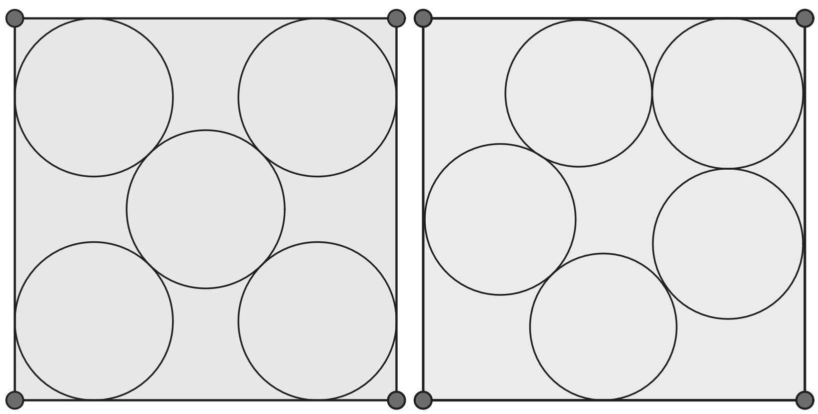
Equal Circle Packings

Summer 2010 REU

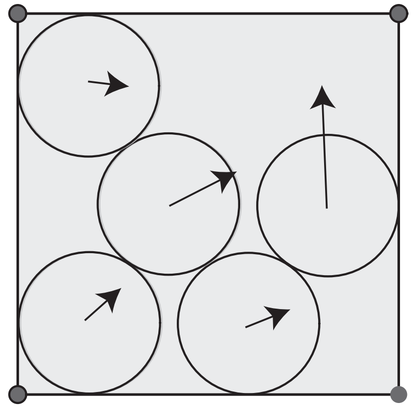
Project Overview

Introduction

The central theme of this project is to try and determine the most dense packing(s) of a fixed number of equal circles into a certain type of domain. For example, suppose we have a unit square (the domain) and we want to put five equal circles into it so that the circles don't overlap each other or the boundary. (This is called a *packing* - the interiors of the circles are disjoint from each other and the boundary, if any.) In the best arrangement, where would the centers of the circles be and what would be the common radius? After exploring this for a while you would probably come up with the following arrangements:



Both of these arrangements look pretty good. There appears to be no way to slide the circles around so that they all lose contact with each other and the boundary. This is the intuitive definition of locally maximally dense. An arrangement is locally maximally dense if there is no way to move the circles a little bit so that they all lose contact with the boundary and each other. (You remember local maximums from calculus, right? A locally maximally dense arrangement is a LOCAL maximum of the density function.) You can think of this differently to see that if there was a way to move the circles a little bit so that they all lose contact with the boundary and each other, then you could move them and then increase the common radius while still being a packing. This would led to a near-by arrangement that was more dense i.e. the first arrangement was not locally maximally dense. For example, the following arrangement of five circles is not locally maximally dense:



Notice that the vectors in the arrangement indicate how to move the circles so that they all lose contact with each other and the boundary. There is a systematic way to find these vectors that we will use when finding them is not so obvious. We will find them using the tool of tensegrity frameworks. The techniques of this area of mathematics will help us determine when an arrangement is locally maximally dense.

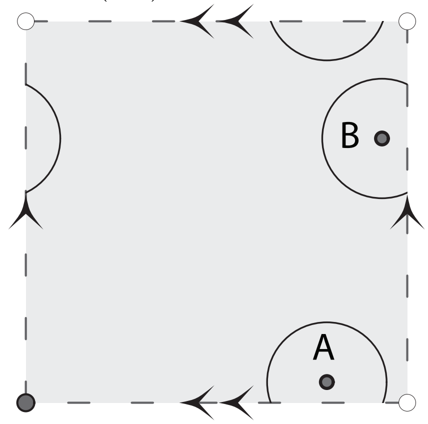
In the opening sentence of this section, I really meant that we want to find the *globally* most dense arrangement in a given domain. Of course the globally most dense arrangement is also locally maximally dense, so determining the locally maximally dense arrangements is a start. Of the two locally maximally dense arrangements pictured above, which is better (i.e. more dense)? After some algebra, it is not too hard to figure out that the common radius of the circles on the right is $\frac{2}{\sqrt{2}+6+\sqrt{4\sqrt{2}+2}} \approx .19643$ and the common radius of the circles on the left is $\frac{1}{2\sqrt{2}+2} \approx .20710$. So the packing on the left covers more of the unit square and is therefore more dense.

Fundamental Questions: Is the arrangement pictured in the above left the *globally* most dense arrangement of five circles in a unit square? Is there *any* other arrangement of five equal circles in a unit square that will be more dense then the arrangement given on the left above?

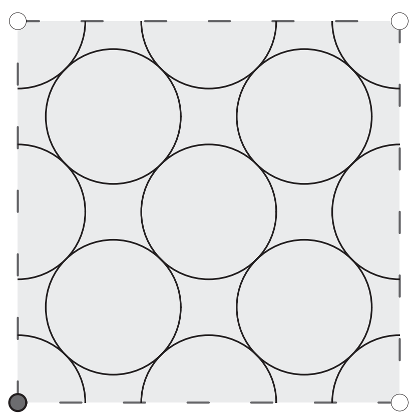
The answer to this last question is no! No matter how much time you spend trying to do better or how clever you are or anyone else is, you will never find a more dense arrangement of five equal circles in a unit square! You'll see a clever proof of this later, but this illustrates the basic goal of this project. In general, the arguments that prove a particular arrangement is globally most dense are tricky and usually don't apply to other arrangements.

Extensions To Flat Tori

The natural extension of the above problem, where instead of five circles you try to find the most dense arrangement of six circles in a hard boundary square, then seven and so on, has been well studied. There are proven globally most dense arrangements up to about 27 circles and conjectured most dense arrangements well into the hundreds of circles. Therefore we are not going to study this hard-boundary square problem, but we are going to extend it in a different way. We are going to change the domain of the packing and in that domain try to find the globally best arrangements for small numbers of circles (say 1–5 circles). We are going to pack circles in domain called a (flat) torus. Consider the following picture:



This new domain is like a hard-boundary square (i.e. the domain considered in the introduction) except the top and bottom, and left and right sides have been identified. So circles can 'go off' one side and 'come on' the opposite side. In the picture above the circle labeled A goes off the bottom and has a piece at the top. The circle labeled B is similar only with the left and right sides. This is a (not very dense!) packing of two circles on a square flat torus. Here is another packing of circles in a flat torus:

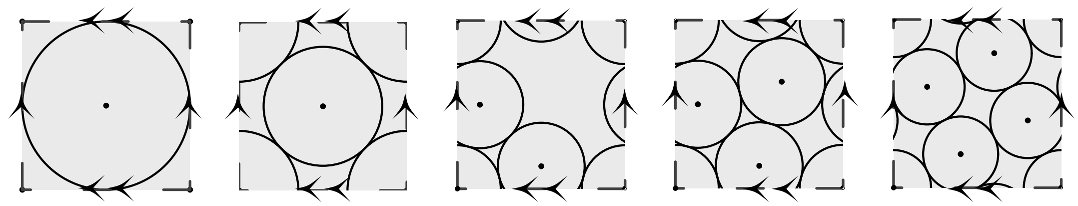


How many circles are packed on this torus? This arrangement looks like it is locally maximally dense, but it is not. Do you see how to improve it? Think about sliding an off-diagonal row of circles, while holding the circle in the lower left fixed.

Notice how this process of identifying the left and right, and top and bottom sides could be applied to any parallelogram. When the parallelogram used to create the torus is a square, the resulting torus is called a Square Flat Torus. When the parallelogram used to create the torus is a rhombus with a 60 degree angle, the resulting torus is called a Triangular Flat Torus. In previous summers we have studied packings of 1-6 circles on these two tori.

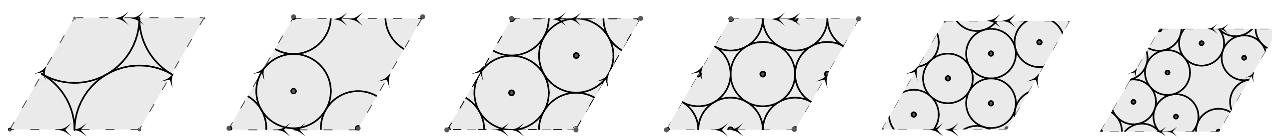
Previous Results

Work from the previous summers (2007 and 2008) can be summed up in the following two figures.



(a) 1 Circle (b) 2 Circles (c) 3 Circles (d) 4 Circles (e) 5 Circles

Figure 1: Globally maximally dense arrangements on the Square Flat Torus.



(a) 1 Circle (b) 2 Circles (c) 3 Circles (d) 4 Circles (e) 5 Circles (f) 6 Circles

Figure 2: Globally maximally dense arrangements on the Triangular Flat Torus.

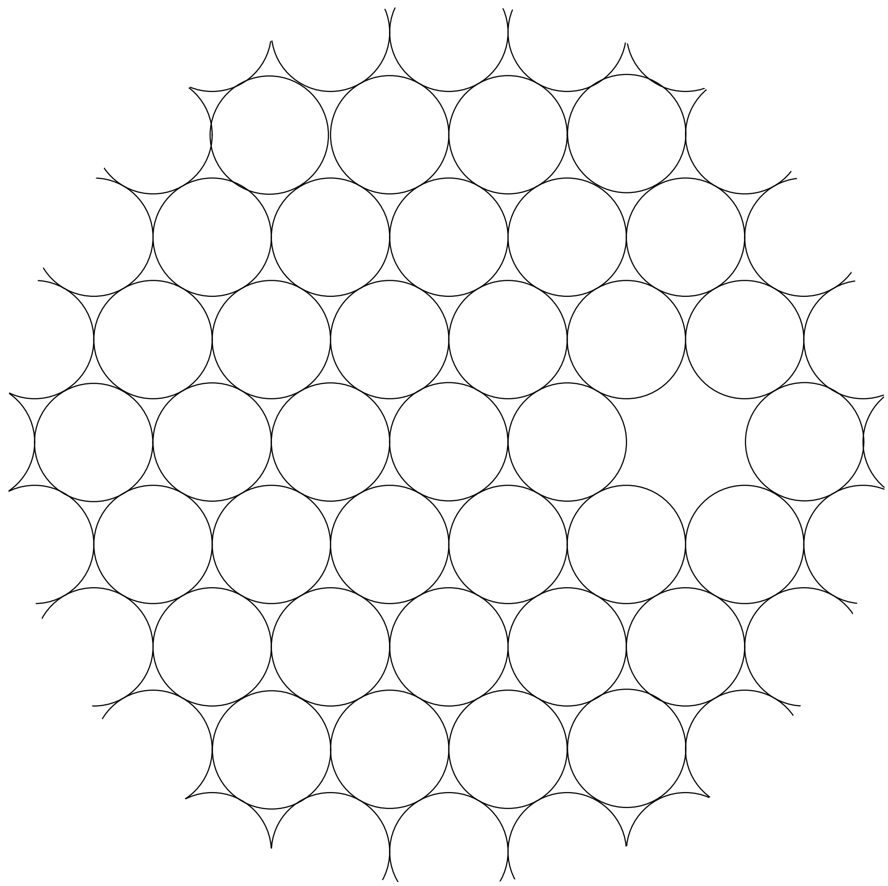
In this project we will build on these results and try to find globally maximally dense arrangements of 1-6 circle on a torus from a rhombus with an angle of 75 degrees. Hopefully this will lead to insights about the globally maximally dense arrangements of 1-6 circles on tori formed from rhombi with angles between 60 and 120 degrees. I would love to be able to see a video of the globally maximally dense arrangement of 5 circles as the angle of the torus changes!

Visualization and Manipulation Tool

Go to the address `http://faculty.gvsu.edu/dickinsw/ezpack/` and play around with packings. Use the toolbar on the right-had side to get started. The directions for each of the tools in the toolbar is explained in the right-hand pane at the bottom of the screen. Use the program to create the torus formed from the rhombus with the 75 degree angle and start to play with it.

L. Fejes Tóth's Conjecture

If things are *wildly* successful this summer we might be able to shed light on an open case of a conjecture of Tóth's. This conjecture starts with the triangular close packing in the plane with one circle removed. That is, you start with the following infinite arrangement of circles:



The open case of this Tóth conjecture states that if you remove a fixed finite number of circles from this arrangement (creating a large blank patch in the packing) and then put ALL the circles back in the packing (i.e. each one is put *anywhere* in the blank patch without overlapping any circle) then the resulting arrangement is exactly like the original except the hole is possibly in a different location. (That is the hole is moved around.)

Formalizing The Previous Results

What does it look like when you try to write these ideas up formally? In the same place that you clicked on the link for this document you can find the paper that as been accepted to a journal for publication pending revisions. Take some time to look over this paper. This will be something that the students selected for this project will read detail at the the start of the program.