What kind of number is . . . ?

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- And the rest ...

The rest

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Please write down a few irrational numbers that come to your mind.

Irrationality of $\sqrt{2}$

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Making new irrational numbers

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Contradiction.

So: $1 + \sqrt{2}$, $2 - \sqrt[3]{3}$, $\frac{5}{3}\sqrt{7}$, $3 + 4\sqrt[3]{5}$ etc. are all irrational.

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So: $\sqrt{\sqrt{2}}, \sqrt[3]{1+\sqrt{2}}$, $\sqrt{\frac{5}{3}\sqrt[4]{5}+3}$, $2+\sqrt[3]{7+\frac{3}{2}\sqrt{5}}$ etc. are all irrational.

And so is:

$$\frac{11}{5}+\sqrt[7]{\frac{1}{2}+\frac{2}{5}\sqrt{4+\sqrt{3}}}$$

Irrationality of e

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Euler: e is irrational, using continued fractions, 1737.

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Fourier: a more elementary proof using the series expansion, 1815.

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proof: Taylor expansion of e^x: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!} + \ldots$

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$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots = \frac{1}{n}$$

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Contradiction.

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Niven: a relatively simple proof following an idea of Hermite, using integrals, 1947

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Properties: 1) $P(x)$ is a polynomial of degree $2n$.

2) For $0 < x < 1$, $0 < P(x) < \frac{1}{n!}$.

3) Any kth derivative of $P(x)$ yields an integer for $x = 0$ and $x = 1$.

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$$N \overset{\text{(by parts)}}{=}$$
$$a^n \Big(-\cos \pi x \big(P(x) - \frac{P''(x)}{\pi^2} + \dots \big) + \sin \pi x \big(\frac{P'(x)}{\pi} - \frac{P^{(3)}(x)}{\pi^3} + \dots \big) \Big) \Big|_0^1$$

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$$= -a^n \cos \pi x \big(P(x) - \frac{P''(x)}{\pi^2} + \ldots \pm \frac{P^{(2n)}(x)}{\pi^{2n}}\big)\Big)\Big|_0^1$$

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$a^n \frac{1}{\pi^{2n}}$ is an integer and the derivative values are integers, so N is an integer.

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Corollary: π is irrational.

Making more irrational numbers

Theorem: (?) If x and y are irrational numbers, then $x + y$ is irrational.

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What about $\pi + \sqrt{2}$? Is this irrational?

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example: $1 + \sqrt{2}$ is algebraic: $p(x) = x^2 - 2x - 1$

example: i is algebraic

Algebraic numbers

There are algebraic numbers which are not expressible by radicals, Abel, 1824

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Closure property of addition and multiplication

Closure property of algebraic numbers: if we add or multiply two algebraic numbers we get another algebraic number.

So: $2\sqrt[3]{2} + \frac{3}{2}\sqrt{7}$, $1 + \sqrt{2} + \sqrt[3]{3} + 5\sqrt{7}\sqrt[3]{130}$ etc. are algebraic.

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- Algebraic numbers: $\sqrt{2}$, $\sqrt[3]{5}$, $1+\sqrt{3}+2\sqrt[3]{13}, \ldots$

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- Rational numbers: ..., 0, 1, 1/2, 2, 1/3, 3, 1/4, 2/3, ...
- Algebraic numbers: $\sqrt{2}$, $\sqrt[3]{5}$, $1+\sqrt{3}+2\sqrt[3]{13}, \ldots$
- And the rest ...

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π transcendental conjectured by Euler, 1755; proved by Lindemann, 1882

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Theorem: If x is algebraic and y is transcendental, then $x + y$ and xy $(x \neq 0)$ are transcendental, and hence irrational.

So: $\pi + \sqrt{2}$, $e - \sqrt[3]{1 + \sqrt{5}}$, $\pi(1 + \sqrt{7}) - \sqrt[3]{2 - \frac{3}{5}\sqrt{11}}$, etc. are all transcendental, and hence irrational.

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Theorem: If x is algebraic, then $\sqrt{x}$ is algebraic.

proof: If x satisfies $ax^2 + bx + c = 0$, then $\sqrt{x}$ satisfies $ax^4 + bx^2 + c = 0$.

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proof: If x satisfies $ax^2 + bx + c = 0$, then $\sqrt{x}$ satisfies $ax^4 + bx^2 + c = 0$.

Theorem: If x is algebraic, then $\sqrt[n]{x}$ is algebraic.

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Theorem: (Equivalent) If x is transcendental, then x^2, x^3, $x^4, \ldots$ are transcendental.

So: π^2, e^5, $\sqrt{2} + \pi^2$, etc. are all irrational.

More transcendence results

Theorem: (Hermite, Lindemann) 1) If α is an algebraic number not equal to 0 or 1, then $\log \alpha$ is transcendental.
2) If α is a non-zero algebraic number, then e^{α} is transcendental.

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Transcendence of π: If π were algebraic, then $i\pi$ is algebraic and $e^{i\pi} = -1$ would have to be transcendental.

Hilbert's 7th problem

Theorem: (Gel'fond, Schneider, 1934) If α is an algebraic number not equal to 0 or 1 and β is a non-rational algebraic number, then α^β is transcendental.

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Also: e^π is transcendental.

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So: $3^{\sqrt{3}}$, $2^{1+\sqrt{2}}$, $\sqrt{2}^{\sqrt{2}}$, etc. are all transcendental.

Also: e^π is transcendental.
If e^π were algebraic, then $(e^\pi)^i = e^{i\pi} = -1$ would have to be transcendental.

Conclusion

MTH 210 conjecture: $\pi + e$ is irrational.

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Open problems:
Are $\pi + e$, π/e, π^e, 2^e, $\ln(\pi)$ irrational?

References

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