that T is clone-free. Note that if x_i, x_j are in different partitions, they are not clones. So assume x_i, x_j are in the same partition.

If $i < j$, we have $x_i \to x_{i+1} \to x_j$ unless $i = 1, j = m$ and m is odd. In that case, $x_m \to y_1 \to x_1$. Thus there are no clones.

Next, suppose $m \geq 3, n = 2$. Let $A = \{x_1, \ldots, x_m\}, B = \{y_1, y_2\}$. Define T as follows:

If m is even, let $A \to y_1, x_m \to x_1$ and $y_2 \to A$. In addition, let $x_i \to x_j$ whenever $i < j < m$ and $i + j$ is odd.

If m is odd, let $A - \{x_1\} \to y_1, y_1 \to x_1, x_m \to x_2, x_i \to x_j$ for all $i < j$ for which $i + j$ is odd and $(i, j) \neq (2, m)$. Further, let $A - \{x_1, x_3\} \to y_2$ and $y_2 \to x_1, y_2 \to x_3$.

It is easily verified that T is tripartite with the following partitions:

$$P_1 = \{x_{2k+1} \mid 0 \leq k \leq \lceil \frac{m}{2} \rceil - 1\}, \;\; P_2 = \{x_{2k} \mid 1 \leq k \leq \lfloor \frac{m}{2} \rfloor\}, \;\; P_3 = \{y_1, y_2\}$$

We need show that $T \in A(2, m)$. We know that $\{x_i, y_1\} \notin \mathcal{C}(T)$, as the arcs between x_i and y_1 are identical to those in the case where $n = 1$. Thus we need only show that $\{x_i, y_2\} \notin \mathcal{C}(T)$ for all i.

If m is even, $y_2 \to x_i \to x_{i+1}$ for $i < m$, and $y_2 \to x_m \to x_1$.

If m is odd, then $x_i \to x_{i+1} \to y_2$ for $i < m$ and $i \neq 2$. Finally, $x_m \to x_2 \to y_2$ and $y_2 \to x_1 \to x_2$.

Thus $x_i \vee y_2$ is nontrivial for all i.

The proof that there are no clones among the elements of A is the same as in the case when $n = 1$. In addition, we have $y_2 \to x_3 \to y_1$ (whether m is even or odd), so T is clone-free.

Finally, suppose $m, n > 2$. Let $A = X_1 \cup X_2 \cup \{x\}$ and $B = Y_1 \cup Y_2 \cup \{y\}$ with $|A| = m, |B| = n$ such that $|X_1|$ and $|X_2|$ differ by at most one, and $|Y_1|, |Y_2|$ differ by at most one. Suppose without loss of generality that $X_1 = \{a_1, \ldots, a_r\}$ and $X_2 = \{b_1, \ldots, b_s\}$, and $Y_1 = \{c_1, \ldots, c_k\}, Y_2 = \{d_1, \ldots, d_\ell\}$.

Define T as follows: $y \to X_1 \to Y_1 \to y \to X_2$, $x \to X_1 \cup X_2 \cup Y_1 \cup Y_2$, and $X_2 \to Y_2 \to y$. Further, $a_i \to b_i \; \forall i$, and $b_i \to a_j$ whenever $i \neq j$. Similarly, let $c_i \to d_i \; \forall i$, and $d_i \to c_j$ whenever $i \neq j$.

Then T is tripartite, with partitions

$$P_1 = X_1 \cup Y_2, \;\; P_2 = X_2 \cup Y_1, \;\; P_3 = \{x\} \cup \{y\},$$

and it is straightforward to show that T is clone-free.

Lastly, we show that T satisfies the anti-bipartite condition. We need to show that, given $x_i \in X_i$, $y_j \in Y_j$ that each of $x_i \vee x_j$, $x \vee y_j$, and $x_i \vee y$ are nontrivial, or, equivalently, that the vertices in each of the above suprema have a 2-path between them. This is easily checked. □

In light of Example 4.3 one may ask whether T must always have a 3-cycle. If not, then what can we say about the cycles of T?

To answer the first question: a 3-cycle is not required. Consider the tripartite tournament in $A(1, 4)$ given by $x_1 \to y_i$ for all i and $y_1 \to y_2 \to y_3 \to y_4 \to y_1$. Here, $A = \{x_1\}$ and $B = \{y_1, y_2, y_3, y_4\}$. However, notice that T does have a 4-cycle, and that all vertices in the 4-cycle are in B.

Theorem 4.6. Suppose that T satisfies the anti-bipartite condition, and that $|T| \geq 3$. If there is no 3-cycle containing vertices from both A and B, then there exists a cycle which either contains only vertices from A or only vertices from B.

Note that the length of the cycle in Theorem 4.6 is at most $\max\{m, n\}$.

Proof. Suppose that there are no 3-cycles containing vertices from both A and B. By Examples 4.2 and 4.3, we can assume that at least one of m and n is at least 3.

We index the elements of A and B as follows. Arbitrarily choose elements $x_{11} \in A$, $y_1 \in B$. Since $\{x_{11}, y_1\} \notin \mathcal{C}(T)$, there must be either some $x_{12} \in A$ or some $y_2 \in B$ with $x_{11} \to x_{12} \to y_1$, $y_1 \to x_{12} \to x_{11}$, or similarly with y_2 in place of x_{12}. Without loss of generality, either $x_{11} \to x_{12} \to y_1$ or $x_{11} \to y_2 \to y_1$.

Let $x_{11}, \ldots, x_{1j_1}$ be the longest sequence of elements in A such that $x_{1j} \to x_{1(j+1)}$ for all $1 \leq j \leq j_1 - 1$ and $x_{1j} \to y_1$ for all $1 \leq j \leq j_1$. Now $\{x_{1j_1}, y_1\} \notin \mathcal{C}(T)$. If $x \in A$, we cannot have $x_{1j_1} \to x \to y_1$ because this would give us a longer sequence. If v is any vertex, we cannot have $y_1 \to v \to x_{1j_1}$, because this would give us a 3-cycle between y_1, v, and x_{1j_1}. We must then have some $y_2 \in B$ with $x_{1j_1} \to y_2 \to y_1$. Note that, by convention, this will still occur if $j_1 = 1$.

We repeat this process with $x_{21} = x_{1j_1}$. We find the longest sequence $x_{21}, \ldots, x_{2j_2}$ with $x_{2j} \to x_{2(j+1)}$ and $x_{2j} \to y_2$. Note that if $x_{2j} = x_{1k}$ then we have the cycle $x_{1k} \to \cdots \to x_{1j_1} = x_{21} \to \cdots \to x_{2j} = x_{1k}$, which has length at most $|A|$. If this occurs, we are done. If not, label $x_{2j_2} = x_{31}$ and repeat the process.

Assuming that a cycle amongst either vertices in A or vertices in B has not been constructed, we can continue this process, with $x_{ij} \in A$, $y_i \in B$, where $x_{ij} \to x_{i(j+1)}$ and $x_{ij} \to y_i$. We have a directed path between x_{ij} and x_{kl} whenever either $i < k$ or $i = k$ and $j < l$, unless $x_{ij} = x_{kl}$ (this would happen, for example, if $j = j_i$, $k = i + 1$, and $l = 1$). Also, there is a path between y_i and y_j whenever $j < i$.

Since A and B are both finite, this process cannot continue indefinitely. So there is a point where we get to the last x_{ij_i}. At this point, we note that $\{x_{ij_i}, y_i\} \notin \mathcal{C}(T)$. We cannot have $y_i \to v \to x_{ij_i}$ for any vertex v, for this would create a forbidden 3-cycle. Thus, we must either have $x_{ij_i} \to x_{kl} \to y_i$ for either $k < i$ or $k = i$ and $k < j_i$ or $x_{ij_i} \to y_k \to y_i$ for some $k < i$. In the first case, we get a cycle amongst elements of A, and in the second case we get a cycle amongst elements of B. This completes the proof. ☐

5 Open Questions

We end with some questions for further inquiry.

1. *Is it generally possible to determine the partitions of T from the lattice $\mathcal{C}(T)$?* We have settled when bipartite tournaments can be determined from $\mathcal{C}(T)$. Can something like this be done for p-partite tournaments for $p \geq 3$? If we know how many partitions T must have, can we determine what they are from $\mathcal{C}(T)$? If not, can we at least determine what the sizes of the partitions of T are?

2. *Is it possible to determine whether or not u and v are clones if $\{u, v\}$ is convex, and if, for all vertices $a \notin \{u, v\}$ we have $u \in v \vee a$ and $v \in u \vee a$?* This is the only case not settled by Theorem 2.4, and it has the highest potential for ambiguity. The

examples in Figure 1 tell us that in this case, we may not be able to tell if clones exist, let alone determine what the clones are.

3. *Given a lattice $\mathcal{L}$, what can we say about the multipartite tournaments T such that $\mathcal{L} \cong \mathcal{C}(T)$?* It would be nice to have a way of constructing all possible T with $\mathcal{L}$ as their lattice of convex subsets. If not, could we at least have some sort of test as to whether there are any multipartite tournaments T with $\mathcal{L} \cong \mathcal{C}(T)$?

References

[HW96] D.J. Haglin and M.J. Wolf, *On convex subsets in tournaments*, SIAM Journal on Discrete Mathematics **9** (1996), 63–70.

[HW99] ———, *An optimal algorithm for finding all convex subsets in tournaments*, Ars Combinatoria **52** (1999), 173–179.

[Pfa71] J.L. Pfaltz, *Convexity in directed graphs*, J. Combinatorial Theory **10** (1971), 143–152.

[PWW] D.B. Parker, R.F. Westhoff, and M.J. Wolf, *On the breadth of lattices of convex sets in multipartite tournaments*, Preprint.

[Var76] J.C. Varlet, *Convexity in Tournaments*, Bull. Societe Royale des Sciences de Liege **45** (1976), 570–586.