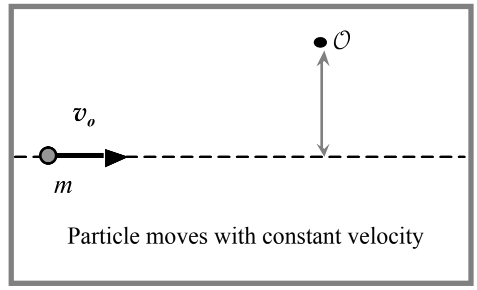
Homework: Angular momentum and Kepler's second law Name ________________________

1. The magnitude of any vector cross product can be written in terms of the magnitudes of the individual vectors and the sine of the angle ϕ between those vectors: $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\phi$.

   a. Explain in words <u>and</u> in diagrams why the magnitude of a vector cross product can be thought of as (i) the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ that is perpendicular to $\vec{A}$, as well as (ii) the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ that is perpendicular to $\vec{B}$.

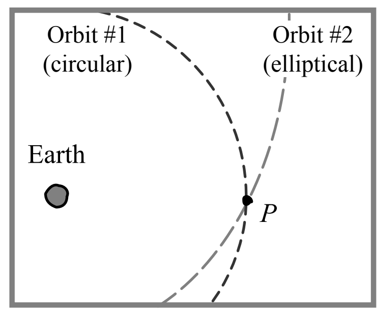
   b. Consider the situation shown at right, in which a particle of mass *m* moves with constant velocity.

   Apply your reasoning from part a to explain why the angular momentum of the particle (measured with respect to point *O*) <u>remains constant</u> while the particle moves along the straight line.



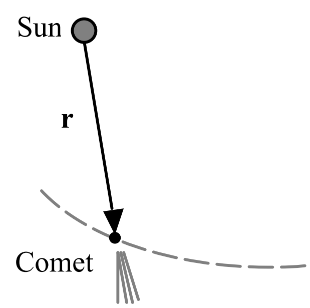
   c. Now consider two satellites, each of mass *m,* orbiting the Earth. One satellite (#1) follows a circular orbit, and the other (#2) follows an elliptical orbit that intersects the orbit of satellite #1 at point *P,* as shown at right. (Ignore the interactions between the satellites themselves.)

   If both satellites have the <u>same angular momentum</u>, which one passes through point *P* with the *greater speed?* Explain how you can apply your reasoning from part a to justify your answer.



2. A comet moves in a counter-clockwise orbit around the Sun. A portion of the orbit is shown below. (Ignore all gravitational forces acting on the comet other than that by the Sun.)

   a. The position vector $\vec{r}$ of the comet at a time *t* is shown in the diagram at right. In the diagram, draw a vector $d\vec{r}$ representing the infinitesimal displacement of the comet between time *t* and time *(t + dt).*



   b. Show that the magnitude of the vector cross-product $\vec{r}\times d\vec{r}$ is directly proportional to the area *dA* swept out by the comet from time *t* to time *(t + dt).* Explain your reasoning in words <u>and</u> with one or more diagrams.

   (Hint: Show that the magnitude of $\vec{r}\times d\vec{r}$ is equal to the area of the parallelogram formed by $\vec{r}$ and $d\vec{r}$. How does the area *dA* swept out by the comet compare to that of the parallelogram?)

   c. On the basis of your work in part b, show that the rate *dA*/*dt* at which the comet sweeps out area along its orbit is directly proportional the angular momentum of the comet, and determine the constant of proportionality between the two quantities.

3. As we found in tutorial, Kepler's second law states that the angular momentum of an orbiting body, measured with respect to the center of the body it is orbiting, does not change.

   For objects with constant mass, the quantity *angular momentum per unit mass,* $\vec{l} = \vec{r} \times \vec{v}$, is also a constant of the motion. The symbol l is often used to represent the magnitude of this quantity.

   a. Recall that in polar coordinates position and velocity are expressed as the following:

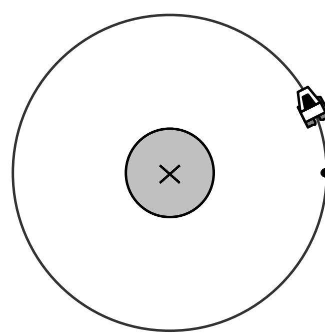
   $$\vec{r} = r\,\hat{e}_r\,; \quad \vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

   Clearly show how the above expression for velocity results from the time derivative of position. (That is, show that taking $\frac{d}{dt}\left(r\hat{e}_r\right)$ directly yields the above expression for velocity.)

   b. Using the above position and velocity expressions, show that $l = r^2\dot{\theta}$. Show all work.

   c. Recall two situations from tutorial in which you showed angular momentum was constant: (i) an object moving with constant velocity (see Case 3 on p. 1 of the tutorial) and (ii) a comet in an elliptical orbit (p. 3 of tutorial). In both your *own words* and *diagrams,* explain what it means for $l = r^2\dot{\theta}$ to be a constant quantity in each situation. (What is r in each case? What is $\dot{\theta}$?)

4. Consider a shuttlecraft that follows a circular orbit around a planet (see diagram at right).



   a. Suppose that, the next time the shuttle passes point *P,* the pilot briefly fires the forward thrusters for a very short time, essentially causing the speed of the shuttle to suddenly increase at point *P.*

      i. Determine whether the sudden increase in the speed of the shuttle causes each of the following quantities to *increase, decrease,* or *remain unchanged.* Explain your reasoning in each case.

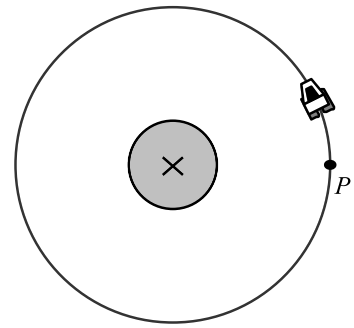
         - the angular momentum of the shuttle
         - the gravitational force on the shuttle at point *P*
         - the (centripetal) acceleration of the shuttle at point *P*
         - the radius of curvature of the orbit as measured at point *P*

      ii. On the diagram, make a qualitatively correct sketch of the new orbit of the shuttlecraft, and clearly indicate whether the latus rectum of the orbit is *larger than, smaller than,* or *the same as* before. (*Hint:* Point *P* will lie along the major axis of the new elliptical orbit.)

(Problem 4 is continued on the next page.)

4. [continued]

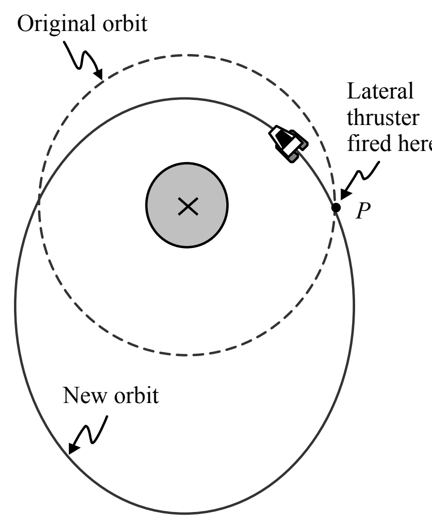
   b. Suppose instead that, the next time the shuttle passes point *P,* the pilot briefly fires a retrothruster for a very short time, essentially causing the speed of the shuttle to suddenly <u>decrease</u> at point *P*.



      i. Answer the same questions from part a(i) for the situation described here in part b. Clearly explain your reasoning.

      ii. On the diagram at right, make a qualitatively correct sketch of the new orbit of the shuttlecraft, and clearly indicate whether the latus rectum of the orbit is *larger than, smaller than,* or *the same as* before.

   c. Suppose instead that, upon passing point *P,* the shuttle pilot were to fire a lateral thruster for a very short time. (Assume that the effect of firing the lateral thruster would be to suddenly add a component of velocity perpendicular to the original velocity.)



      As a result, the shuttle is observed to follow a new elliptical orbit, shown at right. The semi-major axis of the new orbit is perpendicular to the imaginary line from point *P* to the center of the planet.

      i. Did the sudden firing of the lateral thruster cause the speed of the shuttle to *suddenly increase, suddenly decrease,* or *remain unchanged?* Explain.

      ii. Did the sudden firing of the lateral thruster cause the magnitude of the angular momentum of the shuttle to *suddenly increase, suddenly decrease, or remain unchanged?* Explain.

   d. Reflect on your results in parts a – c of this problem. Do your results suggest a relationship between the angular momentum of the shuttle and the latus rectum of the shuttle's orbit? Use your results in parts a – c to defend your answer.

