From Chebyshev to Bernstein: A Tour of Polynomials Small and Large*

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Introduction

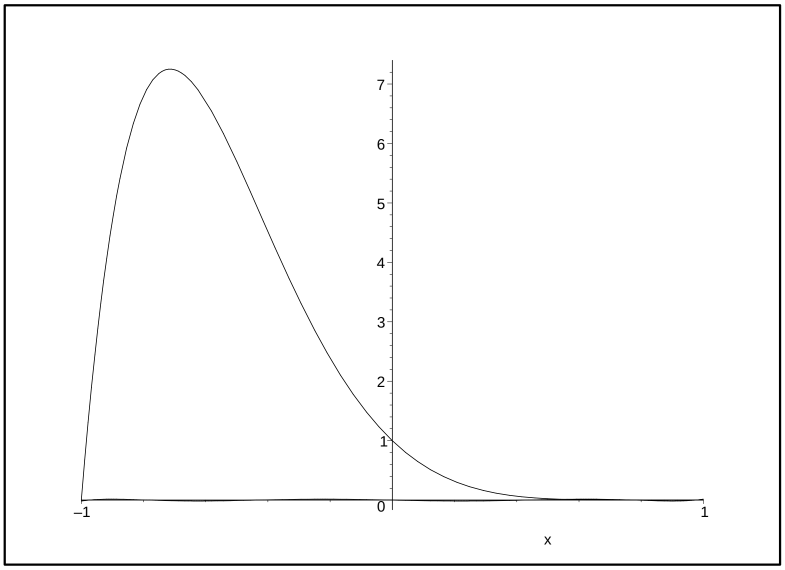

Figure 1: Two monic, degree 7, real polynomials with all real roots in $[-1, 1]$.

Are there really two monic real polynomials of degree exactly 7 in the picture above, both having all real roots in $[-1, 1]$? Indeed there are: one is $T_7(x) = x^7 - \frac{7}{4}x^5 + \frac{7}{8}x^3 - \frac{7}{64}x$, while the other is $B_7(x) = x^7 - 5x^6 + 9x^5 - 5x^4 - 5x^3 + 9x^2 - 5x + 1$. What explains the vast differences in the graphs of these two functions?

Students of mathematics know that for any monic polynomial function, there are two standard representations:

$$p(x) = x_n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

or

$$p(x) = \prod_{i=1}^{n}(x - r_i).$$

In each case, one must select n parameters to uniquely determine p. In the expanded form, we must choose the coefficients $a_{n-1}, \ldots, a_0$, while in factored form it is the roots $r_1, \ldots, r_n$ that generate a formula for the function. We are primarily interested in real polynomials with all real roots, so we initially stipulate that all of the constants a_i and r_i are real.

In this paper, we will focus specifically on how the size (supremum norm) of a polynomial is a function of the location of its roots. Essentially we will study the mapping P that is given by

$$(r_1, \ldots, r_n) \to p_{r_1,\ldots,r_n},$$

where $p_{r_1,\ldots,r_n}(x) = \prod_{i=1}^{n}(x - r_i)$, and examine how the polynomial p is sensitive to changes in the location of one or more of its roots. Several recent papers [1, 2, 3, 5, 12] have shown how the critical numbers of polynomials with all real zeros depend on the location of roots. We will revisit the notion of polyonomial root-dragging to see its implications for the size of a polynomial, and ultimately meet the polynomials who can fairly claim to be the largest and smallest in a natural class of functions.

Background and Preliminaries

The functions T_7 and B_7 in Figure 1 are already prominently known in mathematics. T_7 is the monic degree 7 Chebyshev polynomial, while B_7 is a scaled Bernstein polynomial. Both families of functions have many important applications and a wide range of amazing properties; entire books are devoted to their study [9, 7].

One of the most prominent features of Chebyshev polynomials, defined by

$$T_n(x) = \cos(n \arccos(x)),\ \ n = 0, 1, \ldots$$

is that they are equioscillatory. Indeed, one can show that the value $T_n(x) = \pm 1$ is achieved at each of the $n-1$ critical points of T_n. Since the leading coefficient of T_n is 2^{n-1}, it follows that the monic Chebyshev polynomial of degree n has maximum absolute value 2^{1-n} on $[-1, 1]$. Moreover, a well-known fact (often studied in numerical analysis courses) is that among *all* monic polynomials of degree n, no such function has an absolute extreme value on $[-1, 1]$ as small as the monic Chebyshev polynomial [6]. In this sense, we can reasonably call $2^{1-n}T_n$ the "smallest" monic polynomial of degree exactly n. As such, the fact that T_7 is hiding in Figure 1 is not terribly surprising.

Knowing how small a polynomial can be, it is natural to wonder how large one might be as well. In search of an upper bound for the size of p, we will show that under some reasonable restrictions, certain scaled Bernstein polynomials win the prize as the "largest" polynomial of degree n, and also provide a beautiful formula for this maximum size.

Throughout what follows, given a polynomial p, we use the notation $||p||_{[a,b]}$ to denote the supremum norm of p:

$$||p||_{[a,b]} = \max_{x \in [a,b]} |p(x)|.$$

Initially, we will also assume that p is a real polynomial with all distinct real zeros in $[a, b]$, and $p(a) = p(b) = 0$, with the polynomial scaled so that $-a = b = 1$. Polynomials with *distinct* real zeros are a natural starting point: Chebyshev polynomials (indeed any family of orthogonal polynomials) satisfy this requirement, and there are a variety of interesting and useful recent results on such functions [1, 2, 3, 5, 12]. In addition, it is natural that we place some stipulations on the roots in order to “anchor” the function, for without any restrictions, we can clearly make the norm of p as large as we like. Later in our discussion, we'll consider the case of real polynomials with some complex roots as well.

Polynomial Root-Dragging Revisited

A beautiful result on the critical numbers of polynomials with distinct real zeros is our starting point. In [1], Anderson proved the Polynomial Root-Dragging Theorem:

> If we move some subset of interior zeros of a polynomial with distinct real zeros in one direction, each by at most ϵ, then *all* of the critical numbers must move to the right, and each moves less than ϵ.

In closely examining the behavior of a polynomial when just *one* root is dragged, we noticed some further interesting properties about the relationship between the original function and the new one that results. Consider the graph in Figure 2, and compare the extrema of the original function and the new “dragged” polynomial.

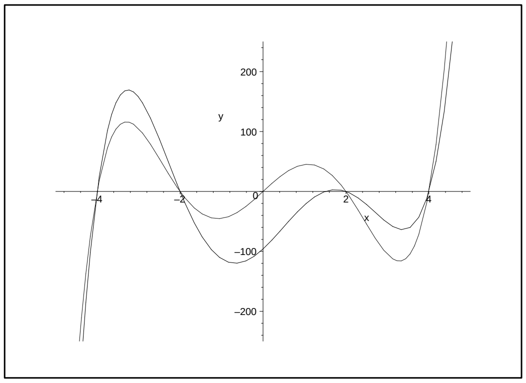

Figure 2: Two degree 5 polynomials with four common roots. The root at $x = 0$ in one was "dragged" to $x = 1.75$ in the other.

Essentially, it appears that when one root is dragged, the polynomial becomes more extreme behind the dragged root, and less extreme ahead of it. That this holds in general is our first result.

Proposition 1. *Let* $p(x) = \prod_{i=1}^{n}(x - r_i)$*, where* $r_1 < r_2 < \ldots < r_n$*, and choose some interior root* r_d *and drag it to the right by* ϵ*, where* $0 < \epsilon < r_{d+1} - r_d$*. Call the resulting new polynomial* $p_\epsilon(x) = \left[\prod_{i \neq d}(x - r_i)\right](x - r_d - \epsilon)$*. Then* $p(x)$ *and* $p_\epsilon(x)$ *are related as follows:*

a. If $x < r_{d-1}$*, then* $|p_\epsilon(x)| \geq |p(x)|$ *(with equality only at the common roots).*

b. If $x > r_{d+1}$*, then* $|p_\epsilon(x)| \leq |p(x)|$ *(again with equality only at the common roots).*

c. Finally, the polynomials p *and* p_ϵ *do not intersect on* (r_{d-1}, r_{d+1})*, and if* $p'(r_{d-1}) > 0$*, then* $p_\epsilon(x) > p(x)$ *for all* $r_{d-1} < x < r_{d+1}$*. The reverse inequality holds if* $p'(r_{d-1})$ *is negative.*

We briefly outline the proof of Proposition 1. There are several key ideas to note: first, it is evident that p and p_ϵ only intersect at values of $x = r_i, i \neq d$, and there both take on the value 0. Furthermore, p and p_ϵ cannot intersect in (r_{d-1}, r_{d+1}), due to the degree of $p - p_\epsilon$. It is also important to observe that for any two polynomials that share zeros at the endpoints of an interval and do not intersect in between, the polynomial with the greater derivative at the left endpoint will have the greater value throughout the interval. This is not hard to prove rigorously by a contradiction argument and using the Taylor series expansion about a of each of the two functions. The details are included in an appendix.

A careful analysis of p' and p'_ϵ now leads to Proposition 1. Specifically, observe that

$$p'(x) = \sum_{j=1}^{n} \prod_{i \neq j} (x - r_i), \text{ and}$$

$$p'_\epsilon(x) = (x - r_d - \epsilon) \sum_{j=1}^{n-1} \prod_{i \neq j,d} (x - r_i) + \prod_{i \neq d} (x - r_i).$$

Evaluating each at r_i, $i \neq d$, we see

$$p'(r_i) = (r_i - r_d) \prod_{j \neq i,d} (r_i - r_j), \text{ while}$$

$$p'_\epsilon(r_i) = (r_i - r_d - \epsilon) \prod_{j \neq i,d} (r_i - r_j).$$

If $i < d$, then $|r_i - r_d| < |r_i - r_d - \epsilon|$, and it follows that $|p'(r_i)| < |p'_\epsilon(r_i)|$, which makes $|p(x)| \leq |p_\epsilon(x)|$ for all $x < r_{d-1}$. Similar reasoning completes the case where $i > d$, and a bit of further care justifies our claim on (r_{d-1}, r_{d+1}).

A corresponding result holds for dragging a root to the left, with appropriate changes in the various inequalities; the result also holds when dragging the rightmost or leftmost root. Proposition 1 even generalizes to polynomials where some of the roots are repeated; the details of this argument are not enlightening, so we omit them here.

We now recall that our overall goal is to understand what makes a polynomials "large." We'll apply the principle of one-root-dragging to see how certain polynomials can have their supremum norm increased.

Dragging Toward Bigger Sup-Norms

Before proceeding, we need to place some reasonable constraints on the zeros of the polynomials under consideration; otherwise it would be possible to make the supremum norm as large as we want, simply by dragging one root off to infinity. Many standard families of polynomials have all of their zeros in the interval $[-1, 1]$, and any polynomial may be scaled to be so. Hence we will assume that all functions under consideration are monic degree n polynomials with n distinct (for now) real zeros in $[-1, 1]$ and fixed roots at -1 and 1. How do we know, given such a function p, if there exists a function q in the same class such that $\|p\|_{[-1,1]} < \|q\|_{[-1,1]}$?

The process is straightforward, according to Proposition 1. Find a critical value that produces the supremum norm of p, and drag an interior root that lies to the right of the extremum even further to the right; this will make the extremum increase. If there are no roots to the right to drag, find a root to the left and drag it further left. This, too, will result in a polynomial with a larger supremum norm. We demonstrate the process through the following example and sequence of graphs. In each plot with two functions, the slightly darker graph is the "updated" function.

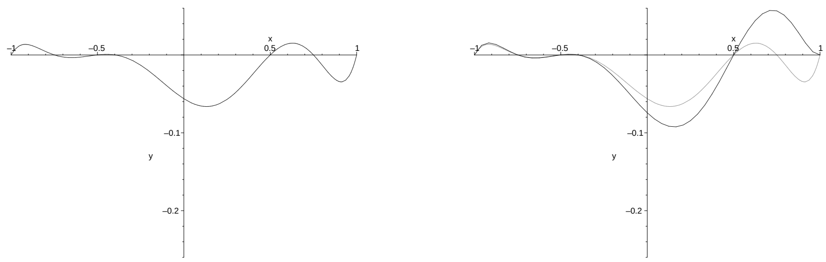

Figure 3: Given the degree 7 polynomial at left above, we choose to drag the root $r_1 = 3/4$ so that $r_1 \to 1$.

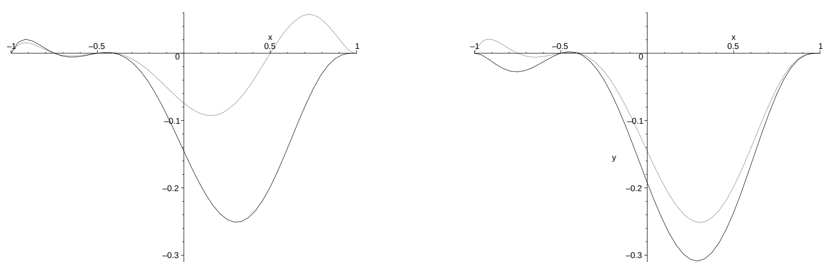

Figure 4: Next, in the updated polynomial at above right in Figure 3, we drag the root $r_2 = 1/2$ so that $r_2 \to 1$. Having completed this, we now take our most recent version of the polynomial, and drag $r_3 = -3/4$ towards -1.

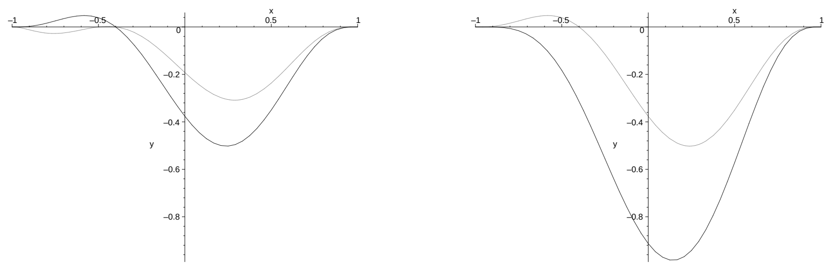

Figure 5: Finally, it only remains to similarly drag the roots at $-1/2$ and $-2/5$ to -1, which results in the final polynomial, $(x-1)^3(x+1)^4$. Note the change of scale on the y-axis as we move from Figure 4 to Figure 5.

While it is not always guaranteed that the extreme value of the updated version of $|p|$ will occur between the same pair of roots, it is clear that we'll ultimately want to drag some subset of the roots to -1, and the remaining roots to $+1$. Formally, this discussion shows that if we begin with a monic polynomial with all distinct real roots in a given interval, we can always find a new monic polynomial, still with all distinct roots in the interval, so that the new function has a greater supremum norm.

Enter Bernstein

This process of increasing the supremum norm leads us to a natural finite collection of polynomials which must contain the largest polynomial having n real zeros in $[-1,1]$, with at least one root at each of ± 1. Particularly, we see from our root-dragging argument above that, in the limit, we arrive at the following functions, all of whose roots lie either at 1 or -1:

$$p_i(x) = (x+1)^i(x-1)^{n-i}, \quad i = 1, \ldots, n-1.$$

These are scaled versions of the aforementioned Bernstein polynomials, traditionally defined on $[0, 1]$ by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

Such polynomials are interesting in their own right, and even have noteworthy patterns in their extrema [8]. We simply want to know which of these polynomials generates the largest supremum norm. While perhaps obvious from their graphs, this question can be rigorously answered with ideas from first semester calculus.

Each p_i will attain its norm at its only critical point interior to $[-1, 1]$; the function's value at all other critical numbers is 0, due to the presence of repeated roots. This critical point that produces the absolute extreme, denoted c_i, is found in the standard way:

$$\begin{aligned} p_i'(x) &= i(x+1)^{i-1}(x-1)^{n-i} + (n-i)(x+1)^i(x-1)^{n-i-1} \\ &= (x+1)^{i-1}(x-1)^{n-i-1}\left[i(x-1) + (n-i)(x+1)\right]. \end{aligned}$$

It follows that c_i is the root of $0 = i(c_i - 1) + (n-i)(c_i + 1)$, and therefore $c_i = \dfrac{2i-n}{n}$. Hence $||p_i||_{[-1,1]} = |p_i(c_i)|$, and

$$\begin{aligned} |p_i(c_i)| &= \left| \left(\frac{2i-n}{n} + 1\right)^i \left(\frac{2i-n}{n} - 1\right)^{n-i} \right| \\ &= \left(\frac{2^i i^i}{n^i}\right) \left| \left(\frac{2^{n-i}(i-n)^{n-i}}{n^{n-i}}\right) \right| \\ &= \left(\frac{2^n}{n^n}\right) i^i (n-i)^{n-i}. \end{aligned}$$

We now must find which value of i yields the greatest supremum norm from among the original $n-1$ polynomials considered. Note that it suffices to maximize $i^i(n-i)^{n-i}$, because $\dfrac{2^n}{n^n}$ is constant with respect to i. We instead maximize the continuous function $g(t) = t^t(n-t)^{n-t}$ on $t \in [1, n-1]$. Moreover, since $g(t)$ and $\ln(g(t))$ will share the same critical points, so we can instead find the critical values of $f(t) = \ln(t^t(n-t)^{n-t}) = t\ln(t) + (n-t)\ln(n-t)$. Observe that

$$\begin{aligned} f'(t) &= 1 + \ln(t) - 1 - \ln(n-t) \\ &= \ln\left(\frac{t}{n-t}\right). \end{aligned}$$

It follows that the only critical number of f (and hence of g) is $t = \frac{n}{2}$. It is straightforward to verify that this value produces a minimum of g on $[1, n-1]$, and thus, in seeking the maximum of g, we turn to the endpoints. In turn, we see that the maximum of g (and therefore of $|p_i(c_i)|$) occurs when $t = 1$ or $t = n-1$ (and these values are identical, because of symmetry in $|p_i(c_i)|$). In particular, the maximum value of $|p_i(c_i)|$ is given when $i = 1$ by

$$|p_1(c_1)| = \frac{2^n}{n^n}(n-1)^{n-1} = \frac{2^n}{n}\left(\frac{n-1}{n}\right)^{n-1}.$$

This proves that for all monic polynomials of degree n with n real zeros in $[-1, 1]$, with roots at 1 and -1,

$$||p||_{[-1,1]} \leq \frac{2^n}{n}\left(\frac{n-1}{n}\right)^{n-1}, \quad n \geq 3. \tag{1}$$

Note that equality is achieved by one of the scaled Bernstein polynomials.

The inequality (1) obviously generalizes to the interval $[a, b]$, where for any monic degree n polynomial $p(x)$ with all real zeros in $[a, b]$ where $p(x)$ has roots fixed at a and b,

$$||p||_{[a,b]} \leq \frac{(b-a)^n}{n}\left(\frac{n-1}{n}\right)^{n-1}.$$

We remark in passing that if $b - a \leq 1$, then $||p||_{[a,b]} \to 0$ as $n \to \infty$, since $\frac{(b-a)^n}{n} \to 0$ and $\left(\frac{n-1}{n}\right)^{n-1} \to e^{-1}$.

When Some Roots are Complex

It is natural as well to wonder what happens if we relax our stipulation that all of the polynomial's roots are real. Is it possible that polynomials with even greater norm result? We continue to examine $\|p\|_{[-1,1]}$, and still assume that p is a real polynomial. Hence, any discussion of complex roots supposes that the roots appear in conjugate pairs. We also assume that any complex roots have modulus at most 1, in parallel to our discussion of the real case. Right away, however, there is a key difference: the Root-Dragging Theorem doesn't hold for complex functions, nor is our one-root-dragging principle applicable. Still, it turns out that the basic idea of dragging roots "outward" is the key.

Let p be a real polynomial with roots at ± 1, and all other roots in the open unit disk, D, where $D = \{z \in \mathbb{C} \ : \ |z| < 1\}$. Assume that p has at least one complex root with non-imaginary part, say $w = a + bi$, so that $\overline{w} = a - bi$ is also a zero of p. Denoting the other roots of p by $r_1, \ldots, r_{n-2}$, we can write[1]

$$p_w(x) = (x - r_1) \cdots (x - r_{n-2}) \cdot (x - w) \cdot (x - \overline{w}).$$

There exists some value $x^* \in [-1, 1]$ such that $|p(x)|$ is maximized by x^*. Fix this value of x^*, as well as all roots of p except for w and $\overline{w}$. Now, we also know that $\|p_w\|_{[-1,1]}$ is given by

$$|p_w(x^*)| = |x^* - r_1| \cdots |x^* - r_{n-2}| \cdot |x^* - w| \cdot |x^* - \overline{w}|. \qquad (2)$$

Here we are viewing the notation $|\cdot|$ as being the modulus of a complex number, while remembering that $p_w(x^*)$ is in fact real. Geometrically, this

[1]Because of the pending emphasis on the role of w, we will now call p "p_w".

value $|p_w(x^*)|$ is measured by the products of the distances from x^* to the various roots of p_w. Allowing ourselves to change the location of certain roots, we observe that there is a direction in which we can move w (and hence $\overline{w}$) that causes $|p_w(x^*)|$ to increase.

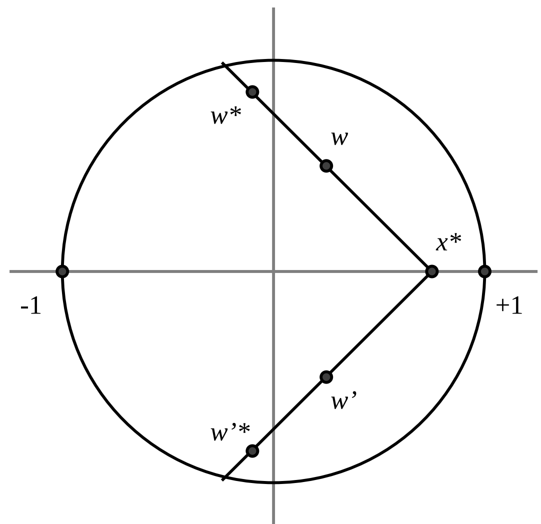

Figure 6: Root w and its conjugate (w') dragged outward towards the unit circle.

In particular, if (as in Figure 6) we "drag" w to w^* along the line from x^* through w (and similarly $\overline{w}$), towards the boundary of D, Equation (2) shows that $|p_w(x^*)|$ must increase. We note specifically that

$$\|p_w\|_{[-1,1]} = |p_w(x^*)| < |p_{w^*}(x^*)| \leq \|p_{w^*}\|_{[-1,1]}.$$

This shows that, given a real monic polynomial of degree n with at least one root in the open unit disk, D, there exists another real monic polynomial, also of degree n, with all its roots in D, having a greater supremum norm. Moreover, to increase the norm as much as possible, we want to drag w to the boundary of the unit circle. Repeating this argument one complex root at a time, this demonstrates that among all real polynomials with (possibly complex) roots in $\overline{D}$, the real polynomial having maximum norm must have all its complex roots on the boundary of the unit circle. Finally, it is apparent that any real roots in $(-1, 1)$ should also be dragged appropriately to ± 1.

This result is analogous to our earlier work with polynomials all of whose zeros are real: there we dragged all interior roots in $[-1, 1]$ outward to the roots at ± 1 in order to increase the supremum norm of the polynomial under consideration. In this case, it remains to consider those real polynomials with roots at ± 1 and any remaining complex zeros w satisfying $|w| = 1$: which of these polynomials has the largest supremum norm? We thus seek the optimum location(s) on the unit circle for the roots we're moving.

The geometry of the situation again informs us. The polynomial p now has form

$$p(x) = (x+1)^j (x-1)^k (x-e^{i\theta_1})(x-e^{-i\theta_1}) \cdots (x-e^{i\theta_{(n-j-k)/2}})(x-e^{-i\theta_{(n-j-k)/2}}).$$

Holding all the complex roots fixed but one (and its conjugate), we can again see how to increase the supremum norm: as before, we know that there is some $x^* \in (-1, 1)$ such that

$$\|p\|_{[-1,1]} = |p(x^*)| = |x^* + 1|^j |x^* - 1|^k |x^* - e^{i\theta_1}| \cdots |x^* - e^{-i\theta_{(n-j-k)/2}}|.$$

Focusing on just the angle related to that root (and its conjugate), say θ_1, it is clear that we want to drag θ_1 to either 0 or π, so that the two complex conjugate roots lie at ± 1. This is again the case because doing so will maximize the quantities $|x^* - e^{i\theta_1}|$ and $|x^* - e^{-i\theta_1}|$, as seen in Figure 7. Repeating this argument, we find that we must drag all of the complex roots on the boundary of the unit circle to one or the other of the roots at -1 and $+1$. This leaves us in precisely the same case as we found ourselves

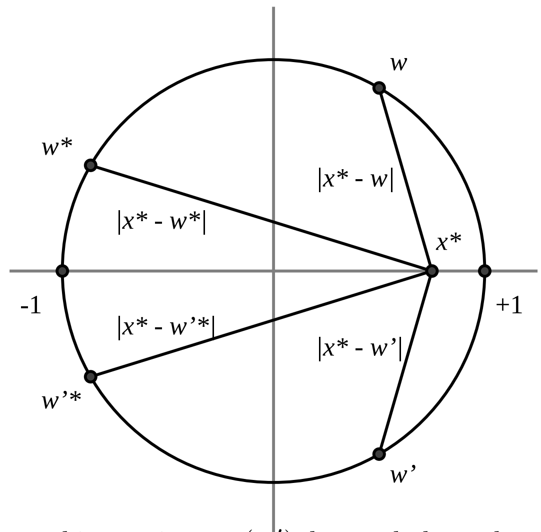

Figure 7: Root w and its conjugate (w') dragged along the unit circle towards -1.

with real polynomials earlier where it became sufficient to consider

$$p_i(x) = (x+1)^i (x-1)^{n-i}, \quad i = 1, \dots, n-1.$$

Thus, we see that the inclusion of complex roots in $\overline{D}$ cannot increase the greatest possible norm of a real, monic polynomial of degree n. In particular, we can now state the more general version of our result:

Theorem 1. *For all monic real polynomials of degree $n \geq 2$ with all zeros in the closed unit disk in $\mathbb{C}$, and roots at 1 and -1,*

$$\|p\|_{[-1,1]} \leq \frac{2^n}{n} \left(\frac{n-1}{n} \right)^{n-1}. \tag{3}$$

Thus, we can reasonably claim that the scaled Bernstein polynomial $p_{1,n}(x) = (x+1)(x-1)^{n-1}$ is the largest polynomial in the class of all real polynomials with zeros at ± 1 and all remaining zeros in the closed unit disk. Being that the monic Chebyshev polynomial $2^{1-n} T_n(x)$ has the smallest supremum norm among *all* degree n polynomials, certainly it is the smallest among those with our standard restrictions to zeros in the unit disk. We note in conclusion that, since the Chebeyshev polynomials are not zero at ± 1, the lower bound can actually be tightened for the functions under consideration, simply by an appropriate transformation of T_n. Ultimately, we can say that if p is a monic degree n polynomial with roots in the unit disk and roots at ± 1, then

$$\frac{1}{2^{n-1}} < \frac{1}{2^{n-1} \cdot \cos^n(\frac{\pi}{2n})} \leq \|p\|_{[-1,1]} \leq \frac{2^n}{n} \left(\frac{n-1}{n}\right)^{n-1}. \tag{4}$$

Returning to Figure 1, we can now say we understand that only the one polynomial appears present not simply because the Chebyshev polynomial is the smallest, but also because the scaled Bernstein polynomial is the largest.

Appendix 1

We here offer a formal proof of a result cited in the justification of the principle of one-root-dragging.

Lemma 1. *Let $f(x)$ and $g(x)$ be polynomial functions on $[a, b]$ and assume that $f(a) = g(a) = 0$ and $f(b) = g(b) = 0$. Moreover, assume that f and g do not intersect on (a, b). If $f'(a) > g'(a)$, then $f(x) > g(x)$ for all $x \in (a, b)$; similarly, if $f'(a) < g'(a)$, then $f(x) < g(x)$ for all $x \in (a, b)$.*

Proof. Let $f(x)$ and $g(x)$ be as above. Since f and g do not intersect anywhere on (a, b), it must be the case that either $f(x) < g(x)$ for all $x \in (a, b)$, or $g(x) < f(x)$ for all $x \in (a, b)$.

Now, assume to the contrary that $f'(a) > g'(a)$ and $f(x)$ is not greater than $g(x)$ for some $x \in (a, b)$. Then since they don't intersect, it must be the case that $f(x) < g(x)$ for all $x \in (a, b)$.

Now write the Taylor series expansion of both $f(x)$ and $g(x)$ about a to get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

and

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \cdots$$

Dividing both sides of the inequality by $(x-a)$, and noticing that both $f(a)$ and $g(a)$ are 0, we get

$$f'(a) + \frac{f''(a)}{2!}(x-a) + \cdots < g'(a) + \frac{g''(a)}{2!}(x-a) + \cdots .$$

Now, as we take the limit as $x \to a$ of both sides of the inequality, we see that $f'(a) \leq g'(a)$. But this is a contradiction, since we assumed that $f'(a) > g'(a)$. Thus, it must be the case that $f(x) > g(x)$ for all $x \in (a, b)$. □

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