1. Although the steady-state motion of a forced harmonic oscillator resembles simple harmonic motion, the amplitude $A$ and phase angle $\phi$ both depend upon the driving frequency $\omega$ as follows.

\[
A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2 \omega^2\right]^{1/2}}; \quad \phi(\omega) = \tan^{-1}\left[\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right]
\]

a. Use differentiation to determine the frequency $\omega_r$ of the driving force that results in a maximum (or resonant) steady-state amplitude. Express $\omega_r$ in terms of $\omega_0$, the frequency of the undamped (ideal) oscillator and $\gamma$, the damping constant.

(Hint: Rather than differentiate and extremize the entire expression for $A(\omega)$, make your job easier by judiciously picking which part of that expression to work with.)

b. Determine expressions for the amplitude $A(\omega = \omega_r)$ and phase difference $\phi(\omega = \omega_r)$ for the case in which the driving frequency is set equal to the resonant frequency (from part a). Show all work.

c. Consider your expression for the phase difference $\phi(\omega = \omega_r)$ from part b in the limit of weak damping ($\gamma \rightarrow 0$). Show that the resulting value of $\phi$ agrees with your result from tutorial.

2. Consider two linear oscillators, $A$ and $B$, that began to move prior to the instant $t = 0$.

• One oscillator ($A$) is an ideal (frictionless) simple harmonic oscillator with amplitude 0.50 m.

• Another oscillator ($B$) is identical to oscillator $A$ except that it has a linear damping force and a sinusoidal driving force applied. The driving force has been adjusted to achieve resonance, so that at $t = 0$ a (resonant) steady-state amplitude of 0.50 m is achieved.

a. Two position versus time graphs, each sinusoidal in form and with amplitude 0.50 m, are shown below.

Identify which graph (1 or 2) represents which oscillator ($A$ or $B$). Explain your reasoning.

(Hint: How does the resonant frequency compare to the frequency of the undamped oscillator?)

b. As oscillator $B$ moves, at which (approximate) location does it experience the greatest magnitude of force by the driver (i) in the $+x$ direction? (ii) in the $-x$ direction? Indicate your answers by clearly labeling the appropriate points on the above graph. Explain your reasoning.

[Problem 2 continues on next page]
2.  [continued]

c.  If each division along the time axis represents 0.1 s, calculate the quantities below. Show all
work.
   i.  the angular frequency \( \omega \) of oscillator B
   ii. the damping constant \( \gamma \) of oscillator B
   iii. the angular frequency \( \omega_0 \) that oscillator B would have if the driving force were removed

3.  A series LRC circuit is connected across the terminals of an AC power supply that produces a voltage
\( V(t) = V_o \exp(i\omega t) \). The “equation of motion” for the charge \( q(t) \) across the capacitor is as follows:

\[
L\ddot{q} + R\dot{q} + \frac{q}{C} = V_o \exp(i\omega t)
\]

The above differential equation will have a steady-state solution of the form:

\[
q(t) = q_o \exp(-i(\omega t + \phi))
\]

[Note: The parameters \( q_o \) and \( \phi \) are actually functions of \( \omega \), the frequency of the AC power supply. However, in this problem you will not have to write out these functions in full.]

a.  In terms of \( q_o \), \( \omega \), \( \phi \), and the relevant coefficients from the differential equation, write down the
following functions:
   i.  the potential difference \( \Delta V_C(t) \) across the capacitor
   ii. the potential difference \( \Delta V_R(t) \) across the resistor
   iii. the potential difference \( \Delta V_L(t) \) across the inductor

b.  Determine the values of \( \alpha \), \( \beta \), and \( \gamma \) (in radians) that satisfy the following Euler relations:

\[
\cdot \quad e^{i\alpha} = i \quad \quad \quad \cdot \quad e^{i\beta} = -1 \quad \quad \quad \cdot \quad e^{i\gamma} = -i
\]

c.  Using your results from part b, rewrite the functions in part a so that each function can be written
as a positive real number times a complex exponential. Use your rewritten functions to answer
the following questions.
   i.  What is the phase difference between \( \Delta V_C(t) \) and \( \Delta V_R(t) \)? Is \( \Delta V_C(t) \) ahead of or behind
\( \Delta V_R(t) \) by this amount? Show all work.
   ii. What is the phase difference between \( \Delta V_R(t) \) and \( \Delta V_L(t) \)? Is \( \Delta V_L(t) \) ahead of or behind
\( \Delta V_R(t) \) by this amount? Show all work.
4. A harmonic oscillator with a restoring force $25m\alpha o^2 x$ is subject to a damping force $3m\alpha o \dot{x}$ and a sinusoidal driving force $F_o \cos(10\alpha o t)$.

a. Write down the differential equation that governs the motion of this oscillator.

b. Show that if the driving force were removed the oscillator would become underdamped, and express the frequency of the oscillator in terms of the given quantities. Explain your reasoning.

c. For any damped oscillator that is driven by a sinusoidal external force, we know that the eventual (steady-state) motion is sinusoidal in nature. However, before the oscillator reaches steady state, its motion can be thought of as the algebraic sum of the steady-state motion plus a transient oscillatory motion whose amplitude dies exponentially with time.

[In fact, the transient component of the motion (considered by itself) could accurately describe the motion of the same oscillator with the driving force turned off (but with the damping still present).]

i. Each $x$ vs. $t$ graph below illustrates the actual motion (transient plus steady-state) of a damped, driven oscillator starting at $t = 0$. For each case, is the frequency of the steady-state motion greater than, less than, or equal to that of the transient motion? Explain.

Graph 1

Graph 2

ii. Identify which graph (1 or 2) would better correspond to the damped, driven oscillator described in parts a–c of this problem. Explain your reasoning.