# INSTRUCTOR NOTES
## *Phase space diagrams: Self-limiting oscillators*

### *Emphasis*

This tutorial is intended to give students experience with self-limiting oscillators as a category of non-linear oscillations. Students analyze qualitatively the equation of motion in order to identify regions of phase space in which the damping force acts against or in the direction of velocity. They then examine the asymptotic behavior of a weakly damped self-limiting oscillator.

### *Prerequisites*

Students should already have studied simple harmonic oscillators and damped oscillators, and they should also have already completed the tutorials *Phase space diagrams: Simple harmonic motion* and *Phase space diagrams: Damped harmonic motion*. Coverage of non-linear oscillations is recommended, although this tutorial can be used to introduce this topic as well.

## TUTORIAL PRETEST

The pretest is designed with the assumption that few students (if any) will have had prior experience with self-limiting oscillators, and so the questions on the pretest probe the ability of students to analyze qualitatively the differential equation of motion for such an oscillator,

$$\ddot{x} + \gamma'\left(\frac{x^2}{A^2} + \frac{\dot{x}^2}{B^2} - 1\right)\dot{x} + \omega_o^2 x = 0$$

In particular, students are given three possible sets of initial conditions (as represented by points in phase space) and are asked in each case to determine the relative directions of the velocity and the damping force at the outset of the motion. A correct answer in each case will depend upon whether the point in phase space representing the initial conditions lies inside, outside, or along the elliptical phase space contour,

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{B^2} = 1$$

Students are also asked in each case whether the total mechanical energy of the oscillator will increase, decrease, or remain the same during a small time interval at the outset of the motion.

## TUTORIAL SESSION

### *Equipment and handouts*

Each group will need a whiteboard and set of markers, or a large sheet of paper. Each student will need a copy of the tutorial handout (no special handouts required).

### *Discussion of tutorial worksheet*

*Section I: Equation of motion*

Students begin by comparing and contrasting the differential equations of motion for a self-limiting oscillator and a linearly damped oscillator. The students first review how the form of the equation for the linearly damped oscillator demonstrates why the damping force always opposes the velocity. Then they are guided through the reasoning necessary to determine under what conditions the self-limiting oscillator experiences a damping force acting in the same direction or in the opposite direction of velocity, and therefore under what conditions the damping force would increase or decrease the total energy of the oscillator. The checkpoint at the end of section I is usually helpful to check on the students' progress, especially given the counter-intuitive idea that a "negative" damping force can increase the total energy.

*Section II: Phase space trajectories of a self-limiting oscillator*

The results from the preceding section lead students to make predictions about the possible phase space trajectories that a (weakly damped) self-limiting oscillator can follow. In particular, they recognize that there exists a region of phase space near the origin in which the damping is "negative." They also find that the boundary between the regions of positive and negative damping traces the trajectory that the oscillator eventually follows after sufficient time as elapsed. At this point students are asked to justify the terms *self-limiting oscillator* and *limit cycle* in light of their results. Finally, students are asked a question to help them recognize that the limit cycle, though elliptical in shape in this case, is not to be confused with a trajectory that the oscillator would take if the damping were removed completely.

## TUTORIAL HOMEWORK

The homework contains a challenging qualitative problem that is designed to help students apply and extend their thinking about non-linear oscillators. The problem forces students to consider three differential self-limiting oscillators that are each governed by a unique differential equation:

$$(1) \qquad \ddot{x} + \gamma'\left(\frac{x^2}{A^2} + \frac{\dot{x}^2}{B^2} - 1\right)\dot{x} + \omega_o^2 x = 0$$

$$(2) \qquad \ddot{x} + \gamma\left(\frac{x^2}{A^2} - 1\right)\dot{x} + \omega_o^2 x = 0$$

$$(3) \qquad \ddot{x} + \gamma\left(\frac{\dot{x}^2}{B^2} - 1\right)\dot{x} + \omega_o^2 x = 0$$

The first type (1) is, obviously, the same category of self-limiting oscillator explored in the tutorial. The other two, however, describe oscillators for which the regions of positive and negative damping are determined solely by the position (2) or velocity (3) of the oscillator. Students must match each of three possible limit cycles to one of the above differential equations of motion and carefully explain the reasoning for their choices.

 