$\zeta(2)$, $\zeta(3)$, and Beyond...

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Undergraduate Thesis

Introduction

Infinite series are fascinating mathematical objects that are first encountered in basic calculus but can still be found at the frontiers of current research. The convergence of infinite series is an issue of analysis, but the series themselves have a kind of intrinsic number-theoretic appeal. For instance, one is led rather naturally to inquire about *the sum of the reciprocals of the squares of the positive integers*,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

The great mathematicians of the 18th century made such an inquiry, knowing that the series converged but not knowing *to what.* The answer was not so obvious, but in 1735 a young Leonhard Euler succeeded in coming up with the correct and surprising value of $\pi^2/6$. So what about the analogous series for *cubes*,

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots ?$$

This one proved beyond the reach of Euler. Although the series for squares is itself no trivial matter, it seems so in comparison with the one for cubes. These two series, today called $\zeta(2)$ and $\zeta(3)$ for reasons we'll see in a moment, are special instances of

$$\sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

When $s \in \mathbb{R}$, a calculus student can deduce from the integral

$$\int_1^{\infty} \frac{1}{x^s}\, dx,$$

that (1) converges if and only if $s > 1$. In fact, if we consider s to be a complex number $a + bi$, with complex exponentiation given by

$$n^s = e^{s \log n} = e^{(a+bi) \log n} = e^{a \log n}(e^{ib \log n}) = n^a[\cos(b \log n) + i \sin(b \log n)],$$

the series in (1) converges in $\mathbb{C}$ for and only for s with real part $a > 1$. In 1859, Bernhard Riemann succeeded in defining a (unique) function $\zeta(s)$ analytic on all of $\mathbb{C}$ except for a simple pole at $s = 1$ and coinciding with the sum (1) for s with real part greater than 1. This function is known as the Riemann zeta function and is of extreme importance in modern number theory. For a lengthy account of this interesting function see [5]. The significance of $\zeta(s)$ is due to its connection with prime numbers. Leonhard Euler was the first to demonstrate such a connection by establishing the so-called Euler product formula,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \tag{2}$$

where the product is taken over all primes p. This identity is valid for all s with real part > 1 and is essentially a consequence of the Fundamental Theorem of Arithmetic: every integer has a unique decomposition into prime factors.

It can be shown that $\zeta(s) = 0$ when s is a negative even integer. The famous Riemann hypothesis concerns the locations of the remaining, 'nontrivial' zeros of $\zeta(s)$. It asserts that they all have real part equal to $1/2$. A proof of the Riemann hypothesis would yield important results surrounding the distribution of prime numbers among the natural numbers. The prime number theorem concerns an approximation of the function $\pi(x)$, the number of primes less than or equal to x. The theorem says that for large x, $\pi(x)$ is approximately the logarithmic integral of x, which is given by

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log t} dt.$$

The precise statement is that $\pi(x)$ is *asymptotic* to $\mathrm{li}(x)$, meaning

$$\lim_{x\to\infty} \frac{\pi(x)}{\mathrm{li}(x)} = 1.$$

This is often phrased equivalently as $\pi(x)$ is asymptotic to $x/\ln x$. The validity of the Riemann hypothesis would yield important information about exactly how well $\mathrm{li}(x)$ approximates $\pi(x)$. Specifically, it would imply that there is a constant c such that, for any x, $\mathrm{li}(x)$ is within $c\sqrt{x}\log x$ of $\pi(x)$. See [8].

Despite the product in (2), Euler's involvement with $\zeta(s)$ wasn't so much centered around primes as it was the sums which arise from $\zeta(s)$ for $s = 2, 3, 4, ...$ Despite their seeming innocence, some of these sums have continued to confound the most brilliant of mathematicians up to this day. Even the case of $s = 2$ proved deceptively challenging, but it was Euler who finally cracked it. In 1735 he offered the shocking and beautiful formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3}$$

As was often the case for the bold Euler, his proof was lacking in mathematical rigor but the end result was correct. In particular, he fearlessly extended properties of polynomials to power series. For some of the history behind this so-called "Basel Problem" and Euler's solution to it, see [4]. We'll now examine some more modern, rigorous proofs of (3).

Apostol's Approach

The following is due to Tom Apostol [2]. We consider the integral

$$I = \int_0^1 \int_0^1 \frac{1}{1-xy} dxdy.$$

It should be noted that the integral is improper, the denominator of the integrand vanishing at the point $(1,1)$. This can be justified by writing

$$\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\int_0^1 \frac{1}{1-xy}dxdy.$$

The integrand can be expanded in a geometric series, so we have

$$I=\lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\int_0^1 \sum_{n=0}^{\infty} x^n y^n dxdy.$$

We'd like to exchange the inner integral and the summation sign. in order to do this, we need $\sum x^n y^n$ to converge uniformly as a function of x on $[0,1]$, with y fixed in $[0,1-\varepsilon]$. The Weierstrass M-Test allows us to easily verify this: With $y\in[0,1-\varepsilon]$ we have $x^n y^n \le 1-\varepsilon$ for all $x\in[0,1]$, and $\sum(1-\varepsilon)^n$ is a convergent geometric series.

$$\begin{aligned}
I &= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\int_0^1 \left(\sum_{n=0}^{\infty} x^n y^n\right) dxdy \\
&= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\sum_{n=0}^{\infty}\left(\int_0^1 x^n y^n dx\right) dy \\
&= \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\sum_{n=0}^{\infty}\frac{y^n}{n+1}dy.
\end{aligned}$$

Again by uniform convergence, we can bring the integral inside the sum:

$$\begin{aligned}
I &= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\int_0^{1-\varepsilon}\frac{y^n}{n+1}dy \\
&= \lim_{\varepsilon\to 0}\sum_{n=0}^{\infty}\frac{(1-\varepsilon)^{n+1}}{(n+1)^2}
\end{aligned}$$

Uniform convergence of the above sum as a function of ε allows us to take the limit termwise:

$$\begin{aligned}
I &= \sum_{n=0}^{\infty}\lim_{\varepsilon\to 0}\frac{(1-\varepsilon)^{n+1}}{(n+1)^2} \\
&= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}.
\end{aligned}$$

This is precisely $\zeta(2)$. Now we evaluate I another way. We introduce the change of variables

$$x = \frac{u-v}{\sqrt{2}}, \quad y = \frac{u+v}{\sqrt{2}},$$

which rotates the coordinate axes clockwise through an angle of $\pi/4$. This transforms the integrand to

$$\begin{aligned}\frac{1}{1-xy} &= \frac{1}{1-(u^2-v^2)/2} \\ &= \frac{2}{2-u^2+v^2}.\end{aligned}$$

The region of integration is transformed into a square in the uv-plane with vertices at $(0,0)$, $(\sqrt{2}/2, 1)$, $(\sqrt{2}, 0)$, and $(\sqrt{2}/2, -1)$. Since map is a rotation, the determinant of its Jacobian matrix is 1. Taking into account the symmetries of the integrand and the region of integration, we have

$$I = 4\int_0^{\sqrt{2}/2} \left(\int_0^u \frac{dv}{2-u^2+v^2} \right) du \quad + \quad 4\int_{\sqrt{2}/2}^{\sqrt{2}} \left(\int_0^{\sqrt{2}-u} \frac{dv}{2-u^2+v^2} \right) du.$$

To evaluate the inner integrals we use the fact that

$$\int_0^x \frac{dt}{a^2+t^2} = \frac{1}{a} \arctan \frac{x}{a}.$$

We have

$$\int_0^u \frac{dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}} \arctan \frac{u}{\sqrt{2-u^2}}$$

and

$$\int_0^{\sqrt{2}-u} \frac{dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}} \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}}.$$

Thus

$$I = 4\int_0^{\sqrt{2}/2} \arctan\left(\frac{u}{\sqrt{2-u^2}} \right) \frac{du}{\sqrt{2-u^2}} \quad + \quad 4\int_{\sqrt{2}/2}^{\sqrt{2}} \arctan\left(\frac{\sqrt{2}-u}{\sqrt{2-u^2}} \right) \frac{du}{\sqrt{2-u^2}}.$$

Let's the call the first integral I_1 and the second I_2. For I_1 we make the substitution

$$u = \sqrt{2}\sin\theta,$$

so that

$$du = \sqrt{2}\cos\theta d\theta = \sqrt{2-u^2}d\theta$$

and

$$\tan\theta = \frac{u}{\sqrt{2-u^2}}.$$

This yields

$$I_1 = 4\int_0^{\pi/6}\theta d\theta = 2\left(\frac{\pi}{6}\right)^2.$$

For I_2 we let

$$u = \sqrt{2}\cos 2\theta,$$

whereby

$$du = -2\sqrt{2}\sin 2\theta d\theta = -2\sqrt{2}\sqrt{1-\cos^2 2\theta}d\theta = -2\sqrt{2}\sqrt{1-u^2/2}d\theta = -2\sqrt{2-u^2}d\theta$$

and

$$\frac{\sqrt{2}-u}{\sqrt{2-u^2}} = \frac{\sqrt{2}(1-\cos 2\theta)}{\sqrt{2-2\cos^2 2\theta}} = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} = \tan\theta,$$

where we've used the double angle formula for cosine. This gives

$$I_2 = 8\int_0^{\pi/6}\theta d\theta = 4\left(\frac{\pi}{6}\right)^2.$$

Thus we have

$$I = I_1 + I_2 = 6\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}.$$

Three of the following four methods for evaluating $\zeta(2)$ as well as some others can be found in [7].

Another Double Integral Approach

We start off by considering the sum of the squares of the *even* positive integers. Observe that

$$\begin{aligned}
\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots &= \sum_{k=1}^{\infty}\frac{1}{(2k)^2} \\
&= \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} \\
&= \frac{1}{4}\zeta(2).
\end{aligned}$$

Thus we conclude

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{3}{4}\zeta(2).$$

Now observe that

$$\begin{aligned}
\int_0^1 x^{2k}dx &= \left[\frac{x^{2k+1}}{2k+1}\right]_0^1 \\
&= \frac{1}{2k+1}.
\end{aligned}$$

From this we get

$$\begin{aligned}
\int_0^1\int_0^1 x^{2k}y^{2k}dxdy &= \int_0^1 x^{2k}dx\int_0^1 y^{2k}dy \\
&= \frac{1}{(2k+1)^2}.
\end{aligned}$$

Thinking back to the odd power terms in our sum we see that

$$\frac{3}{4}\zeta(2)=\sum_{k=0}^{\infty}\left(\int_0^1\int_0^1 x^{2k}y^{2k}dxdy\right).$$

Because the convergence is uniform, we can integrate termwise:

$$\frac{3}{4}\zeta(2)=\int_0^1\int_0^1\left(\sum_{k=0}^{\infty}x^{2k}y^{2k}\right)dxdy.$$

Now observe that

$$\begin{aligned}\sum_{k=0}^{\infty} x^{2k}y^{2k} &= \sum_{k=0}^{\infty}(x^2x^2)^k \\ &= \frac{1}{1-x^2y^2}.\end{aligned}$$

So

$$\frac{3}{4}\zeta(2) = \int_0^1\int_0^1 \frac{1}{1-x^2y^2}dxdy.$$

Now we make a change of variables by letting

$$x = \frac{\sin u}{\cos v}, \quad y = \frac{\sin v}{\cos u}.$$

A bit of calculation will show that

$$\frac{1}{1-x^2y^2}dxdy = dudv.$$

The new region of integration in the uv-plane turns out to be a triangle in the first quadrant with vertices at $(0,0)$, $(0,\pi/2)$, and $(\pi/2,0)$. The area of this triangle is $\pi^2/8$. Thus

$$\frac{3}{4}\zeta(2) = \frac{\pi^2}{8},$$

and so

$$\zeta(2) = \frac{\pi^2}{6}.$$

Residue Calculus Approach

This approach to evaluating Euler's sum is fueled by the following theorem from complex analysis.

Cauchy's Residue Theorem. If C is a simple closed positively oriented contour and f is analytic inside and on C except at the points $z_1, z_2, ..., z_n$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f; z_k).$$

Here $\text{Res}(f; z_k)$ denotes the residue of f at z_k. That is, it is the coefficient of $(z - z_k)^{-1}$ in the Laurent expansion for f around the singularity z_k.

We will be considering the function

$$f(z) = \frac{\cot(\pi z)}{z^2}.$$

C_n will denote a rectangular contour centered at the origin with sides formed by the lines

$$\Re(z) = \pm(n + \frac{1}{2}) \text{ and } \Im(z) = \pm n.$$

Now $\cot(\pi z)$ is bounded on C_n, that is $|\cot(\pi z)| \leq M$ for all z on C_n and some positive constant M. Also $|z| \geq n$ for all z on C_n, and so

$$|f(z)| \leq \frac{M}{n^2}$$

for all z on C_n. The length of C_n is $8n + 2$ and so

$$\left| \oint_{C_n} f(z)dz \right| \leq \frac{M}{n^2}(8n + 2).$$

This shows

$$\lim_{n\to\infty} \oint_{C_n} f(z)dz = 0.$$

We also know that the integral of f along C_n is equal to $2\pi i$ times the sum of the residues of f inside of C_n. Now, f is singular at each of the integers along the real axis. As n increases, C_n encloses more and more singularities and so more and more residues contribute to the sum that is the value of the integral along C_n. So in the limit as $n \to \infty$, we have that 0 equals $2\pi i$ times some infinite series. Hence this infinite series converges to zero.

So the task before us is to compute the residues of f. Say we have expanded f in a Laurent Series about one of its singularities $k \in \mathbb{Z}, k \neq 0$:

$$f(z) = \sum_{j=-1}^{\infty} a_j (z-k)^j.$$

That there are no powers of $z - k$ less than -1 will be justified by what we are about to do. So we have

$$\begin{aligned}(z-k)f(z) &= \sum_{j=-1}^{\infty} a_j (z-k)^{j+1} \\ &= a_{-1} + \sum_{j=0}^{\infty} a_j (z-k)^{j+1}.\end{aligned}$$

Letting $z \to k$ we should have

$$\begin{aligned}a_{-1} &= \lim_{z \to k} (z-k)f(z) \\ &= \lim_{z \to k} (z-k)\frac{\cot(\pi z)}{z^2} \\ &= \lim_{z \to k} \frac{z-k}{\sin(\pi z)} \cdot \frac{\cos(\pi z)}{z^2}.\end{aligned}$$

If we apply L'Hospital's rule to the left factor and evaluate the limit of the factor on the right, we get that

$$a_{-1} = \frac{1}{\pi k^2}.$$

The fact that this limit is finite explain why the Laurent series of f around k has no powers of $z - k$ less than -1. If it did, then the series for $(z - k)f(z)$ would contain negative powers of $z - k$, and so could not have a limit at $z = k$. We must resort to a different method to determine the residue at 0. The series expansions of the cosine and sine functions about the origin are

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots .$$

and

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots .$$

Thus we have

$$\begin{aligned} z\cot(\pi z) &= z\frac{\cos(\pi z)}{\sin(\pi z)} \\ &= \frac{1 - \pi^2 z^2/2 + \pi^4 z^4/24 - \cdots}{\pi - \pi^3 z^2/6 + \pi^5 z^4/120 - \cdots}. \end{aligned}$$

We use long division to obtain the first few terms of this power series:

$$z\cot(\pi z) = \frac{1}{\pi} - \frac{\pi}{3}z^2 - \frac{\pi^3}{45}z^4 + \cdots .$$

Dividing through by z^3 we obtain

$$\frac{\cot(\pi z)}{z^2} = \frac{1}{\pi}z^{-3} - \frac{\pi}{3}z^{-1} - \frac{\pi^3}{45}z + \cdots .$$

Thus the residue of f at 0 is $-\frac{\pi}{3}$. We now have all the residues of f at our disposal and can conclude that

$$\lim_{n\to\infty} \oint_{C_n} f(z)dz = 2\pi i \left[-\frac{\pi}{3} + 2\sum_{k=1}^{\infty} \frac{1}{\pi k^2} \right] = 0.$$

Setting the expression inside the brackets equal to zero yields our result.

Real Fourier Analysis Approach

Suppose $f : \mathbb{R} \to \mathbb{R}$ is periodic with period $2L$ and is piecewise smooth with at most "jump" discontinuities. The Fourier Series of f is defined as

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right),$$

where

$$\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \\
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n \geq 1, \\
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx.
\end{aligned}$$

Fourier's Theorem asserts that the Fourier series of f converges for every x. At points x where f is continuous, the Fourier series of f converges to $f(x)$. At points of discontinuity x_0, it converges to

$$\frac{1}{2}\left[\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right].$$

We wish to compute the Fourier Series of the function f where

$$f(x) = 1 \text{ if } 0 < x < \pi, \quad -1 \text{ if } -\pi < x < 0,$$

and we extend this in the natural way to a periodic function defined on all of $\mathbb{R}$ with period 2π. Notice that we need not specify the value of f at 0 because the Fourier series will converge to 0 regardless of our choice. Now we compute the coefficients of the Fourier series:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0.$$

For $n \geq 1$,

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx.$$

Since f is an odd function and $\cos(nx)$ is an even function, their product is odd. Hence $A_n = 0$ for $n \geq 1$. Lastly,

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Since both f and $\sin(nx)$ are odd functions their product is even. Thus

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx \\ &= -\frac{2}{n\pi}[\cos(nx)]_0^{\pi} \\ &= \frac{2}{n\pi}[1-\cos(n\pi)] \\ &= \frac{4}{n\pi} \text{ if } n \text{ is odd}, \quad 0 \text{ if } n \text{ is even.} \end{aligned}$$

Hence the Fourier series of f is

$$\frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \cdots = \frac{4}{\pi}\left[\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \cdots\right].$$

Now, according to a theorem about Fourier series, we can integrate f by termwise integration of its Fourier series. Thus

$$\int_0^X f(x)dx = \frac{4}{\pi}\left[-\cos(x) - \frac{1}{9}\cos(3x) - \frac{1}{25}\cos(5x) - \cdots\right]_0^X.$$

If $X \in [-\pi, \pi]$ this equation becomes

$$X = \frac{4}{\pi}\left[\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) - \left(\cos(X) + \frac{1}{3^2}\cos(3X) + \frac{1}{5^2}\cos(5X) + \cdots\right)\right].$$

Evaluating at $X = \pi/2$, all the cosine terms vanish and we obtain

$$\frac{\pi}{2} = \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right],$$

which yields

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

But since

$$\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{1}{2^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{1}{4}\zeta(2),$$

we have shown that

$$\frac{3}{4}\zeta(2) = \frac{\pi^2}{8}.$$

Thus

$$\zeta(2) = \frac{\pi^2}{6}.$$

Complex Fourier Analysis Approach

Here we consider the vector space over $\mathbb{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$ with the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt.$$

The set

$$\{e^{0it}(= 1), e^{\pm 1it}, e^{\pm 2it}, \cdots\}$$

forms an orthonormal basis for this inner product space. So if

$$f(t) = \cdots + a_{-2}e^{-2it} + a_{-1}e^{-it} + a_0 + a_1e^{it} + a_2e^{2it}, \quad a_i \in \mathbb{C},$$

we can compute the coefficients by orthogonal projection:

$$\begin{aligned} a_k &= f \cdot e^{kit} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-kit}dt. \end{aligned}$$

A fact from functional analysis known as Parseval's identity, an analog of the Pythagorean theorem for inner product spaces, tells us that

$$\langle f, f \rangle = \cdots + |a_{-2}|^2 + |a_{-1}|^2 + |a_0|^2 + |a_1|^2 + |a_2|^2 + \cdots . \tag{4}$$

Consider the function $f(t) = t$ on $[-\pi, \pi)$, extended to $\mathbb{R}$ in the natural way to make it 2π-periodic. From the definition of the inner product we have

$$\begin{aligned} \langle f, f \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt \\ &= \frac{1}{2\pi} \frac{t^3}{3} |_{-\pi}^{\pi} \\ &= \frac{\pi^2}{3}. \end{aligned}$$

As indicated above, we also can compute $f \cdot f$ using the coordinates of f with respect to our orthonormal basis. Recall now that

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} te^{-kit} dt.$$

Employing integration by parts we see that

$$a_k = \pm \frac{i}{k}$$

for $k \neq 0$, and

$$a_0 = 0.$$

Thus, using (4) above,

$$f \cdot f = 2\zeta(2).$$

Equating this with $\pi^2/3$ yields our result.

Results concerning $\zeta(s)$ for $s = 3, 4, 5, ...$

Euler also determined $\zeta(2k)$ for $k = 2, 3, ...$ The general formula for the value of ζ at positive even integers involves the sequence $\{B_n\}$ of Bernoulli numbers. One way to define them is by the identity

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The general formula that Euler discovered is

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!},$$

displaying the fact that $\zeta(2k)$ is always a rational multiple of π^{2k}. For a proof of this formula that extends an argument for $\zeta(2)$, see [1].

The next logical place to go seems to be $\zeta(2k+1)$ for $k = 1, 2, 3, \ldots$, starting with

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Euler went on to consider this sum, but found that the methods he employed for $\zeta(2k)$ were not adaptable. He did not make any progress on $\zeta(3)$. For hundreds of years, all mathematicians that followed encountered the same fate. In 1978, Roger Apéry finally managed to shed a bit of light on the problem. He succeeded in proving that $\zeta(3)$ is an irrational number. It wasn't an exact answer like everyone was hoping for, but it was *something.* Apéry's proof was elementary but complicated, and initially not everyone was convinced of its validity. There turned out to be no errors though, and all skepticism ended when F. Beukers [3] published a simplified version of Apéry's proof. He first presents an irrationality proof for $\zeta(2)$ which demonstrates the technique used to attack $\zeta(3)$. Huylebrouck [6] later published an article explaining the general method involved in these proofs and applied them to π and $\ln 2$. Before delving into Beuker's irrationality proofs for $\zeta(2)$ and $\zeta(3)$, we'll examine the general method as laid out by Huylebrook.

The General Idea

The proofs boil down to a contradiction argument. It is shown that the rationality of a number ξ

implies the existence of a sequence $\{A_n/B_n\}$ of *nonzero* rational numbers such that

$$\lim_{n\to\infty} A_n = 0.$$

The numerators A_n are all nonzero, so this cannot happen.

The method for producing the sequence $\{A_n/B_n\}$ involves producing an integrable function $f : [0,1] \to \mathbb{R}$ such that for every $j \in \mathbb{N}$, the integral

$$\int_0^1 x^j f(x)dx$$

yields a rational functional in ξ over the integers. Thus if ξ is a rational number, every such integral produces a rational number C_j/D_j. Then if we take a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

where the coefficients a_j are integers, we have

$$\int_0^1 P(x)f(x)dx = \int_0^1 \sum_{j=0}^{n} a_j x^j f(x)dx = \sum_{j=0}^{n} a_j \int_0^1 x^j f(x)dx = \sum_{j=0}^{n} a_j \frac{C_j}{D_j} = \frac{A}{B},$$

where $A, B \in \mathbb{Z}$. We will take a sequence of polynomials $\{P_n(x)\}$, whereby we produce a sequence of nonzero rational numbers

$$\int_0^1 P_n(x)f(x)dx = \frac{A_n}{B_n}.$$

To produce the contradictory fact that $A_n \to 0$ we show

$$\left| B_n \int_0^1 P_n(x)f(x)dx \right| \to 0.$$

The sequence of polynomials we will want to consider will be the sequence of Legendre polynomials, defined for every $n \in \mathbb{N}$ by

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1-x)^n].$$

Lemma 1. $P_n(x) \in \mathbb{Z}[x]$.

Proof. From the binomial theorem we have that

$$x^n(1-x)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n+k}.$$

Thus the coefficients of $d^n/dx^n[x^n(1-x)^n]$ are of the form

$$\pm\binom{n}{k}(n+k)(n+k-1)\cdots(k+1) = \pm\binom{n}{k}\frac{(n+k)!}{k!}.$$

If we divide by $n!$ we get

$$\pm\binom{n}{k}\frac{(n+k)!}{n!k!} = \pm\binom{n}{k}\binom{n+k}{n}.$$

This shows that $P_n(x) \in \mathbb{Z}[x]$. $\square$

The reason for choosing $P_n(x)$ lies in the susceptibility of $\int_0^1 P_n(x)f(x)dx$ to integration by parts.

Lemma 2. $\int_0^1 P_n(x)f(x)dx = (-1)^n \int_0^1 \frac{1}{n!}x^n(1-x)^n \frac{d^n f(x)}{dx^n}dx$.

Proof. Integrating by parts we have

$$\begin{aligned}
\int_0^1 P_n(x)f(x)dx &= \int_0^1 \frac{1}{n!}\frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}[x^n(1-x)^n]\right) f(x)dx \\
&= \left[\frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}[x^n(1-x)^n]f(x)\right]_0^1 - \int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}[x^n(1-x)^n]\frac{df(x)}{dx}dx.
\end{aligned}$$

Now, $x^n(1-x)^n$ has 0 and 1 as roots of order n. Therefore its derivatives up to the order of $n-1$ will also have 0 and 1 as roots. Thus the expression on the left above outside of the integral will vanish. Thus we see that

$$\int_0^1 \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1-x)^n]f(x)dx = (-1)\int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}[x^n(1-x)^n]\frac{df(x)}{dx}dx.$$

Hence we see inductively that if we perform integration by parts a total of n times, we will arrive at

$$(-1)^n \int_0^1 \frac{1}{n!}x^n(1-x)^n \frac{d^n f(x)}{dx^n}dx.$$

$\square$

Now recall that, when our number ξ is rational, we have

$$\int_0^1 P_n(x)f(x)dx = \frac{A_n}{B_n} \neq 0$$

for every $n \in \mathbb{N}$, and we want to have

$$|A_n| = \left| B_n \int_0^1 P_n(x)f(x)dx \right| \to 0$$

to obtain our contradiction. To this end, the function f is chosen so that

$$\left|(-1)^n \int_0^1 \frac{1}{n!} x^n (1-x)^n \frac{d^n f(x)}{dx^n} dx.\right| = \left| \int_0^1 \frac{1}{n!} [g(x)]^n h(x) dx \right|,$$

where the maximum value M of g on $[0, 1]$ is small enough to ensure that

$$\left| B_n \int_0^1 \frac{1}{n!} [g(x)]^n h(x) dx \right| \leq \left| B_n M^n \int_0^1 h(x) dx \right| \to 0$$

as $n \to \infty$, yielding the desired contradiction that

$$\lim_{n\to\infty} \left| B_n \int_0^1 P_n(x)f(x) \right| = \lim_{n\to\infty} |A_n| = 0.$$

Let's now see this in action.

Demonstration with π

We shall prove the irrationality of π using this technique. The function to use is $f(x) = \sin(\pi x)$.

It is not difficult to show using integration by parts that

$$\int_0^1 x^j \sin(\pi x) dx = a_0 + \frac{a_1}{\pi} + \frac{a_2}{\pi^2} + \cdots + \frac{a_j}{\pi^j},$$

for some integers $a_0, ..., a_j$. If π were a rational number a/b, we would have

$$\int_0^1 x^j \sin(\pi x) dx = a_0 + a_1 \left(\frac{b}{a}\right) + a_2 \left(\frac{b}{a}\right)^2 + \cdots + a_j \left(\frac{b}{a}\right)^j = \frac{A}{a^j}$$

for some integer A. From this it would follow that

$$\int_0^1 P_n(x)\sin(\pi x)dx = \frac{A_n}{a^n},$$

for some $A_n \in \mathbb{Z}$ The integrand is nonnegative and not identically zero on $[0,1]$, so $A_n > 0$. Using Lemma 2, we then have

$$0 < |A_n| = \left|a^n \int_0^1 P_n(x)\sin(\pi x)dx\right| = \left|a^n \int_0^1 \frac{1}{n!}x^n(1-x)^n \frac{d^n}{dx^n}(\sin(\pi x))dx\right|.$$

Since

$$\frac{d^n}{dx^n}(\sin(\pi x)) = \pm\pi^n \sin(\pi x) \quad \text{or} \quad \pm\,\pi^n\cos(\pi x),$$

we have

$$0 < |A_n| \leq \left|a^n \int_0^1 \frac{1}{n!}x^n(1-x)^n\pi^n dx\right|$$

The maximum value of $x(1-x)$ is $1/4$, so

$$0 < |A_n| \leq \left|a^n \frac{1}{n!}\left(\frac{1}{4}\right)^n \pi^n\right| = \left|\frac{(a\pi/4)^n}{n!}\right|.$$

But for any fixed k, $k^n/n! \to 0$ since $e^k = \sum k^n/n!$. Thus

$$\lim_{n\to\infty} |A_n| = 0,$$

proving that π is irrational.

Irrationality Proofs for $\zeta(2)$ and $\zeta(3)$

The following proofs for the irrationality of $\zeta(2)$ and $\zeta(3)$ are presented in the style of Beukers [], but they fit the general format described above. For $\zeta(2)$, the appropriate function f is

$$f(x) = \int_0^1 \frac{(1-y)^n}{1-xy}dy.$$

For $\zeta(3)$ it is

$$f(x) = \int_0^1 \frac{P_n(y)}{1-xy} \ln xy \; dy.$$

We first establish some important lemmas for later use. Here d_n shall denote for us the least common multiple of the numbers $1, 2, ..., n$. Both proofs make use of the following standard result from number theory:

Lemma 3. $d_n < 3^n$ for sufficiently large n.

Proof. We first note that

$$d_n = \prod_{p \le n} p^{[\log_p n]},$$

where the product is over all primes p less than or equal to n, and $[x]$ means the greatest integer less than or equal to the real number x. We then see that

$$d_n < \prod_{p \le n} p^{\log_p n} = \prod_{p \le n} n,$$

and this will give us simply n raised to the number of primes less than or equal to n, so

$$d_n < n^{\pi(n)}.$$

Now we turn to the prime number theorem, which tells us that

$$\lim_{n \to \infty} \frac{\pi(n)}{\left(\frac{n}{\ln n}\right)} = 1.$$

Observe that

$$\frac{\pi(n)}{\left(\frac{n}{\ln n}\right)} = \frac{\pi(n)}{n} \ln n = \ln n^{\pi(n)/n} \approx 1,$$

for n sufficiently large. Thus

$$n^{\pi(n)/n} \approx e,$$

and so

$$d_n < n^{\pi(n)} \approx e^n < 3^n,$$

for n sufficiently large. $\square$

The next lemma establishes the vital connection between rational numbers and $\zeta(2)$.

Lemma 4. Let $r, s \in \mathbb{Z}$ be non-negative. if $r > s$, then

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dxdy$$

is a rational number with denominator dividing d_r^2. If $r = s$, then

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dxdy = \zeta(2) - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{r^2}\right).$$

If $r = 0$, we let the sum in parentheses vanish.

Proof. Let $\sigma \in \mathbb{R}$ be non-negative and consider

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} dxdy.$$

Expanding $(1-xy)^{-1}$ into the geometric series $\sum_{k=0}^{\infty} x^k y^k$ and noting uniform convergence, we find

$$\begin{aligned}
\int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^{k+r+\sigma} y^{k+s+\sigma} dxdy &= \int_0^1 \sum_{k=0}^{\infty} \left(\int_0^1 x^{k+r+\sigma} y^{k+s+\sigma} dx \right) dy \\
&= \int_0^1 \sum_{k=0}^{\infty} \frac{y^{k+s+\sigma}}{k+r+\sigma+1} dy \\
&= \sum_{k=0}^{\infty} \int_0^1 \frac{y^{k+s+\sigma}}{k+r+\sigma+1} dy \\
&= \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)}.
\end{aligned}$$

Suppose that $r > s$. Observe, then, that we have the partial fraction decomposition

$$\frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} = \frac{1}{r-s}\left(\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1}\right).$$

When we sum from $k = 0$ to $k = \infty$ the series telescopes and we're left with

$$\int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^{k+r+\sigma} y^{k+s+\sigma} dxdy = \frac{1}{r-s} \left(\frac{1}{s+1+\sigma} + \frac{1}{s+2+\sigma} + \cdots + \frac{1}{r+\sigma} \right).$$

Now putting $\sigma = 0$ we have discovered that

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dxdy = \frac{1}{r-s} \left(\frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{r} \right).$$

So the integral is rational with a denominator that divides $r - s$ times the least common multiple of $s + 1, s + 2, ..., r$. Hence the denominator divides d_r^2. □

Lemma 5.

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} dxdy = \frac{A_n + B_n \zeta(2)}{d_n^2},$$

for some $A_n, B_n \in \mathbb{Z}$.

Proof. Let

$$I = \int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} dxdy. \tag{5}$$

The numerator is a polynomial of degree $2n$ in x and y with integer coefficients, say

$$\sum_{i,j=0}^{n} a_{ij} x^i y^j,$$

for some collection $\{a_{ij} \in \mathbb{Z} : i, j = 0, 1, ..., n\}$. Then

$$I = \sum_{i,j=0}^{n} a_{ij} \int_0^1 \int_0^1 \frac{x^i y^j}{1-xy} dxdy. \tag{6}$$

Lemma 4 allows us to say something about these individual integrals. Whenever $i \neq j$, we get a rational number with denominator dividing d_k^2, where $k = \max\{i, j\}$. Since $k \leq n$, this denominator necessarily divides d_n^2. So in this case we can write

$$\int_0^1 \int_0^1 \frac{x^i y^j}{1-xy} = \frac{b_{ij}}{d_n^2},$$

where $b_{ij} \in \mathbb{Z}$. When $i = j$, we have

$$\int_0^1 \int_0^1 \frac{x^i y^j}{1-xy} = \zeta(2) - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{i^2}\right)$$
$$= \zeta(2) + \frac{c_i}{d_n^2},$$

for some $c_i \in \mathbb{Z}$. Thus the integral in (6) can be expressed as

$$\sum_{i,j=0, i\neq j}^{n} a_{ij} \frac{b_{ij}}{d_n^2} \quad + \quad \sum_{i=0}^{n} a_{ii} \left(\zeta(2) + \frac{c_i}{d_n^2}\right),$$

which can be written as

$$\frac{A_n + B_n \zeta(2)}{d_n^2}$$

for some $A_n, B_n \in \mathbb{Z}$. □

Theorem 1. $\zeta(2)$ is irrational.

Proof. We return to the integral in (5) and consider

$$\int_0^1 \frac{P_n(x)}{1-xy} dx = \frac{1}{n!} \int_0^1 \frac{(d^n/dx^n)[x^n(1-x)^n]}{1-xy} dx.$$

We integrate by parts with $u = (1-xy)^{-1}$ and $dv = P_n(x)dx = \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1-x)^n]dx$. This transforms the integral into

$$\left[\frac{\frac{d^{n-1}}{dx^{n-1}} x^n(1-x)^n}{n!(1-xy)}\right]_0^1 - \frac{y}{n!} \int_0^1 \frac{(\frac{d^{n-1}}{dx^{n-1}})[x^n(1-x)^n]}{(1-xy)^2} dx = -\frac{y}{n!} \int_0^1 \frac{(\frac{d}{dx})^{n-1} x^n (1-x)^n}{(1-xy)^2} dx.$$

Performing integration by parts a second time yields

$$-\frac{y}{n}\left[\frac{(\frac{d}{dx})^{n-2} x^n(1-x)^n}{n!(1-xy)^2}\right]_0^1 + \frac{y^2}{n!} \int_0^1 \frac{(\frac{d}{dx})^{n-2} x^n(1-x)^n}{(1-xy)^3} dx = (-1)^2 \frac{y^2}{n!} \int_0^1 \frac{(\frac{d}{dx})^{n-2} x^n(1-x)^n}{(1-xy)^3} dx.$$

Becoming aware of the pattern, we see that after integrating like this a total of n times we will have

$$\int_0^1 \frac{P_n(x)}{1-xy} dx = (-1)^n y^n \int_0^1 \frac{x^n(1-x)^n}{(1-xy)^{n+1}} dx.$$

Thus,

$$I = (-1)^n \int_0^1 \int_0^1 \frac{y^n(1-y)^n x^n (1-x)^n}{(1-xy)^{n+1}} dxdy. \tag{7}$$

A straightforward computation in multivariable calculus shows that

$$\frac{y(1-y)x(1-x)}{1-xy} \leq \left(\frac{\sqrt{5}-1}{2}\right)^5 \qquad \text{for all } (x,y) \in [0,1]^2.$$

Thus

$$\begin{aligned}
\left|(-1)^n \int_0^1 \int_0^1 \frac{y^n(1-y)^n x^n(1-x^n)}{(1-xy)^{n+1}} dxdy\right| &= \int_0^1 \int_0^1 \left[\frac{y(1-y)x(1-x)}{1-xy}\right]^n \frac{1}{1-xy} dxdy \\
&\leq \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \int_0^1 \int_0^1 \frac{1}{1-xy} dxdy \\
&= \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2).
\end{aligned}$$

Since the integrand in (7) is nonnegative and not identically zero on $[0,1]^2$, from Lemma 5 it now follows that

$$0 < |I| = \left|\frac{A_n + B_n\zeta(2)}{d_n^2}\right| < \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2).$$

Recalling from Lemma 3 that $d_n < 3^n$ for n sufficiently large, this yields

$$0 < |A_n + B_n\zeta(2)| < d_n^2 \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2) < 9^n \left(\frac{\sqrt{5}-1}{2}\right)^{5n} \zeta(2)$$

for n sufficiently large. Now, it is seen numerically that

$$9\left(\frac{\sqrt{5}-1}{2}\right)^5 < \frac{5}{6},$$

and so

$$0 < |A_n + B_n\zeta(2)| < \left(\frac{5}{6}\right)^n \tag{8}$$

for sufficiently large n. Now if $\zeta(2) = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $p, q > 0$, then for any integers A and B satisfying $|A + B\zeta(2)| > 0$ we have

$$|A + B\zeta(2)| = \left|A + B\frac{p}{q}\right| = \left|\frac{Aq + Bp}{q}\right| \geq \frac{1}{q}$$

since $Aq + Bp > 0$ is an integer. This is in contradiction to (8) above. Thus $\zeta(2)$ is irrational. $\square$

The proof for $\zeta(3)$ closely parallels that of $\zeta(2)$. Recall the variable σ from the proof of Lemma 4 that seemed to serve no purpose whatsoever. Well now it comes in quite handy as the means of connecting the $\zeta(2)$ proof to the $\zeta(3)$ proof. The key idea is that differentiating the reciprocal of a square yields the reciprocal of a cube. Lemma 6 that follows is the $\zeta(3)$ analog of Lemma 4, establishing the connection between $\zeta(3)$ and certain rational numbers.

Lemma 6. Let $r, s \in \mathbb{Z}$ be non-negative. If $r > s$, then

$$\int_0^1 \int_0^1 -\frac{\ln xy}{1-xy} x^r y^s dxdy$$

is a rational number with denominator dividing d_r^3. If $r = s$, then

$$\int_0^1 \int_0^1 -\frac{\ln xy}{1-xy} x^r y^s dxdy = 2\left[\zeta(3) - \left(\frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{r^3}\right)\right]$$

If $r = 0$, the let the sum in parentheses vanish.

Proof. Let $\sigma \in \mathbb{R}$ be non-negative. Suppose $r > s$ and recall the equation

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} dxdy = \frac{1}{r-s}\left(\frac{1}{s+1+\sigma} + \frac{1}{s+2+\sigma} + \cdots + \frac{1}{r+\sigma}\right)$$

from the proof of Lemma 4. We differentiate both sides with respect to σ, applying Leibniz's Rule:

Let $f(x,t)$ and $\partial f/\partial t(x,t)$ be continuous on $[a,b]^2$. Then

$$\frac{\partial}{\partial t} \int_a^b f(x,t)dx = \int_a^b \frac{\partial f}{\partial t}(x,t)dx.$$

So on the left side, we differentiate the integrand:

$$\frac{d}{d\sigma} \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} = \frac{x^r y^s}{1-xy} \frac{d}{d\sigma}(xy)^\sigma = \frac{\ln xy}{1-xy} x^r y^s (xy)^\sigma.$$

Also differentiating on the right side, we arrive at

$$\int_0^1 \int_0^1 \frac{\ln xy}{1-xy} x^r y^s (xy)^\sigma dxdy = -\frac{1}{r-s}\left(\frac{1}{(s+1+\sigma)^2} + \frac{1}{(s+2+\sigma)^2} + \cdots \frac{1}{(r+\sigma)^2}\right).$$

Setting $\sigma = 0$ we have

$$\int_0^1 \int_0^1 -\frac{\ln xy}{1-xy} x^r y^s = \frac{1}{r-s}\left(\frac{1}{(s+1)^2} + \frac{1}{(s+2)^2} + \cdots \frac{1}{(r)^2}\right).$$

This is a rational number with denominator dividing d_r^3.

Next suppose $r = s$ and recall from the proof of Lemma 4 the equation

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy} dxdy = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}.$$

We again differentiate both sides with respect to σ. On the right, uniform convergence in σ of the sum allows us to differentiate termwise. We obtain

$$\int_0^1 \int_0^1 \frac{\ln xy}{1-xy} x^r y^r (xy)^\sigma dxdy = \sum_{k=0}^{\infty} \frac{-2}{(k+r+\sigma+1)^3}.$$

Putting $\sigma = 0$ we have

$$\begin{aligned} \int_0^1 \int_0^1 -\frac{\ln xy}{1-xy} x^r y^r dxdy &= 2\sum_{k=0}^{\infty} \frac{1}{(k+r+1)^3} \\ &= 2\left[\zeta(3) - \left(\frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{r^3}\right)\right], \end{aligned}$$

as desired. ☐

Over two and a half centuries after Euler's victory over $\zeta(2k)$ for every positive integer k, we are now in a position to break the seal on $\zeta(3)$.

Theorem 2. $\zeta(3) \notin \mathbb{Q}$.

Proof. Consider the integral

$$J = \int_0^1 \int_0^1 \frac{-\ln xy}{1-xy} P_n(x) P_n(y) dxdy,$$

where again $n!P_n(x) = (d/dx)^n x^n(1-x)^n$. Using observations similar to those in the proof of Lemma 5 for $\zeta(2)$, we see that this equals

$$\frac{A_n + B_n\zeta(3)}{d_n^3},$$

for some $A_n, B_n \in \mathbb{Z}$. Now observe that

$$\begin{aligned}\int_0^1 \frac{1}{1-(1-xy)z}dz &= \frac{-1}{1-xy}\left[\ln(1-(1-xy)z)\right]_{z=0}^{z=1} \\ &= \frac{-\ln xy}{1-xy}.\end{aligned}$$

Thus

$$J = \int_0^1\int_0^1\int_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)z}dxdydz.$$

Now we perform integration by parts n times on the integral with respect to x, and get

$$J = \int_0^1\int_0^1\int_0^1 \frac{(xyz)^n(1-x)^nP_n(x)}{[1-(1-xy)z]^{n+1}}dxdydz.$$

Next we make the substitution

$$w = \frac{1-z}{1-(1-xy)z},$$

whereby

$$1-w = \frac{xyz}{1-(1-xy)z},$$

and

$$\frac{dw}{1-(1-xy)w} = \frac{-dz}{1-(1-xy)z}.$$

Hence, taking into account the change in the limits of integration, our integral now becomes

$$J = \int_0^1\int_0^1\int_0^1 (1-x)^n(1-w)^n\frac{P_n(y)}{1-(1-xy)w}dxdydw.$$

We again perform an n-fold integration by parts, this time with respect to y. Doing so we obtain

$$\int_0^1\int_0^1\int_0^1 \frac{x^n(1-x)^ny^n(1-y)^nw^n(1-w)^n}{[1-(1-xy)w]^{n+1}}.$$

Multivariable calculus will verify for us that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leq (\sqrt{2}-1)^4 \quad \text{for all } (x,y,w) \in [0,1]^3.$$

Thus

$$\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{[1-(1-xy)w]^{n+1}} &\leq (\sqrt{2}-1)^{4n} \int_0^1 \int_0^1 \int_0^1 \frac{1}{1-(1-xy)w} dxdydw \\
&= (\sqrt{2}-1)^{4n} \int_0^1 \int_0^1 \frac{-\ln xy}{1-xy} dxdy \\
&= 2(\sqrt{2}-1)^{4n}\zeta(3).
\end{aligned}$$

Since the integrand is non-negative and non constantly zero on $[0,1]^3$,we have

$$0 < |J| = |A_n + B_n\zeta(3)|d_n^{-3} < 2\zeta(3)(\sqrt{2}-1)^{4n}.$$

Hence, recalling Lemma 3, for sufficiently large n we have

$$0 < |A_n + B_n\zeta(3)| < 2\zeta(3)(3^n)^3(\sqrt{2}-1)^{4n}.$$

We can check numerically that

$$27(\sqrt{2}-1)^4 < \frac{4}{5}.$$

Thus

$$0 < |A_n + B_n\zeta(3)| < \left(\frac{4}{5}\right)^n$$

for sufficiently large n. As in the proof for $\zeta(2)$, this implies the irrationality of $\zeta(3)$. □

Some Recent Results

As said earlier, no none has succeeded in determining explicitly the value of the Riemann Zeta function at any odd integer greater than 2. We can now say with confidence that the number $\zeta(3)$, sometimes called Apéry's constant, is irrational. However, currently there is no other odd zeta value that can be singled out as possessing this property. We know more irrationals are out there though; lots more. In 2000, T. Rivoal proved that $\zeta(2n+1)$ is irrational for infinitely many positive integers n. Working off of this result, V. V. Zudilin showed in 2001 that among the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$, at least one of them is irrational. It is fascinating how impenetrable these numbers are. There are still questions in mathematics that are simple to ask but terribly difficult to answer. It is not hard to envision the eventual development of new regions of mathematics for which the irrationality of $\sum_{n=1}^{\infty} 1/n^s$ for $s = 2, 3, 4, ...$ might be a trivial matter. On the other hand, it is still conceivable that someone could one day stumble upon a rational number $\zeta(2n+1)$ for some $n \in \mathbb{N}$. Either way, the fact remains that a lot of interesting mathematics is waiting to be discovered.

References

[1] Apostol, T. M., Another Elementary Proof of Euler's Formula for $\zeta(2n)$, *Am. Math. Monthly*, **80** (1973), 425-431.

[2] Apostol, T. M., A Proof that Euler Missed: Evaluating $\zeta(2)$ the Easy Way, *Math. Intelligencer*, **5** (1983), no. 3, 59-60.

[3] Beukers, F., A Note on the Irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.*, **11** (1979), 268-272.

[4] Dunham, W., *Euler: Master of Us All*, The Mathematical Association of America, 1999.

[5] Edwards, H., *Riemann's Zeta Function*, Academic Press, New York, 1974.

[6] Huylebrouck, D., Similarities in Irrationality Proofs for π, $\ln(2)$, $\zeta(2)$, and $\zeta(3)$, *Am. Math. Monthly*, **108** (2001) No. 3, 222-231.

[7] Kalman, D., Six Ways to Sum a Series, *Coll. Math. J.*, **24** (1993), 402-421.

[8] Klee, V. and Wagon, S., *Old and New Unsolved Problems in Plane Geometry and Number Theory*, Math. Assoc. Am., 1991.