An Overview of the Contour Integral

Math 402, Winter 2001

Suppose that f is a complex valued function and that γ is a contour in the domain of f. Then the contour integral $\int_\gamma f(z)dz$ is defined using Riemann sums.

- Theorem, p. 116: If the complex-valued function $f(t)$ is continuous on $[a,b]$ and $F'(t) = f(t)$ for all t in $[a,b]$, then

$$\int_{t=a}^{t=b} f(t)dt = F(b) - F(a).$$

  While we don't use this theorem very much, it was very useful for proving the next result.

- The Integration of Powers Theorem: If $R > 0$, then

$$\int_{|z-z_0|=R} (z-z_0)^n dz = \begin{cases} 0, & \text{if } n \neq -1; \\ 2\pi i, & \text{if } n = -1. \end{cases}$$

- The Pólya Field Interpretation of the Contour Integral: Let $\mathbf{F} = \langle u(x,y), -v(x,y) \rangle$ denote the Pólya Field of $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Then

$$\int_\gamma f(z)dz = \int_\gamma \mathbf{F}\cdot\mathbf{T}ds + i\int_\gamma \mathbf{F}\cdot\mathbf{N}ds.$$

  In this equation, $\mathbf{T}$ denotes the unit tangent vector, $\mathbf{N}$ denotes the unit normal vector (the unit tangent rotated by $\frac{\pi}{2}$ radians clockwise), and ds denotes the arclength differential. This interpretation of the contour integral is useful for a variety of reasons.

  1. First, it allows us to numerically estimate certain contour integrals just by inspecting the vector field and the contour.
  2. The Pólya Field interpretation shows us that reversing the direction of integration along a contour has the effect of negating the value of the contour integral.
  3. Most importantly, the Pólya Field interpretation gives us physical interpretations of the contour integral. For example, if $\mathbf{F}$ represents some type of fluid flow, then this interpretation tells us that the real part of $\int_\gamma f(z)dz$ measures how much of the fluid flows against our back as we travel along γ and that the imaginary part of $\int_\gamma f(z)dz$ measures how much of the fluid flows against our left side as we travel along γ. Another example arises when $\mathbf{F}$ represents an electrostatic field in the plane. In this case, two of the four Maxwell's Equations from physics (specifically Faraday's Law of Induction and Gauss' Law for Electricity) tell us that

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma \mathbf{F}\cdot\mathbf{T}ds + i\int_\gamma \mathbf{F}\cdot\mathbf{N}ds \\ &= -\frac{d\Phi_B}{dt} + i4\pi q.\end{aligned}$$

     Here Φ_B denotes the magnetic flux of the field through γ and q denotes the total charge enclosed inside γ. An analagous result holds when $\mathbf{F}$ represents a magnetic field in the plane. In this case, the other two of Maxwell's Equations (Ampere's Law and Gauss' Law for Magnetism) yield a similar type of interpretation.

- The Complex Fundamental Theorem of Calculus (Theorem 6, p. 124): Suppose that the function $f(z)$ is continuous on the domain D and has an antiderivative $F(z)$ throughout D. (This implies that $F(z)$ is analytic in D.) Then for any contour γ lying in D, with initial point z_1 and terminal point z_T, we have

$$\int_\gamma f(z)dz = F(z_T) - F(z_1).$$

This theorem is analagous to the second part of the Real Fundamental Theorem of Calculus. Note that if γ is a closed loop, the intitial and final points of γ are the same, whence the integral above is zero. This is Corollary 2 on page. 127.

- The Complex Antiderivative Theorem (Theorem 7, p. 125): I will not state this theorem in its entirety. One important aspect of this theorem is that it introduces the notion of path independence. Secondly, the proof of the theorem gives us a way of "cooking up" antiderivatives. Specifically, if $f(z)$ is continuous in the domain D, if γ_z is any path in D connecting z and the fixed point a, and if path independence holds, then the function

$$F(z) = \int_{\gamma_z} f(w)dw$$

is an antiderivative of $f(z)$. This result is analagous to the first part of the Real Fundamental Theorem of Calculus.