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Mapping Properties of Complex-Valued Functions #1

Here we consider the function $f(z) = e^{2i}z$ and some of its properties.

The domain of $f(z) = e^{2i}z$ is $\mathbb{C}$. The function *f(z)* can be broken down into two functions, *u* and *v*, where *u* is a function that describes the real part of *f* and *v* is a function that describes the complex part of *f*. Therefore we can describe *f(z)* as

$$f(z) = f(x+iy) = u(x,y) + i*v(x,y) .$$

For our function,

$$f(x+iy) = e^{2i}(x+i*y)$$

$$= [\cos(2)+i*\sin(2)](x+i*y)$$
$$= \cos(2)x + i\cos(2)y + i\sin(2)x + i^2\sin(2)y$$
$$= (\cos(2)x - \sin(2)y) + i(\sin(2)x + \cos(2)y),$$

where *u(x,y)* = $(\cos(2)x - \sin(2)y)$ and *v(x,y)* = $i(\sin(2)x + \cos(2)y)$.

Let us examine how *f* maps two particular inputs from $\mathbb{C}$, *f*(1+2i) and *f*(-1-3i), into the range of our function. Using the functions *u* and *v* for the first example we find that:

$$f(1+2i) = (\cos(2)*1 - \sin(2)*2) + i(\sin(2)*1 + \cos(2)*2)$$

$$= (\cos(2) - 2\sin(2)) + i(\sin(2) + 2\cos(2))$$

and

$$f(-1-3i) = (\cos(2)*(-1) - \sin(2)*(-3)) + i(\sin(2)*(-1) + \cos(2)*(-3))$$

$$= (-\cos(2) + 3\sin(2)) - i(\sin(2) + 3\cos(2)) .$$

We now determine how our function maps the domain into the codomain. The first set we experiment with, using our function $f(z) = ze^{2i}$, is the set $|z| \le 2$ and $\text{Arg}(z) = \frac{k\pi}{3}$, where $k = 0,1,2,3,4,5$. In other words, we are looking at radial segments less then or equal to two with the following orientations: $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$. Below is an example of our domain. The point *A* corresponds to $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Pay attention to what happens to this point as we apply our function to the domain.

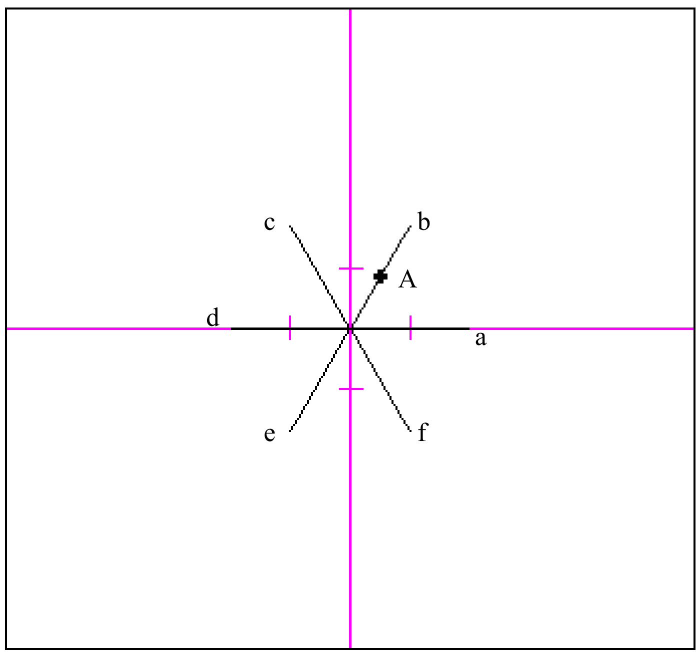

Figure 1 - Domain of First Example

Applying the function to this domain results in Figure 2. Notice that the function rotates each point in the domain two radians counterclockwise. Particularly, notice that $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ has been mapped to $f(z) = \cos\left(\frac{\pi}{3} + 2\right) + \sin\left(\frac{\pi}{3} + 2\right)$.

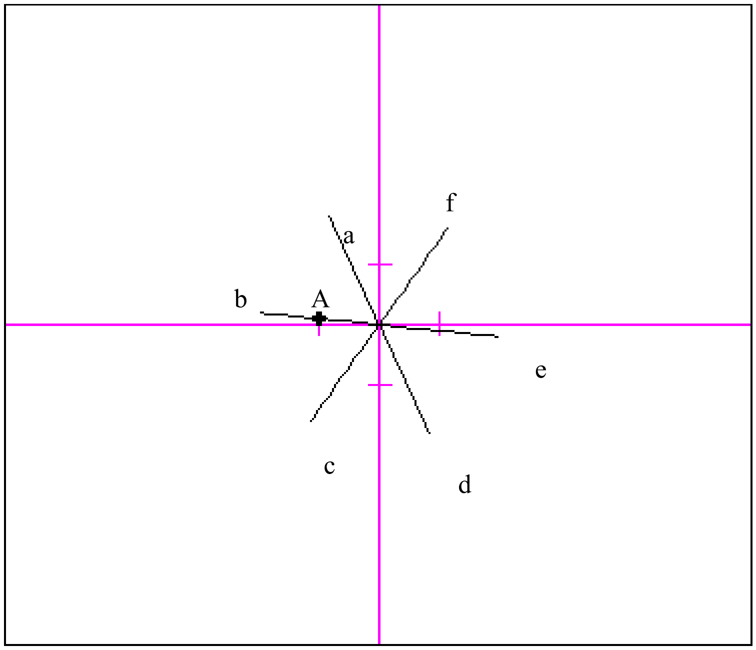

Figure 2 - Range of First Example

Instead of choosing a different range for our next examples, we instead tweak the function that we are given. First of all, we consider the generalization of our function, namely $f(z) = ze^{\alpha i}$ where α is real. This function will just rotate the points in the domain α radians. If α is positive, the rotation will be counterclockwise; however, if α is negative, the rotation will be clockwise.

Now consider the function $f(z) = 3ze^{\alpha i}$. Using our original domain and $\alpha = 2$, the function range is show in Figure 3. As you can see, the function succeeds in rotating the points in the domain 2 radians counterclockwise and tripling the modulus of each point in the domain.

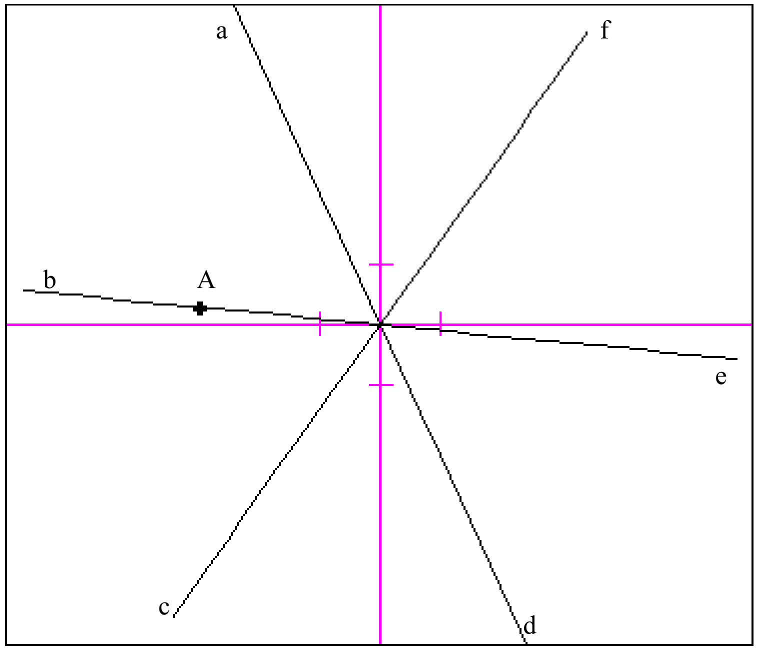

Figure 3 - Dilation Function

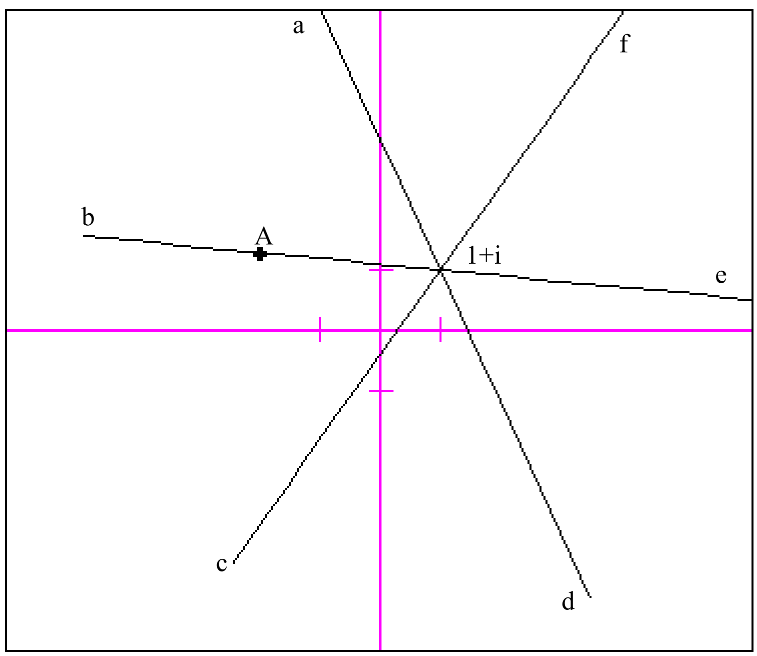

Figure 4 - Shift Function

The last function we will consider is $f(z) = 3ze^{\alpha i} + (1+i)$. Using our original domain and $\alpha = 2$, the function range is show in Figure 4 (above). As you can see, the function succeeds in rotating the points in the domain 2 radians counterclockwise, dilating each point in the domain by a factor of 3, and shifting the center of the domain one unit to the right and 1 unit upward.

Notice that the range of this function, $f(z) = e^{2i}z$, is $\mathbb{C}$. This implies that *f* is onto (the codomain is equal to the range). Recall that *f* maps a point (*x*+iy) to a new point that has been rotated counterclockwise by two radians. Thus, for a particular output, $w = f(z)$, there is a unique input, *z*, two radians clockwise from *w* that *w* is mapped from. Because of this, $f(z) = e^{2i}z$ is one-to-one. Because *f* is one-to-one and onto, we would expect *f* to have an inverse, and it does have one. The inverse function of *f* is the following function:

$$f^{-1}(z) = e^{-2i}z .$$

A simple check shows this is true. Consider $f(f^{1}(z))$.

$$\begin{aligned} f(f^{1}(z)) &= f(e^{-2i}z) \\ &= e^{2i}e^{-2i}z \\ &= z. \end{aligned}$$

In summary, this function succeeds in rotating, dialating, and shifting points in a given domain. Also, the function is one-to-one and onto and has an inverse.