Optimal Packings Of Up To 5 Equal Circles On A Square Flat Torus

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1 Introduction

We prove that the globally maximally dense packing of 5 equal circles on the quotient of the Euclidean plane by a lattice generated by two unit perpendicular vectors (a square flat torus) is the arrangement (up to reflected variants) conjectured in [5] and [15]. This answers question 7.3.2 of Melissen in [15]. Packings on the square flat torus are considered in section 2.2.5 of [15], where dense arrangements for 1 to 19 equal circles are presented. The arrangements for 1 to 4 circles are proven to be globally maximally dense using a technique that partitions a fundamental domain into different regions using a lower bound on the globally best radius. In [11], similar techniques are used to determine the globally maximally dense packings of 1 to 4 equal circles in a rectangular flat torus. In two related articles, circle packings on the torus are explored. In [17], packings of two equal circles on a torus are explored, however the torus is not fixed, but is constrained to contain a closed geodesic of length one. In [14], large numbers (50-10,000) of equal circles packed on a square torus (among other domains) are explored using a billiards algorithm implemented on a computer and large scale patterns are discussed. For similar explorations, see [6] for circle packings from a physics point of view. (Articles [18] and [10] explore optimal packings of squares on the square flat torus.)

In this article, we use techniques that are fundamentally different. They involve discerning the properties of packing graphs associated to locally

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maximally dense arrangements of equal circles, enumerating all the 2-cell imbeddings of certain graphs onto a torus and applying results from rigidity theory. The power of these techniques are first demonstrated on arrangements of 1 to 4 circles and then they are used to prove the global optimality of an arrangement of 5 equal circles on the square flat torus. These results also prove that there are no locally maximally dense arrangements of 1 to 5 circles on the square torus other than the globally maximally dense ones. Finally, we establish the local maximally of a lattice arrangement of $n = a^2 + b^2$ ($a > b > 0$ and $\gcd(a, b) = 1$) circles on the square flat torus.

2 Definitions and Basic Notions

The quotient of the Euclidean plane by a lattice generated by two independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is called a **flat torus**. A **fundamental domain** of a flat torus is the set of points in the Euclidean plane, $\{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid 0 \leq t_1, t_2 < 1\}$. When a basis for the lattice consists of two unit perpendicular vectors, the quotient torus is called the **square flat torus**. (In the literature this is sometimes denoted $\mathbb{R}^2/\mathbb{Z}^2$ or $[0, 1)^2$.) The **standard basis** for the square flat torus is the one where $\mathbf{v}_1 = \langle 1, 0\rangle$ and $\mathbf{v}_2 = \langle 0, 1\rangle$. For a point, p, a **lift** of it is another point q in the Euclidean plane that is equivalent to p.

For a given flat torus, an arrangement of equal circles forms a **packing** on the torus if the interiors of the circles are disjoint. The **density** of a packing is the the ratio of the area of the circles to the area of the flat torus. We define two packings with the same number of circles to be **ϵ-close** if there is a one-to-one correspondence between the circles, so that corresponding circles have centers that are all within ϵ of each other. We define a packing $\mathcal{P}$ to be **locally maximally dense** if there exists an $\epsilon > 0$ so that all ϵ-close packings of equal circles have a packing density no larger than that of $\mathcal{P}$. A packing $\mathcal{Q}$ is **globally maximally dense** if it is at least as dense as all other locally maximally dense packings. Rather than searching directly for the globally maximally dense packings, our techniques allow us to determine all the locally maximally dense arrangements of a fixed number of circles on a given flat torus. This allows us to easily determine the globally maximally dense packing(s).

The main structure that allows us to form a list of all the locally maximally dense packings for a fixed number of circles on a flat torus is the graph of a packing. Given a packing $\mathcal{P}$ on a flat torus, the **graph associated to $\mathcal{P}$**, denoted $G_{\mathcal{P}}$, has vertices and edges defined as follows. The center of each circle in the packing is associated to a vertex (with a location in the flat torus) of $G_{\mathcal{P}}$ and two vertices of $G_{\mathcal{P}}$ are connected with an edge if and only if the corresponding circles are tangent to each other. Thus each packing of equal circles on a flat torus is naturally associated to an imbedding of a graph on a flat torus where all the edges are equal in length. The next section will illuminate some of the properties of packing graphs associated to locally maximally dense packings on a flat torus.

For n circles on a flat torus we can find an upper bound the diameter (or common edge length of the associated packing graph) by using the L. Fejes Tóth-Thue Theorem ([8], [20]) that states that the densest packing of equal circles in the Euclidean plane is uniquely achieved by the triangular close packing, where each circle is tangent to six others. The triangular close packing has density $\frac{\pi}{\sqrt{12}}$and the packing density on the torus cannot exceed this bound. In the case of the square flat torus we have the following result.

Proposition 2.1 (Diameter Upper Bound)**.** *The common diameter of a packing of n equal circles on a square flat torus may not exceed* $\frac{2}{\sqrt{n\sqrt{12}}}$.

3 Results From Rigidity Theory

What is the minimum number of edges that the graph associated to a circle packing must contain in order for the packing to be locally maximally dense? Connelly has answered this in [2], [3], and [4]. The answer comes from the study of tensegrity frameworks and determining when such a framework is rigid. For the convenience of the reader, several definitions and theorems are repeated here. The presentation here is the specialization of a much broader theory to the cases needed for the study of circle packings on a flat torus, for more details about the broader theory see [2] and the references it contains.

The **strut (tensegrity) framework** associated to a circle packing is the graph associated to it where each edge in the graph is designated as a strut. A **strut** is an edge that is not allowed to decrease in length as its endpoint vertices move in a flat torus. This is appropriate because as we move the vertices of a circle packing graph to try and improve the density, we want the length of the edges to either increase or remain unchanged, in order to increase or maintain the density. If the location of the vertices of a strut tensegrity framework with n vertices are denoted $p_1, p_2, \ldots, p_n$ then a **flex** of the framework is a collection of n continuous functions $\{p_i(t) \mid i = 1 \ldots n\}$ from the interval $[0, 1]$ to the flat torus, where

1. $p_i(0) = p_i$ for all i, and

2. for each pair (i, j) where $\overline{p_i p_j}$ is a strut (i.e. edge) in the framework, $|p_i(t) - p_j(t)|$ is not a decreasing function on $[0, 1]$.

A strut tensegrity framework is considered **rigid** if the only flex of the framework is a trivial flex. If each $p_i(t)$ is obtained by restricting the same rigid motion of the torus to each p_i then we say the motion is a **trivial flex**.

In the context of strut frameworks, there is another notion of rigidity that is easily checked and, in our case, is equivalent to the definition of rigidity given above. An **infinitesimal flex** of a framework is a collection of n vectors $\{\mathbf{p}'_i \mid i = 1 \ldots n\}$ where for each pair (i, j) where $\overline{p_i p_j}$ is a strut

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \geq 0. \tag{1}$$

A strut tensegrity framework is considered **infinitesimally rigid** if the only infinitesimal flex of the framework is a trivial infinitesimal flex. In the context of a flat torus, the only **trivial infinitesimal flex** of a framework is the one where, for a fixed non-zero vector $\mathbf{v}$, $\mathbf{p}'_i = \mathbf{v}$ for all i. In [2], Connelly proves that if a tensegrity framework consists only of struts then the framework is rigid if and only if it is infinitesimally rigid. Restating the rigidity theory results in terms of circle packing, we have the following.

Proposition 3.1. *If strut framework associated to a circle packing $\mathcal{P}$ is (infinitesimally) rigid then the circle packing $\mathcal{P}$ is locally maximally dense.*

We are now in a position to state the main result from [2] (specialized to the context of flat tori) which is almost a converse of Proposition 3.1.

Theorem 3.1. *Let $\mathcal{P}$ be a packing that is locally maximally dense on a flat torus, then the strut framework associated to the packing graph $G_{\mathcal{P}}$ is (infinitesimally) rigid or $G_{\mathcal{P}}$ contains a subgraph $G_{\mathcal{Q}}$ (corresponding to sub-packing $\mathcal{Q}$) whose associated strut framework is (infinitesimally) rigid and $\mathcal{P}$ contains at least one circle that is free to move.*

There exist locally maximally dense packings of equal circles which contain an individual circle that is free to move (i.e. not held fixed by its neighbors), but the common diameter of all the circles cannot increase. For example, this occurs in the globally maximally dense arrangement of 7 circles packed into a hard-boundary square. If we remove any circles, in a locally maximally dense arrangement, that are free to move, called **free circles** (also known as **floaters** or **rattlers**), then we obtain a locally maximally dense packing for fewer circles in the flat torus. In this article, we will determine all the locally maximally dense arrangements for 1 to 5 circles without free circles. Therefore, for the remainder of this article, we assume that all of our graphs are connected. It turns out that none of these arrangements admit free circles, so we will have created an exhaustive list of locally maximally dense packings.

Now we observe that we can find a lower bound on the number of edges (and their arrangement) incident to a vertex in the packing graph associated to a locally maximally dense packing with no free circles.

Proposition 3.2. *Let $\mathcal{P}$ be a locally maximally dense packing of circles with no free circles, then no circle in $\mathcal{P}$ has its points of tangency contained in a closed semi-circle. In particular, every circle is tangent to at least three circles.*

Proof. If there were such a circle in a locally maximally dense packing, then we would have a non-trivial infinitesimal flex of the associated strut framework contradicting Theorem 3.1. The infinitesimal flex would be given by assigning a non-zero vector to the vertex (corresponding to the circle) that points directly away from the closed semi-circle and the zero vector to remaining vertices. □

Finally, we answer the question raised at the start of this section. Using Inequalities (1), Connelly proves in [3] and [4] a lower bound on the number of edges that a packing graph must contain in order for the associated packing to be locally maximally dense.

Proposition 3.3 (Minimum Edges)**.** *Let $\mathcal{P}$ be a locally maximally dense packing of n circles on a flat torus with no free circles, then the packing graph associated to $\mathcal{P}$ contains at least $2n - 1$ edges.*

This lower bound is powerful and allows us to determine all the locally maximally dense packings of 1 to 4 circles on the square torus.

4 Packings Of 1 To 4 Circles

The only locally maximally dense packing of one circle with diameter 1 is obviously the only one (see Figure 1), it is also the only packing where a circle is tangent to itself. Therefore we turn our attention to packings of two circles.

Proposition 4.1. *The only locally maximally dense packing of two circles on the square torus (up to translated variants) is the packing pictured in Figure 2. The common diameter is $\frac{\sqrt{2}}{2}$.*

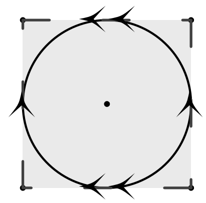


Figure 1: The only locally maximally dense packing of 1 circle on the square torus and therefore the globally maximally dense packing.

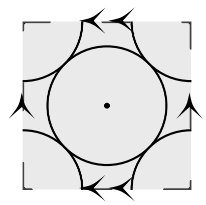


Figure 2: The only locally maximally dense packing of 2 circles on the square torus and therefore the globally maximally dense packing.

Proof. Using translations we may fix the first circle at the origin and by minimum edges proposition we must place the second circle in a location where it is tangent to the first circle in at least 3 ways. If we place this circle anywhere except at the center of this fundamental domain, then a maximum of two tangencies are formed. We can use Inequalities (1) and Proposition 3.1 to show that this packing is locally maximally dense. □

Notice that if there is a pair of circles that are tangent to each other in at least three different ways in a square torus then the above argument implies that the diameter of the circles must be $\frac{\sqrt{2}}{2}$ and the diameter upper bound (Proposition 2.1) implies that $n < 3$. Therefore we observe that if an equal circle packing on the square torus contains a pair of circles that are tangent to each other in three (or more) different ways, then the packing contains 1 or 2 circles.

Proposition 4.2. *The only locally maximally dense packing of three circles on the square torus (up to reflected and translated variants) is the packing pictured in Figure 3. The common diameter is* $\frac{\sqrt{6}-\sqrt{2}}{2}$.

Proof. By the minimum edges proposition we must place three circles in

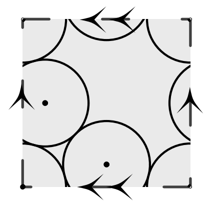


Figure 3: The only locally maximally dense packing of 3 circles on the square torus and therefore the globally maximally dense packing.

locations where they are tangent to each other in least 5 ways. By our previous observation each pair of circles can be tangent in at most 2 different ways, further as there are only three different circles, there must be one circle that is tangent to the other two circles in two different ways. Translating this circle to the origin and using the geometry of the situation implies that the other two circles must be on the perpendicular bisectors of the lines segments connecting the origin and $(0, 1)$ and the origin and $(1, 0)$. These two circles must be tangent to each other and this leads to the arrangement given in Figure 3. We can use Inequalities (1) and Proposition 3.1 to show that this packing is locally maximally dense. □

Having established the only locally maximally dense packing of three circles on the square torus, the globally maximally dense packing for four circle easily follows.

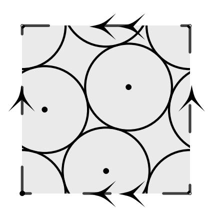


Figure 4: The only locally maximally dense packing of 4 circles on the square torus and therefore the globally maximally dense packing.

Proposition 4.3. *The only locally maximally dense packing of four circles on the square torus (up to reflected and translated variants) is the packing pictured in Figure 4. The common diameter is* $\frac{\sqrt{6}-\sqrt{2}}{2}$.

Proof. Observe that in the locally maximally dense packing for three circles a fourth circle can be located at $(\frac{\sqrt{3}}{1+\sqrt{3}}, \frac{\sqrt{3}}{1+\sqrt{3}})$. The resulting packing must be globally maximally dense. Demonstrating that there are no other locally maximally dense arrangements requires a little more work. The minimum edges proposition implies that there must be at least 7 tangencies. As there are only 4 circles and hence $\binom{4}{2} = 6$ possible single tangencies, there must be at least one pair of circles tangent to each other in two different ways (doubly tangent). One of these circle can be translated to the origin and we can use symmetries so that the other lies on the perpendicular bisector of the line segment between the origin and $(1, 0)$. This forces the location of the remaining two circles and leads to a one-parameter family of packings whose diameters increase until the packing becomes the one pictured in Figure 4. Therefore there is only one locally maximally dense packing of 4 circles in the square torus. □

This proof leads to an observing about packings containing a doubly tangent circle. If two circles are doubly tangent we know that in the Euclidean plane the second circle must be tangent to the first and a lift of the first. As the minimal distance from a circle to a lift of itself is one in the case of a square torus, the triangle inequality implies that the common diameter of the circles is $\frac{1}{2}$ or larger. However, the diameter upper bound (Proposition 2.1) is greater than $\frac{1}{2}$ only if $1 \leq n \leq 4$, so the equal circle packing contains at most 4 circles.

Collecting the results from the rigidity theory section and the observations from the discussion about the packings of 1 to 4 circles and translating them into the language of the packing graphs yields the following proposition. This proposition will help us list all of the possible packing graphs of locally maximally dense arrangement of 5 circles on the torus using the techniques of topologically graph theory.

Proposition 4.4. *Given a locally maximally dense packing, $\mathcal{P}$, of $n > 4$ equal circles (without any free circles) on a torus, the packing graph $G_{\mathcal{P}}$ satisfies the following conditions.*

1. *It contains at least $2n - 1$ and at most $3n$ edges.*
2. *It contains no loops and no multi-edges.*
3. *Every vertex is connected to at least three and at most six others.*

5 Results From Topological Graph Theory

Our approach to the study of packings of n ($n \geq 5$) circles on the square torus requires us to answer the following questions.

1. For a fixed n, how many abstract graphs satisfy the properties given in Proposition 4.4?

2. How many ways can those abstract graphs from Question 1 above be imbedded on a flat torus?

The first question will be discussed in the next section and the second question has been studied by topological graph theorists. One powerful tool for enumerating all the imbeddings of a graph on surfaces is called Edmonds' permutation technique ([7]) which we outline below.

Let G be a connected graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. The set of neighbors of $i \in V(G)$ is given by $N(i) = \{j \in V(G) \mid [i, j] \in E(G)\}$. A **rotation at i** is a cyclic permutation $\rho_i : N(i) \to N(i)$, and a set $\rho = \{\rho_i\}_{i \in V(G)}$ is called a **rotation scheme on G**. Each rotation scheme on G determines a 2-cell imbedding of G onto a surface of genus g and there is a one-to-one correspondence between oriented and labeled imbeddings of G and rotation schemes for G (see [21, Sec. 6.6]). The proof of this correspondence leads naturally to the algorithm below that computes the number of faces determined by the imbedding given by a rotation scheme, ρ, on G. This algorithm takes as input a rotation scheme, ρ and a list, L, of the oriented edges of G (each edge of G corresponds to two oriented edges) that are initially unmarked.

Face Tracing Algorithm

```
faceCount ← 0
for e in unmarked edges of L do
    mark e
    nextEdge ← [0,0]
    tempEdge ← e
    while e ≠ nextEdge do
            nextEdge ← [j, ρ_j(i)] (where tempEdge = [i,j])
            mark nextEdge
            tempEdge ← nextEdge
    faceCount ← faceCount +1
output faceCount
```

This is uses the fact that for each oriented edge $[i, j]$ the next edge (as determined by the rotation scheme ρ) in that face is $[j, \rho_j(i)]$. Therefore the while loop in the above algorithm cycles through the edges of the face containing the edge e until it returns to e. Oriented edges are used because each edge belongs to two faces with a different orientation, so once all oriented edges are marked the algorithm terminates and all the faces determined by the rotation scheme have been traced. The number of vertices, unoriented edges and the number of faces along with the Euler formula determine the genus of the surface determined by ρ.

The number of rotation schemes for a given graph is $\prod_{i=1}^{n}(|N(i)|-1)!$. Searching all of these rotation schemes for torus (genus 1) imbeddings using the algorithm above determines all of the possible oriented labeled 2-cell torus imbeddings of a graph G. However many of these are essentially the same: some are merely relabelings of others, some are identical except for the choice of orientation or both. As circle packing graphs do not come with natural labels or orientations, we are going to regard two rotation schemes, ρ and σ, on G as equivalent if there exists an α in the automorphism group of G such that $\alpha\rho_i\alpha^{-1} = \sigma_{\alpha(i)}$ for all i (the same graph with two different labelings are the same) or $\alpha\rho_i^{-1}\alpha^{-1} = \sigma_{\alpha(i)}$ for all i (the same graph with opposite orientations are the same). See [16] for more details.

This gives us an easily programable method for determining all the possible unlabeled, unoriented 2-cell torus imbeddings of a given abstract graph onto a topological torus. Once we have a complete answer to Question 1 above, this gives us a way to determine a list of *all* potential packing graphs (i.e. 2-cell imbeddings of a graph onto a torus). Among these imbedded graphs, the packing graphs associated to *all* locally maximally dense packings must appear, so long as they are 2-cell imbeddings, as the following proposition guarantees.

Proposition 5.1. *If a packing of circles on a flat torus is locally maximally dense without any free circles then the associated packing graph is a 2-cell imbedding.*

Proof. Suppose there was a locally maximally dense packing whose associated packing graph was not 2-cell imbedded on a flat torus. Consider the region determined by the graph that is not a 2-cell. One component of the boundary of this region must have a vertex that is furthest into the region when lifted into the Euclidean plane. By assigning this vertex a non-zero vector pointing into the region and the zero vector to all other vertices, we obtain a non-trivial infinitesimal flex of the associated strut framework contradicting Theorem 3.1. □

6 Packings Of 5 Circles

Packing 5 circles onto a flat torus requires at least 9 and at most 10 tangencies by Propostion 4.4. This means, abstractly, that the packing graph of any locally maximally dense packing is isomorphic to the complete graph on 5 vertices, denoted K_5, or to K_5 with an edge removed. Using the ideas outlined in Section 5, we have determined that there are 462 labeled, oriented 2-cell imbeddings, which reduce to 6 unoriented, unlabeled 2-cell imbeddings of K_5 on a flat torus. Further, there are 206 labeled, oriented 2-cell imbeddings, which reduce to 14 unoriented, unlabeled 2-cell imbeddings of K_5 minus an edge on a flat torus. (The numbers for K_5 agree with those derived in [16] and [9], K_5 minus edge was not studied in either of these references.) This means that there are 20 potential packing graphs and one or more of them must correspond to locally maximally dense packings on the square torus. Some of these twenty 2-cell imbeddings are pictured in Figures 5, 7, and 8. The software by Kocay ([12]) implementing the torus imbedding algorithm given in [13] was invaluable in drawing these imbeddings on the torus and calculating the automorphism groups of abstract graphs.

6.1 Eliminating Potential Packing Graphs

From these 20 potential packing graphs, we can eliminate 12 of them using the following proposition. This leaves the ones pictured in Figure 7 and 8.

Proposition 6.1. *If a graph imbedded on a torus contains a vertex surrounded by any one of the following face patterns then the graph cannot be the graph associated to a locally maximally dense equal circle packing. The forbidden face patterns are (1) two triangles and a polygon, (2) three triangles and a polygon, (3) five triangles, (4) four triangles and a quadrilateral, (5) six polygons with at least one non-triangle, (6) a triangle, a quadrilateral and a polygon, (7) two triangles and two quadrilaterals, (8) three quadrilaterals, or (9) seven (or more) polygons.*

Proof. Some of these face patterns can eliminated by observing that an equal circle packing graph is equilateral and all angles are at least 60 degrees. For example, two triangles and a quadrilateral cannot surround a vertex because then there would be at most 240 degree around that vertex.

Sometimes, if the imbedded graph is forced to be equilateral and correspond to an equal circle packing then at least one new edge would be forced. For example, if a vertex is surrounded by three quadrilaterals (rhombi) then as the angles in a rhombus in a equal circle packing graph are at most 120 degrees, all three angle around the vertex must be 120 degrees. Therefore each rhombus has a 60 degree angle and there is a pair of vertices (in each rhombus) forced to be the common edge length apart. However, in this case, the circles corresponding to those vertices must be tangent, but are not connected with an edge in the graph.

The other observation is that while certain imbeddings could be equilateral, they cannot be associated to a locally maximally dense equal circle packing. For example, a vertex can't be surrounded by three (equilateral) triangles and a polygon (with 5 or more sides) because the tangencies at that vertex would be contained in a closed semi-circle violating Proposition 3.2. See Figure 5. The other cases follow similarly. □

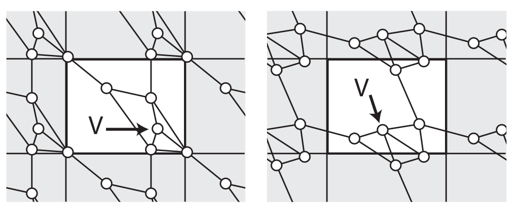


Figure 5: Two imbedding of K_5 minus an edge that are not the packing graph of any local maximally dense equal circle packing. The imbedding on the left cannot be equilateral because then the vertex V would have at most 240 degrees surrounding it. The imbedding on the right is not the packing graph associated to a locally maximally dense packing because the vertex V would correspond to a circle whose tangencies are restricted to a closed semi-circle.

In order to help eliminate 6 of remaining 8 imbeddings we use the following proposition.

Proposition 6.2. *If the packing graph of a locally maximally dense packing of 5 circles contains a triangle, then lifts of the same vertex connected with a chain of 3 edges are a unit length apart.*

Proof. Suppose the packing graph associated to a locally maximally dense packing of five equal circles on the square torus contains a triangle. Using the symmetries of square torus, we may assume that one vertex of the triangle is at the origin in the standard fundamental domain and that line connecting the vertex at the origin to the midpoint of the opposite side makes an angle α with the vector $\langle 1, 0\rangle$ and that $0 \leq \alpha \leq \frac{\pi}{4}$. Let the locations of the vertices of the triangle be $p_1(= (0, 0))$, p_2 and p_3 and the common edge length in the packing graph be d, the diameter of the circles in the packing. A **disk of exclusion for circle C** is a disk of *radius* d centered at C. By the definition of a circle packing, another circle cannot have its center located interior to another circle's disk of exclusion. Let E_1, E_2 and E_3 be the disks of exclusion for the circles with centers p_1, p_2 and p_3. Since we have three tangencies, by the minimum edges proposition,we must place two more circles in the fundamental domain to create at least six additional tangencies.

Examining the case when d is between the maximum of $\frac{2}{\sqrt{5\sqrt{12}}}$ and $\frac{1}{\sqrt{5}}$ and $0 \leq \alpha \leq \frac{\pi}{4}$, we can show that the diameter of the region outside of $E_1 \cup E_2 \cup E_3$ inside the square torus is strictly less than d and therefore is too small for two new circles. These are the unshaded regions in Figure 6. Hence, when the packing graph contains a triangle, the diameter of the circles in the packing must be less than $\frac{1}{\sqrt{5}}$.

Two lifts of the same vertex in a packing graph (in the Euclidean plane) must differ by an element of the square lattice. Suppose two lifts are separated by a chain of 3 edges, then the smallest possible value for the diameter is the length of the lattice vector divided by 3. If the lattice vector has length $\sqrt{2}$ (or larger) then as $\frac{\sqrt{2}}{3} > \frac{1}{\sqrt{5}}$ this is impossible, so the lifts must be separated by a shorter lattice vector, that is, a lattice vector of length one. $\square$

Notice that all the imbeddings in Figure 7 contain a triangle. Further observe that in each imbedding there is a vertex, V, connected to six different lifts of V by chains of length 3. By Proposition 6.2 this vertex is a unit length from six of its lifts. This is impossible in a square lattice, where a vertex can have only four unit distance lifts. This eliminates these 6 imbeddings and implies that the only possible packing graphs that lead to locally maximally dense packings on the square torus are the imbedding in Figure 8.

However, the imbedding in Figure 8(b) is not the packing graph of a locally maximally dense arrangement on the square torus. An equilateral imbedding of this type forces an edge between V and U. Therefore, the only remaining imbedding (Figure 8(a)) must be the packing graph associated to a locally maximally dense packing.

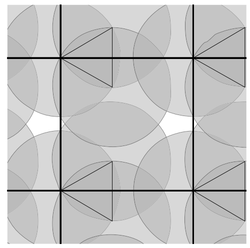
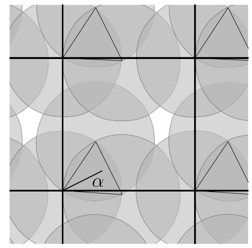
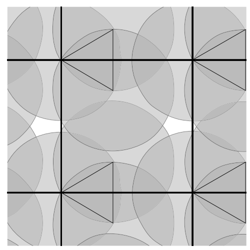
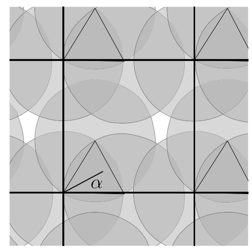
	$\alpha = 0$	$0 < \alpha \leq \frac{\pi}{4}$
$d = \frac{1}{\sqrt{5}}$		
$\frac{1}{\sqrt{5}} < d \leq \frac{2}{\sqrt{5\sqrt{12}}}$		

Figure 6: Representative arrangements of the disks of exclusion E_1, E_2, and E_3, on the square torus for various values of d and α. In each of the cases the white area outside of the disks of exclusion has diameter strictly less than d, the common diameter.

6.2 The Locally Maximally Dense Packing of 5 Circles

Theorem 6.1. *There exists a unique locally maximally dense equal circle packing of 5 circles on the square torus. It has packing graph homotopic to imbedding in Figure 8(a) and the common diameter of the circles is* $\frac{\sqrt{5}}{5}$.

Proof. Using Inequalities (1) on all of the edges in the associate packing graph, we can prove that the packing pictured in Figure 9 is a locally maximally dense equal circle packing.

It is possible that there could be another basis for the square torus in which imbedding in Figure 8(b) is realized as a locally maximally dense equal circle packing graph. (That is, the edge chains in the imbedding of Figure 8(a) could be in different homotopy classes on the torus if a different basis for the square lattice is used, but the face arrangement and degree is the same.) However, by observing that if the imbedding pictured in Figure 8(a) is forced to be equilateral (with common edge length d), then the distance from V_1 to its lift $\overline{V}_1$ is at most $\sqrt{7}d$ because the largest the angle at V_3 in the chain of 3 edges from V_1 to $\overline{V}_1$ is 120 degrees. Using the diameter upper bound (Proposition 2.1), we notice that $\sqrt{7}d < \sqrt{7}(\frac{2}{\sqrt{5\sqrt{12}}}) < \sqrt{2}$. As V_1 and $\overline{V}_1$ are connected with a lattice vector and the lattice vectors of the square lattice have lengths $\{1, \sqrt{2}, 2, \sqrt{5}, \ldots\}$, they must be connected with a length one lattice vector. The similar argument applies to V_1 and $\overline{\overline{V}}_1$. Hence an equilateral imbedding of Figure 8(a) can be realized only on the standard basis for the square torus. An argument similar to the proof of Theorem 4 in [1] shows that this locally maximally dense packing in this homotopy class is unique. The infinitesimal rigidity of the strut framework guarantees a non-zero stress on every strut (see [19, Thm 5.2]) which leads to an energy function with a unique maximum, so any other equilateral infinitesimally rigid strut framework must be congruent to the packing graph associated to the packing in Figure 9. □

The proof of the locally maximality in the first part of this proposition generalizes to the following.

Proposition 6.3. *If $n = a^2 + b^2$ with integers $a > b > 0$ and $\gcd(a, b) = 1$ then there exists a locally maximally dense equal circle packing of n circles on the square torus whose packing graph is the union of squares. The common diameter is $\frac{\sqrt{n}}{n}$.*

Proof. Let the centers of the circles lie on a superlattice (of the square lattice of the torus) generated by the equi-length perpendicular vectors $\mathbf{u}_1 = \langle \frac{a}{a^2+b^2}, \frac{b}{a^2+b^2} \rangle$ and $\mathbf{u}_2 = \langle -\frac{b}{a^2+b^2}, \frac{a}{a^2+b^2} \rangle$. The condition on the greatest common divisor of a and b guarantees we can find appropriate chains of edges (containing all the centers of the circles) from the circle at $(0, 0)$ to a lift this circle in two different directions (one by following the centers at the integer multiples of $\mathbf{u}_1$ and the other by following integer multiples of $\mathbf{u}_2$). Using the $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ basis for the vectors in Inequality (1) greatly simplify the computation. We can show that these inequalities only have the zero solution and hence the arrangement is locally maximally dense. □

This supports the conjectured globally maximally dense packing of $10 = 3^2 + 1^2$ circles on a square torus in [15, Fig. 2.12 l]. A slight modification of this argument shows that the conjectured globally maximally dense packings of 12 and 15 circles on a square torus in [15, Fig. 2.12 n and q] are locally maximally dense.

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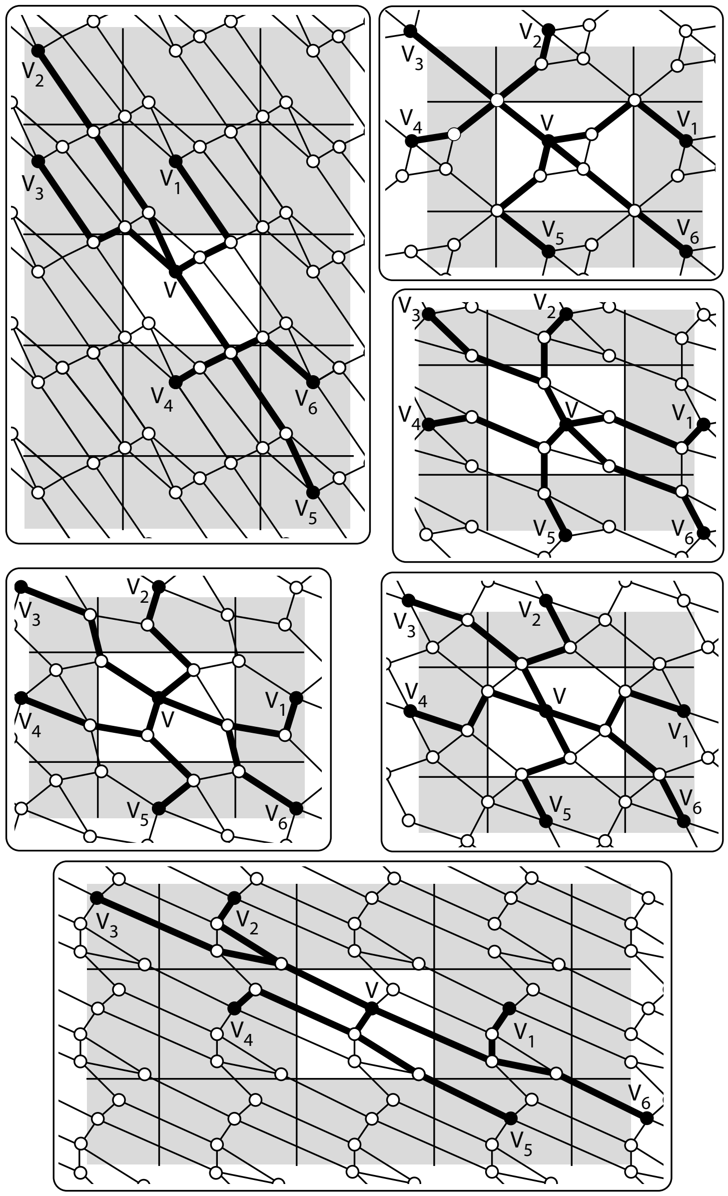
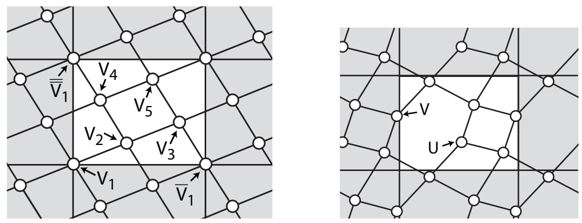


Figure 7: Six of the 20 inequivalent imbeddings of K_5 or K_5 minus edge on a flat torus. For each imbedding a vertex, V, and chains of 3 edges connecting six of the lifts of V to V are highlighted in boldface.



(a) Packing graph of a locally maximally dense arrangement. (b) Not the packing graph of any packing.

Figure 8: The remaining potential packing graphs after Propositions 6.1 and 6.2 are used to eliminate the 18 of the original 20 possibilities.

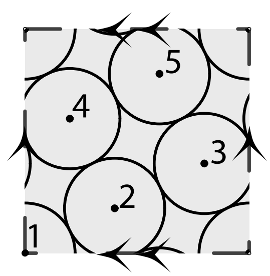


Figure 9: The only locally maximally dense packing of 5 circles on the square torus and therefore the globally maximally dense packing.