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MTH 408 HW 6

1. Prove that $\lim_{x \to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$.

We first choose $\varepsilon > 0$. Then, let $\delta = \min\left(1, \frac{2}{3}\varepsilon\right)$. Suppose that $0 < |x - 1| < \delta$. Then, applying the reverse triangle inequality and that $|x - 1| < \delta \leq 1$, we see

$$\begin{aligned} |x+1| &= |x - 1 - (-2)| \\ &\geq ||x-1| - |-2|| \\ &= |2 - |x-1|| \\ &> 1. \end{aligned}$$

Then, taking recipricols

$$\left|\frac{1}{x+1}\right| < 1. \tag{1}$$

Further, notice that $|x-1| < \delta \leq 1$ implies that $0 < x < 2$. Then, it follows easily that $-1 < 2x - 1 < 3$ and that

$$|2x - 1| < 3. \tag{2}$$

Our goal is to show that $\left|f(x) - \frac{1}{2}\right| < \varepsilon$. So, considering $f(x)$ and (**??**) and (**??**) above, we see

$$\begin{aligned} \left|f(x) - \frac{1}{2}\right| &= \left|\frac{x^2 - x + 1}{x + 1} - \frac{1}{2}\right| \\ &= \left|\frac{2x^2 - 3x + 1}{2(x+1)}\right| \\ &= \frac{|x-1||2x-1|}{2|x+1|} \\ &< \frac{|x-1||2x-1|}{2} \\ &< \frac{\delta \cdot 3}{2} \\ &< \varepsilon. \end{aligned}$$

Thus, $\lim_{x \to 1} \frac{x^2 - x + 1}{x + 1} = \frac{1}{2}$.

2. For some $c \in \mathbb{R}$, prove that $\lim_{x \to c} x^3 = c^3$.

We will perform a proof by contradiction. We assume that there exists $c \in \mathbb{R}$ such that $\lim_{x \to c} x^3 \neq c^3$. Then, by the *Sequential Criterion for Limits*, we know that there exists a sequence $\{x_n\}$ that converges to c and is not equal to c for all n and that the function sequence $\{(x_n)^3\}$ does not converge to c^3. Considering this sequence, which is guaranteed to exist, and applying some algebra of limits, we see

$$\begin{aligned} \lim_{n \to \infty} x_n^3 &= \left(\lim_{n \to \infty} x_n \right)^3 \\ &= c^3. \end{aligned}$$

This, of course, is a contradiction because we are assuming that the function sequence does not converge to c^3. Thus, for all $c \in \mathbb{R}$, $\lim_{x \to c} x^3 = c^3$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{x \to 0} f(x) = L$. For fixed $a > 0$, $g(x) = f(ax)$.

   a. The graphical relationship between f and g is that the the graph of f is scaled in the horizontal direction by $\frac{1}{a}$. It is important to note that the scaling is "centered" at the origin. That is, the horizontal distance from a given point to the origin is scaled by this quantity. This idea is seen in the next two graphs.

      Unfortunately, graphics are beyond the scope of this introductory course. Instead, I'll include a matrix and a table instead!

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \text{ is a triangular matrix.}$$

	Genotype of Parents					
Genotype of Offspring	AA-AA	AA-Aa	AA-aa	Aa-Aa	Aa-aa	aa-aa
AA						
Aa						
aa						

   b. We will show that $\lim_{x \to 0} g(x) = L$ is also true. First assume that $\lim_{x \to 0} f(x) = L$ and assume that a is fixed with $a > 0$. By the *Sequential Criterion for Limits*, we know that if $\{x_n\}$ is a sequence satisfying that $\{x_n\}$ converges to zero and is not equal to zero for all n, then $f(x_n)$ converges to L. We will then perform a proof by contradiction. That is, we assume that $\lim_{x \to 0} g(x) \neq L$. Then, by the *Sequential Criterion for Limits*, we know that there must exist a sequence $\{\hat{x}_n\}$ such that $\{\hat{x}_n\}$ converges to zero and is not equal to zero for all n and $g(\hat{x}_n)$ does not converge to L. Then, consider the sequence $\{a\hat{x}_n\}$, which converges to zero by algebra of limits and is never equal to zero. Then, using the sequential argument for f earlier, we know that $\{f(a\hat{x}_n)\}$ converges to L. But, $\{f(a\hat{x}_n)\} = \{g(\hat{x}_n)\}$, which we are assuming does not converge to L. Thus, we have reached a contradiction and $\lim_{x \to 0} g(x) = L$.

4. Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ DNE by using the *Sequential Criterion for Limits.*

That is, we must show that two different sequences, x_n and $\hat{x}_n$, both converge to zero with $x_n \neq 0$ and $\hat{x}_n \neq 0$ for all n. Further, we must show that their function sequences, $f(x_n)$ and $f(\hat{x}_n)$, converge to L_1 and L_2 with $L_1 \neq L_2$. Consider

$$\begin{aligned}\{x_n\} &= \frac{1}{\sqrt{\frac{\pi}{2} + 2\pi n}} \\ \{f(x_n)\} &= \sin\left(\frac{\pi}{2} + 2\pi n\right) \\ &= 1\end{aligned}$$

and

$$\begin{aligned}\{\hat{x}_n\} &= \frac{1}{\sqrt{\pi + 2\pi n}} \\ \{f(\hat{x}_n)\} &= \sin\left(\pi + 2\pi n\right) \\ &= 0.\end{aligned}$$

Clearly, x_n and $\hat{x}_n$ satisfy the needed properties above. Further, it is obvious that $f(x_n)$ and $f(\hat{x}_n)$ converge to one and zero respectively. Thus, we have constructed two different sequences such that their function sequences converge to different limits. And, by *Sequential Criterion for Limits*, we have shown that $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ DNE.

5. Let $\mathbb{Q}$ denote the set of rational numbers, and define the function

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Consider $f(x) = x \cdot \chi_{\mathbb{Q}}(x)$.

a. The graph of this function is plotted below.

b. We will next prove that $\lim_{x \to 0} f(x) = 0$. Choose $\varepsilon > 0$. Then, let $\delta = \varepsilon$. Suppose that $0 < |x - 0| < \delta$. Then, notice that $\chi_{\mathbb{Q}}(x) \leq 1$ for all x, by definition. So,

$$\begin{aligned} |f(x) - 0| &= |x \cdot \chi_{\mathbb{Q}}(x)| \\ &= |x| \cdot |\chi_{\mathbb{Q}}(x)| \\ &< \delta \cdot 1 \\ &= \varepsilon. \end{aligned}$$

Thus, we see that $|f(x) - 0| < \varepsilon$. Then, we have shown that for all $\varepsilon > 0$, there exists a δ such that $0 < |x - 0| < \delta$ implies $|f(x) - 0| < \varepsilon$. And, by definition, we see that $\lim_{x \to 0} f(x) = 0$.

c. For the case when $x_0 \neq 0$, we will show that $\lim_{x \to x_0} f(x)$ DNE. To do this, we will consider two different sequences. The first, $\{x_n\}$, will be a sequence that converges to $x_0 \neq 0$ along a set of rational numbers and for all n, $x_n \neq x_0$. The second, $\{\hat{x}_n\}$, will be a sequence that converges to $x_0 \neq 0$ along a set of irrational numbers and for all n, $\hat{x}_n \neq x_0$. It is pretty easy to see that the limits of the function sequences are given by

$$\lim_{n \to \infty} f(\{x_n\}) = x_0$$

$$\lim_{n \to \infty} f(\{\hat{x}_n\}) = 0.$$

Then, since $x_0 \neq 0$, we have found two sequences in which their function sequences converge to differing values. Then, by the *Sequential Criterion for Limits*, the limit of $f(x)$ when $x_0 \neq 0$ does not exist.