A Note on PIPCIRs[*]

Matthew R. Boelkins[†] Matthew J. Wells[‡]

October 27, 2004

Abstract

PIPCIRs are Polynomials whose Inflection Points Coincide with their Interior Roots. In this note, we demonstrate that PIPCIRs are also orthogonal polynomials on a subspace of $C[-1,1]$, that they can be generated by a 3-term recurrence relation, and that every derivative of a PIPCIR is a Jacobi polynomial.

---

[*] This work was supported by a GVSU Summer Undergraduate Research Program Grant.

[†] Grand Valley State University

[‡] The University of Kentucky

1 An Overview of PIPCIRs

PIPCIRs are Polynomials whose Inflection Points Coincide with their Interior Roots. The existence of such functions was first noted by Anderson in [1] where he examined possible locations of the inflection points of certain polynomials relative to their roots and critical numbers. Since his discovery, Andrews and Belinsky [2] have demonstrated two key links between PIPCIRs and classical orthogonal polynomials. In this note, we explore further how PIPCIRs behave like orthogonal polynomials themselves.

A priori, a PIPCIR is a polynomial with n distinct real roots. We stipulate that each PIPCIR has its least and greatest roots at ± 1, and thus all interior roots in $(-1, 1)$ are also inflection points. Anderson proved that for each degree there exists a unique monic PIPCIR; following the notation in [2], we denote this function by $\tilde{Q}_n(x)$, $n \geq 2$, while letting $Q_n(x)$ represent any scalar multiple of $\tilde{Q}_n(x)$.

By definition [1], PIPCIRs are solutions to the differential equation

$$(x^2 - 1)Q_n''(x) = n(n-1)Q_n(x). \tag{1}$$

There are at least two different ways to solve this equation. Anderson assumed a polynomial form for $Q(x)$ and equated like coefficients in (1) to generate a descending recurrence relation for the coefficients of $Q_n(x)$; Andrews and Belinsky solved for $Q_n(x)$ by manipulating (1) to a form of Jacobi's differential equation.

2 PIPCIRs and the Link to Jacobi Polynomials

We first recall that for any $\alpha, \beta > -1$, the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ are, by definition, a set orthogonal on $C[-1, 1]$ with respect to the inner product

$$\langle f, g \rangle_{\alpha,\beta} = \int_{-1}^{1} f(x) \cdot g(x) \cdot (1-x)^{\alpha}(1+x)^{\beta}\, dx. \tag{2}$$

Note we must assume that $\alpha, \beta > -1$ to ensure that $(1-x)^{\alpha}(1+x)^{\beta}$ is integrable. It turns out that each Jacobi polynomial is also a solution to Jacobi's differential equation,

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0, \tag{3}$$

where n is a non-negative integer and α, $\beta \in \mathbb{R}$. These and other standard facts about orthogonal polynomials can be found in Szegö's classic text [4]. When $\alpha = \beta = 0$, Equation (3) is usually called Legendre's differential equation, with solution $P_n(x) = P_n^{(0,0)}(x)$.

By rearranging Jacobi's differential equation in insightful ways, Andrews and Belinski were able to deduce the following key results in [2]:

> Theorem 1. [Andrews and Belinsky] The derivative, $Q_n'(x)$, of a degree n PIPCIR is a solution to the degree $n-1$ Legendre differential equation, and therefore, $Q_n'(x) = c_n P_{n-1}(x)$, where P_{n-1} is the $(n-1)$st Legendre polynomial and $c_n \in \mathbb{R}$.

> Theorem 2. [Andrews and Belinsky] If we factor $Q_{n+2}(x)$ by writing $Q_{n+2}(x) = (1-x^2)q_n(x)$, then $q_n(x)$ is a solution to the differential equation
>
> $$(1-x^2)q_n'' - 4xq_n' + n(n+3)q_n = 0,$$
>
> which is a special case of (3) with $\alpha = \beta = 1$. It follows that $q_n(x) = k_n P_n^{(1,1)}(x)$ for some $k_n \in \mathbb{R}$.

Each of the above two theorems assiduously avoid values of α and β that equal -1; but what one is really tempted to do with Jacobi's differential equation is simply substitute $\alpha = \beta = -1$. By doing so, the coefficient of y' vanishes, and we're left with

$$(1-x^2)y'' + n(n-1)y = 0,$$

which is (1), the defining property of PIPCIRs. How does this substitution shed further light on the connections between PIPCIRs and Jacobi polynomials? This question motivates our discussion here.

3 Are PIPCIRs Orthogonal Polynomials?

The standard Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are not defined when $\alpha = \beta = -1$; this is, of course, due to the fact that the weighting function used in the inner product (2) is not integrable in this case. Nonetheless, Jacobi's differential equation still has polynomial solutions (which are obviously PIPCIRs) when $\alpha = \beta = -1$. There do exist functions termed "generalized" Jacobi polynomials that satisfy Jacobi's differential equation and share certain properties with the standard Jacobi polynomials; more on these functions can be found in [4, p. 62ff].

Based on these observations and the key results in [2], it is apparent that PIPCIRs have much in common with orthogonal polynomials: are these functions a class of orthogonal functions themselves? That is, is there a weighting function $w(x)$ such that the PIPCIRs $Q_n(x)$ form a family of orthogonal polynomials in $C[-1, 1]$? The immediate answer, unfortunately, is no. A classic result on orthogonal polynomials states that for an inner product defined on $C[a, b]$ with weighting function $w(x)$, the zeros of the associated orthogonal polynomials are real, distinct, and located *in the interior* of $[a, b]$ [4, Theorem 3.3.1]. Because we consider PIPCIRs on $[-1, 1]$ and stipulate that they always have first and last roots at the endpoints of this interval, it follows that they cannot be orthogonal in $C[-1, 1]$.

It turns out, however, that we can view the PIPCIRs as being orthogonal on a certain subspace of $C[-1, 1]$; this enables us to easily generate the sequence $\{\tilde{Q}_n(x)\}$ via a three-term recurrence relation (in a nearly identical way to that which generates standard orthgonal polynomials). We will also observe an interesting relationship among the derivatives of Jacobi polynomials that sheds further light on how PIPCIRs are antiderivatives of Legendre polynomials and second antiderivatives of the Jacobi polynomials $P_n^{(1,1)}(x)$.

4 A Recurrence Relation for PIPCIRs

Given an inner product $\langle \cdot, \cdot \rangle$ on $C[-1,1]$, it is well known that one can generate a family of orthogonal polynomials, $\{p_n(x)\}$, by applying the Gram-Schmidt process to the standard basis $\{1, x, x^2, \ldots\}$ [3, p.174]. We will assume that $\langle \cdot, \cdot \rangle$ is defined by $\langle f, g \rangle = \int_{-1}^{1} f(x) \cdot g(x) \cdot w(x)\, dx$, where the weighting function $w(x)$ is assumed to be non-negative and integrable on $[-1,1]$. Taking $w(x) = (1-x)^\alpha (1+x)^\beta$ (with $\alpha, \beta > -1$) leads to the aforementioned Jacobi polynomials. Furthermore, the orthogonal polynomials generated by Gram-Schmidt satisfy a three-term recurrence relation given by $p_0(x) = 1,\ p_1(x) = x - a_1$, and

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x),\ n \geq 2, \tag{4}$$

where

$$a_n = \frac{\langle x p_{n-1}(x), p_{n-1}(x) \rangle}{\langle p_{n-1}(x), p_{n-1}(x) \rangle}\ (n \geq 1) \text{ and } b_n = \frac{\langle x p_{n-2}(x), p_{n-1}(x) \rangle}{\langle p_{n-2}(x), p_{n-2}(x) \rangle}\ (n \geq 2).$$

We note that in the event that $w(x)$ is even (as it will be in much of what follows), the coefficients a_n vanish in (4). This is true because, as is stated in [4, p. 29], when given an inner product on an interval that is symmetric about the origin (such as $[-1,1]$), and a weighting function $w(x)$ that is even, the corresponding orthogonal polynomials p_n are even or odd according to the parity of n. From this we observe if $p_{n-1}(x)$ is odd, then $x p_{n-1}(x)$ is even, and therefore when we compute the numerator of a_n, we are integrating an odd function over $[-1,1]$, so $a_n = 0$. The same is true when $p_{n-1}(x)$ is even.

The form of the recurrence relation also guarantees that the sequence $\{p_n(x)\}$ will be monic. We remark that the standard normalization of Jacobi polynomials is done so that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ [4, (4.1.1)]; monic Jacobi polynomials will here be denoted $\tilde{P}_n^{(\alpha,\beta)}(x)$.

We now use the known recurrence relation for Jacobi polynomials, together with Theorem 2 of Andrews and Belinski to generate a similar relationship satisfied by PIPCIRs.

Theorem 3. *The monic PIPCIRs $\tilde{Q}_n(x)$, $n \geq 2$, satisfy a three term* *recurrence relation given by* $\tilde{Q}_2(x) = x^2 - 1,\ \tilde{Q}_3(x) = x^3 - x$ *and*

$$\tilde{Q}_{n+2}(x) = x\tilde{Q}_{n+1}(x) - b_n\tilde{Q}_n(x),\ n \geq 2,\ \textit{where } b_n = \frac{\langle x\tilde{Q}_n, \tilde{Q}_{n+1}\rangle_{-1}}{\langle \tilde{Q}_n, \tilde{Q}_n\rangle_{-1}},$$

with $\langle \cdot, \cdot \rangle_{-1}$ *the inner product** *defined by* $\langle f, g\rangle_{-1} = \int_{-1}^{1} f(x) \cdot g(x) \cdot \frac{1}{1-x^2}\ dx.$

***Remark.** The functional $\langle \cdot, \cdot \rangle_{-1}$ is not an inner product on $C[-1,1]$ because the weighting function $(1-x^2)^{-1}$ is not integrable on $[-1,1]$. However, $\langle \cdot, \cdot \rangle_{-1}$ is an inner product on the subspace $\mathcal{H} = \{f \in C[-1,1] : (1-x^2)^{-1}f(x) \text{ is integrable on } [-1,1]\}$. Since all PIPCIRs belong to $\mathcal{H}$, the singularity will vanish in every case that we apply $\langle \cdot, \cdot \rangle_{-1}$.

Proof. By Theorem 2, we can write $\tilde{Q}_{n+2}(x) = (x^2-1)\tilde{q}_n(x)$, where $\tilde{q}_n(x) = \tilde{P}_n^{(1,1)}(x)$. We generate the polynomials $\{\tilde{q}_n(x)\}$ via the standard recurrence relation using the inner product with $w(x) = (1-x)^1(1+x)^1$, which is

$$\langle f, g\rangle_1 = \int_{-1}^{1} f(x) \cdot g(x) \cdot (1-x^2)\, dx.$$

Since $w(x)$ is even, it follows that $\tilde{q}_0(x) = 1$, $\tilde{q}_1(x) = x$, and

$$\tilde{q}_n(x) = x\tilde{q}_{n-1}(x) - b_n\tilde{q}_{n-2}(x),\ \text{where } b_n = \frac{\langle x\tilde{q}_{n-2}, \tilde{q}_{n-1}\rangle_1}{\langle \tilde{q}_{n-2}, \tilde{q}_{n-2}\rangle_1}. \tag{5}$$

Writing $\tilde{Q}_{n+2}(x) = (x^2-1)\tilde{q}_n(x)$ and applying the recurrence relation for $\tilde{q}_n(x)$ given in (5), we find $\tilde{Q}_2(x) = x^2 - 1$, $\tilde{Q}_3(x) = x(x^2-1)$, and for $n \geq 2$,

$$\tilde{Q}_{n+2}(x) = x\tilde{Q}_{n+1}(x) - b_n\tilde{Q}_n(x). \tag{6}$$

This gives one form of a recurrence relation, but (6) still computes the coefficients b_n in terms of the Jacobi polynomials $\tilde{q}_n(x)$, rather than the PIPCIRs themselves. To that end, we note that

$$\langle \tilde{q}_n, \tilde{q}_n\rangle_1 = \int_{-1}^{1} \tilde{q}_n(x)\tilde{q}_n(x)(1-x^2)\, dx = \int_{-1}^{1} \frac{\tilde{Q}_{n+2}(x)}{1-x^2}\frac{\tilde{Q}_{n+2}(x)}{1-x^2}(1-x^2)\, dx.$$

Simplifying this last expression, we see that

$$\langle \tilde{q}_n, \tilde{q}_n\rangle_1 = \int_{-1}^{1} \tilde{Q}_{n+2}(x) \cdot \tilde{Q}_{n+2}(x) \cdot \frac{1}{1-x^2}\, dx = \langle \tilde{Q}_{n+2}, \tilde{Q}_{n+2}\rangle_{-1}.$$

Similar reasoning allows us to write $\langle x\tilde{q}_{n-1}, \tilde{q}_{n-1}\rangle_1 = \langle x\tilde{Q}_{n+1}, \tilde{Q}_{n+1}\rangle_{-1}$ and $\langle x\tilde{q}_{n-1}, \tilde{q}_{n-2}\rangle_1 = \langle x\tilde{Q}_{n+1}, \tilde{Q}_n\rangle_{-1}$. Substituting appropriately in the expression for b_n in (5), we have proved that $\tilde{Q}_2(x) = x^2 - 1$, $\tilde{Q}_3(x) = x(x^2-1)$, and, for $n \geq 2$,

$$\tilde{Q}_{n+2}(x) = x\tilde{Q}_{n+1}(x) - b_n\tilde{Q}_n(x), \text{ where } b_n = \frac{\langle x\tilde{Q}_n, \tilde{Q}_{n+1}\rangle_{-1}}{\langle \tilde{Q}_n, \tilde{Q}_n\rangle_{-1}}.$$

□

It is now an easy exercise to generate the PIPCIRs $\tilde{Q}_n(x)$:

$$\begin{aligned}
\tilde{Q}_4(x) &= x^4 - \frac{6}{5}x^2 + \frac{1}{5} \\
\tilde{Q}_5(x) &= x^5 - \frac{10}{7}x^3 + \frac{3}{7}x \\
\tilde{Q}_6(x) &= x^6 - \frac{5}{3}x^4 + \frac{5}{7}x^2 - \frac{1}{21} \\
\tilde{Q}_7(x) &= x^7 - \frac{21}{11}x^5 + \frac{35}{33}x^3 - \frac{5}{33}x \\
&\vdots
\end{aligned}$$

5 Derivatives of PIPCIRs

Theorems 1 and 2 also show a fascinating link between classical orthogonal polynomials and the derivatives of PIPCIRs: the first shows that

$$\frac{d}{dx}[Q_n(x)] = c_{1n}P_{n-1}^{(0,0)}(x),$$

for some constants c_{1n}. The second proposition and the defining property of PIPCIRs demonstrate that

$$\frac{d^2}{dx^2}[Q_n(x)] = n(n-1)q_{n-2}(x) = c_{2n}P_{n-2}^{(1,1)}(x).$$

A natural conjecture is that

$$\frac{d^m}{dx^m}[Q_n(x)] = c_{mn}P_{n-m}^{(m-1,m-1)}(x), \text{ for all } m \leq n. \tag{7}$$

Essentially we can read this statement as saying that the zeros of each derivative of a PIPCIR are themselves zeros of some associated Jacobi polynomial.

This is indeed the case, due to the following result mentioned briefly in Szegö's text [4] (4.21.7):

$$\frac{d}{dx}[P_n^{(\alpha,\beta)}(x)] = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

Here, we present a slightly weaker result, showing simply that the zeros of the derivative of $P_n^{(\alpha,\beta)}(x)$ must be the same as the zeros of $P_{n-1}^{(\alpha+1,\beta+1)}(x)$ by working with Jacobi's differential equation and manipulating it in a similar fashion to some of the arguments in [2].

Let y be a solution of Jacobi's differential equation (3), and differentiate both sides of the equation with respect to x. Doing so and regrouping terms yields

$$(1-x^2)y''' + [\beta-\alpha-(\alpha+\beta+4)x]y'' + \{n(n+\alpha+\beta+1)-(\alpha+\beta+2)\}y' = 0.$$

It is straightforward to verify that

$$n(n+\alpha+\beta+1)-(\alpha+\beta+2) = (n-1)(n+(\alpha+1)+(\beta+1)).$$

Noting that $\beta-\alpha-(\alpha+\beta+4)x = (\beta+1)-(\alpha+1)-((\alpha+1)+(\beta+1)+2)x$, it follows that we can rewrite the most recent differential equation as

$$(1-x^2)y'''+[(\beta+1)-(\alpha+1)-((\alpha+1)+(\beta+1)+2)x]y''+\{(n-1)(n+(\alpha+1)+(\beta+1))\}y' = 0.$$

Comparing this last equation to (3), we see that y' is a solution to Jacobi's differential equation in the case where n has been changed to $(n-1)$ and α and β have been respectively replaced by $\alpha+1$ and $\beta+1$. In particular, if y is a solution to Jacobi's equation (so that $y = k_n P_n^{(\alpha,\beta)}(x)$), then it follows that y' is also a solution to a related Jacobi equation, specifically $y' = k_n' P_{n-1}^{(\alpha+1,\beta+1)}(x)$. As the constants k_n and k_n' depend on α and β, we have shown that

$$\frac{d}{dx}\left[P_n^{(\alpha,\beta)}(x)\right] = c_n^{(\alpha,\beta)} P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

for all $\alpha, \beta > -1$ and all $n = 1, 2, \ldots$, where $c_n^{(\alpha,\beta)}$ is some constant.

Note particularly that (7) holds by induction on m. This shows that the PIPCIRs $Q_n(x)$ live at the "top" of a chain of orthogonal polynomials that are connected by differentiation (modulo a constant multiplier):

$$Q_n(x) \to P_{n-1}^{(0,0)} \to P_{n-2}^{(1,1)} \to P_{n-3}^{(2,2)} \to \cdots ,$$

where each arrow represents the process of differentiation.

6 The Orthogonality of PIPCIRs

Near the end of the proof of Theorem 3 we saw how the orthogonality of the functions $\tilde{q}_n(x) = \tilde{P}_n^{(1,1)}(x)$ on $C[-1,1]$ can be used to compute the coefficients of PIPCIRs $\tilde{Q}_n(x)$ through a recurrence relation. Here we remark formally that these ideas demonstrate that PIPCIRs are orthogonal on a certain subspace of $C[-1,1]$. In particular, for $i \neq j$ and $i, j \geq 2$,

$$\begin{aligned}
\langle \tilde{Q}_i, \tilde{Q}_j \rangle_{-1} &= \int_{-1}^{1} \tilde{Q}_i(x)\tilde{Q}_j(x)(1-x^2)^{-1}\, dx \\
&= \int_{-1}^{1} \tilde{P}_{i-2}^{(1,1)}(x)(1-x^2) \cdot \tilde{P}_{j-2}^{(1,1)}(x)(1-x^2)(1-x^2)^{-1}\, dx \\
&= \int_{-1}^{1} \tilde{P}_{i-2}^{(1,1)}(x) \cdot \tilde{P}_{j-2}^{(1,1)}(x) \cdot (1-x^2)\, dx \\
&= \langle \tilde{P}_{i-2}^{(1,1)}, \tilde{P}_{j-2}^{(1,1)} \rangle_1 \\
&= 0,
\end{aligned}$$

where the last equality follows from the known orthogonality of the Jacobi polynomials. This shows that the PIPCIRs are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{-1}$ on $\mathcal{H}$, the subspace of functions in $C[-1,1]$ that are integrable against $(1-x^2)^{-1}$.

That PIPCIRs also satisfy a three-term recurrence relation and have derivatives that are Jacobi polynomials suggests that the PIPCIRs rightly deserve inclusion with any treatment of the classical orthogonal polynomials.

References

[1] B. Anderson, Where the Inflection Points of a Polynomial May Lie, Math Magazine **70** (1997), 32–39.

[2] R. Belinsky and P. Andrews, PIPCIRs – Polynomials Whose Inflection Points Coincide with their Interior Roots, Math Magazine **73** (2000), 285–292.

[3] G. W. Stewart, *Afternotes on Numerical Analysis*, SIAM 1996.

[4] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. 1967.