Homework: Forced harmonic motion Name ________________________

1. Although the steady-state motion of a forced harmonic oscillator resembles simple harmonic motion, the amplitude A and phase angle ϕ both depend upon the driving frequency ω as follows.

$$A(\omega) = \frac{F_o/m}{\left[\left(\omega_o^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right]^{1/2}}; \qquad \phi(\omega) = \tan^{-1}\left[\frac{2\gamma\omega}{\left(\omega_o^2 - \omega^2\right)}\right]$$

a. Use differentiation to determine the frequency ω_r of the driving force that results in a maximum (or *resonant*) steady-state amplitude. Express ω_r in terms of ω_o, the frequency of the undamped (ideal) oscillator and γ, the damping constant.

(*Hint:* Rather than differentiate and extremize the *entire* expression for $A(\omega)$, make your job easier by judiciously picking *which part* of that expression to work with.)

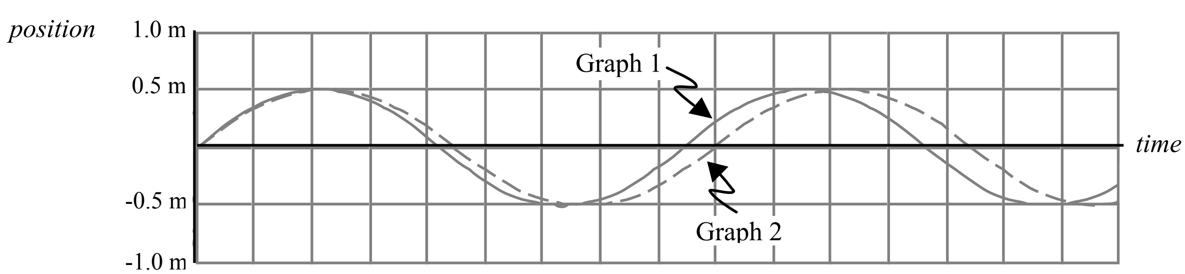
b. Determine expressions for the amplitude $A(\omega = \omega_r)$ and phase difference $\phi(\omega = \omega_r)$ for the case in which the driving frequency is set equal to the resonant frequency (from part a). Show all work.

c. Consider your expression for the phase difference $\phi(\omega = \omega_r)$ from part b in the limit of *weak* damping ($\gamma \rightarrow 0$). Show that the resulting value of ϕ agrees with your result from tutorial.

2. Consider two linear oscillators, A and B, that began to move *prior to* the instant $t = 0$.

- One oscillator *(A)* is an ideal (frictionless) simple harmonic oscillator with amplitude 0.50 m.
- Another oscillator *(B)* is identical to oscillator A *except* that it has a linear damping force *and* a sinusoidal driving force applied. The driving force has been adjusted to achieve resonance, so that at $t = 0$ a (resonant) steady-state amplitude of 0.50 m is achieved.

a. Two position versus time graphs, each sinusoidal in form and with amplitude 0.50 m, are shown below.



Identify which graph *(1* or *2)* represents which oscillator *(A* or *B)*. Explain your reasoning. (*Hint:* How does the resonant frequency compare to the frequency of the undamped oscillator?)

b. As oscillator B moves, at which (approximate) location does it experience the *greatest magnitude* of force by the driver (i) in the $+x$ *direction?* (ii) in the $-x$ direction? Indicate your answers by clearly labeling the appropriate points on the above graph. Explain your reasoning.

[Problem 2 continues on next page]

2. [continued]

   c. If each division along the time axis represents 0.1 s, calculate the quantities below. Show all work.

      i. the angular frequency ω_r of oscillator *B*

      ii. the damping constant γ of oscillator *B*

      iii. the angular frequency ω_d that oscillator *B* would have if the driving force were removed

3. A series LRC circuit is connected across the terminals of an AC power supply that produces a voltage $V(t) = V_o \exp(i\omega t)$. The "equation of motion" for the charge $q(t)$ across the capacitor is as follows:

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = V_o\, e^{i\omega t}$$

   The above differential equation will have a steady-state solution of the form:

$$q(t) = q_o\, e^{-i(\omega t + \phi)}$$

   [*Note:* The parameters q_o and ϕ are actually functions of ω, the frequency of the AC power supply. However, in this problem you will not have to write out these functions in full.]

   a. In terms of q_o, ω, ϕ, and the relevant coefficients from the differential equation, write down the following functions:

      i. the potential difference $\Delta V_C(t)$ across the capacitor

      ii. the potential difference $\Delta V_R(t)$ across the resistor

      iii. the potential difference $\Delta V_L(t)$ across the inductor

   b. Determine the values of α, β, and γ (in radians) that satisfy the following Euler relations:

      - $e^{i\alpha} = i$
      - $e^{i\beta} = -1$
      - $e^{i\gamma} = -i$

   c. Using your results from part b, rewrite the functions in part a so that each function can be written as a *positive real* number times a complex exponential. Use your rewritten functions to answer the following questions.

      i. What is the phase difference between $\Delta V_C(t)$ and $\Delta V_R(t)$? Is $\Delta V_C(t)$ *ahead of* or *behind* $\Delta V_R(t)$ by this amount? Show all work.

      ii. What is the phase difference between $\Delta V_R(t)$ and $\Delta V_L(t)$? Is $\Delta V_L(t)$ *ahead of* or *behind* $\Delta V_R(t)$ by this amount? Show all work.

4. A harmonic oscillator with a restoring force $25m\alpha_o^2 x$ is subject to a damping force $3m\alpha_o\dot{x}$ and a sinusoidal driving force $F_o\cos(10\alpha_o t)$.

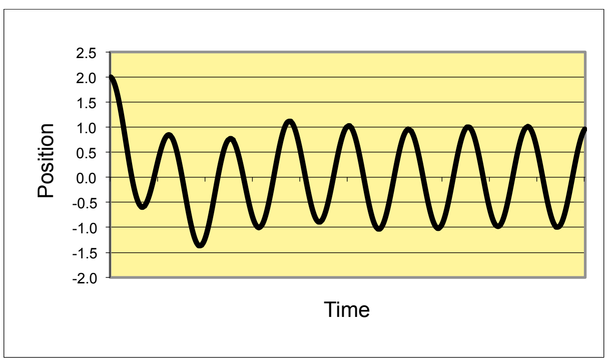
   a. Write down the differential equation that governs the motion of this oscillator.

   b. Show that if the driving force were removed the oscillator would become *underdamped,* and express the frequency of the oscillator in terms of the given quantities. Explain your reasoning.

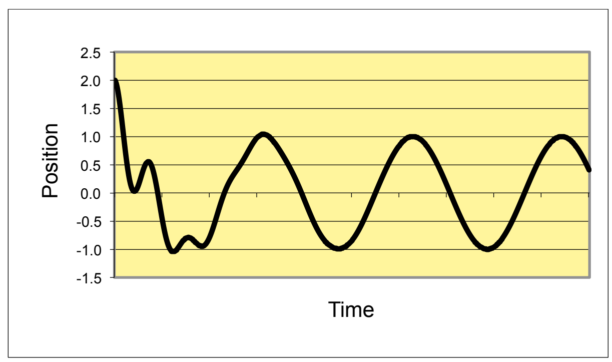
   c. For any damped oscillator that is driven by a sinusoidal external force, we know that the eventual (steady-state) motion is sinusoidal in nature. However, before the oscillator reaches steady state, its motion can be thought of as the algebraic sum of the steady-state motion plus a *transient* oscillatory motion whose amplitude dies exponentially with time.

   [In fact, the transient component of the motion (considered by itself) could accurately describe the motion of the same oscillator with the driving force turned off (but with the damping still present).]

   i. Each *x vs. t* graph below illustrates the actual motion (transient plus steady-state) of a damped, driven oscillator starting at $t = 0$. For each case, is the frequency of the steady-state motion *greater than, less than,* or *equal to* that of the transient motion? Explain.



Graph 1



Graph 2

   ii. Identify which graph (1 or 2) would better correspond to the damped, driven oscillator described in parts a–c of this problem. Explain your reasoning.

