Research Description

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My scholarly interests are somewhat eclectic. I attribute much of this to my first job out of graduate school, where most of the faculty were not active in research. Moreover, while I enjoyed living in northern Minnesota, I was geographically isolated from other mathematicians in my area. This led me, an algebraist, into a collaboration with an algebraic geometer and a computer scientist on a problem in graph theory.

Since then, I have collaborated with other graph theorists, developed problems for research with undergraduates, and worked with an engineer on a problem in electro-optics. The variety has been stimulating, giving me a chance to indulge in my love of all mathematics, not just the area in which I was trained.

Here are my specific research agendas.

Domination & Lights Out Let G be a graph with vertex set V and edge set E. The classical problem in domination theory is to find a set $S \subseteq V$ (usually of minimum size) such that every vertex in V is either in S or adjacent to some vertex in S. Such a set S is called a *dominating set*. Another way of stating this is the following. For $v \in V$, let $N[v] = \{v\} \cup \{w \in V : vw \in E\}$. Then S is a dominating set if $N[v] \cap S \neq \emptyset$ (equivalently $|N[v] \cap S| \neq 0$) for every $v \in V$.

Similar problems have been devised by placing other requirements on $|N[v] \cap S|$ for each $v \in V$. For instance, suppose we label each vertex in V as either odd or even. We say that S is a *parity dominating set* for this labeling if $|N[v] \cap S|$ is odd when v is odd and even when v is even. This problem was studied in [AS96], [ACS98], and [ASZ02]. In particular, they studied graphs that have a parity dominating set for all possible labelings of V. Such graphs are called *all-parity realizable (APR) graphs*.

Now consider the following "Lights Out" game on G. As above, we begin with a labeling of the vertices. In this game, the initial label of each vertex is either "on" or "off". We play the game by toggling a vertex. Each time a vertex is toggled, the labels of the vertex toggled and all adjacent vertices are reversed. We win the game when each vertex is labeled "off". Two of my students, Alexander Giffen and Jake Trochelman, studied graphs in which the game can be won regardless of the initial labeling. We call such graphs *always winnable (AW) graphs*.

If these problems sound similar to you, it is not your imagination. In fact, both problems are equivalent. If we write $V = \{v_1, v_2, \cdots, v_n\}$, then the solution to each problem hinges on whether the $n \times n$ matrix $N = [n_{ij}]$ is invertible over $\mathbb{Z}_2$, where

$$n_{ij} = \begin{cases} 1, & i = j \text{ or } v_i v_j \in E \\ 0, & \text{otherwise} \end{cases}$$

In fact, Alexander and Jake extended their scope to consider Lights Out games in which the vertices are labeled with elements of $\mathbb{Z}_k$, $k \geq 2$. In this game, each time a vertex is toggled, the labels of that vertex and all adjacent vertices increase by one. The game is won when all vertices have label zero. In [GP], Alexander classified all *AW* paths, cycles, and stars.

He also devised a constructive method for generating all non-AW caterpillar graphs in the case $k = p^m$, where p is a prime.

More recently, a group of REU students, supervised by Stephanie Edwards and Darin Stephenson, proved some nice results (see [EEJ$^+$]). Specifically, they characterized all AW spider graphs and theta graphs. In addition, they devised a constructive method for generating all AW trees when $k = p^n$ for p a prime.

Convexity in Multipartite Tournaments Convex sets have been a subject of interest for a long time. Relatively recently, convexity has been generalized to the notion of a *convexity space* (see [vdV93]). One of the contexts in which we can study convexity is graphs and directed graphs.

Let G be a (directed) graph, and let $\mathcal{P}$ be a collection of (directed) paths of G. We call a set $A \subseteq V$ a $\mathcal{P}$-*convex subset* if, whenever $v, w \in A$, any (directed) path in $\mathcal{P}$ that originates at v and ends at w involves only vertices in A. This is analogous to convexity in $\mathbb{R}^n$, where the vertices of G are analogues of points in $\mathbb{R}^n$ and $\mathcal{P}$ is the analogue to the set of line segments in $\mathbb{R}^n$. If $F \subseteq V$, then the convex hull of F, denoted $C(F)$, is the smallest convex subset containing F.

My coauthors and I have been studying different notions of independence related to convexity and the convex invariants that are associated with them. For instance, we say a set $F \subseteq V$ is *convexly independent* if $v \notin C(F - \{v\})$ for all $v \in F$. This is analogous to a linearly independent subset of a vector space, where convex hull is the analogue of span. The *rank* of G is the size of a largest convexly independent set. Similar notions of independence give rise to the *Helly number*, the *Radon number*, and the *Caratheodory number*. We have studied these convex invariants for multipartite tournaments when $\mathcal{P}$ is the set of all 2-paths. Our results are in [PWW09], [PWW08], [PWW], and [PWW06].

With another set of co-authors, I have studied some lattice-theoretic aspects of convexity. Given the lattice of convex subsets, we were able to determine certain properties of the underlying multipartite tournament (see [ADEP05]).

Miscellaneous In addition to another graph theory project (see [ADD$^+$06]), I have had the pleasure of doing some completely different work with Stephanie Edwards of Hope College and Bradley Duncan of the School of Engineering at the University of Dayton.

We have been working on a system in which an infrared beam is directed through two prisms. This is to be applied to an anti-missile defense system for an aircraft. Currently, a system of mirrors is used to direct an infrared beam toward a missile with the purpose of disabling it. It is hoped that a prism system will be less bulky and less expensive. Dr. Duncan and his colleagues Philip Boss and Vassili Sergan of Kent State University and California State University at Sacramento, respectively, have developed the prisms to be used. Stephanie and I were brought in to develop an algorithm that optimally orients the prisms so that the beam can follow the path of an incoming missile. This project is ongoing.

References

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