

THE THEORY OF LEBESGUE MEASURE

In our early discussion of the Riemann integral, we learned that the Dirichlet Function

$$\chi_Q(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

is not Riemann integrable on any interval $[a,b]$. However, in an intuitive sense, because there are only countable many rational numbers, the function χ_Q is equal to 1 “most of the time.” If χ_Q was Riemann integrable on $[a,b]$, we would then expect that

$$\int_{x=a}^{x=b} \chi_Q(x) dx = b - a .$$

This situation illustrates the limitations of the Riemann integral. Mathematicians in the late 1800’s were well aware of this fact, and it was the French mathematician Henri Lebesgue who developed a new type of integral, known as the Lebesgue integral, which allowed the integration of highly discontinuous functions such as χ_Q above.

Later in this course, we will study the Lebesgue integral. Before we do, however, we need to study a separate, but related topic known as Lebesgue measure. The idea of Lebesgue measure is quite simple: it is merely a generalization of the notion of length. What we’re looking for in this project is a new type of function, which we’ll call m (for measure) that satisfies four properties:



Henri Lebesgue
(1875-1941)

- I. If IR^+ denotes the set of nonnegative real numbers and if $P(IR)$ denotes the power set of the real numbers, in other words, the set of all subsets of the real numbers, then $m : P(IR) \rightarrow IR^+$ is a function. In particular, this says that for each subset E of the real numbers, $m(E)$, which we’ll call the “Lebesgue measure of E ,” is defined and equal to a nonnegative real number.
- II. The measure of the interval I should be its length, i.e. $m(I) = \ell(I)$.
- III. If $\{E_n\}$ is a sequence of pairwise disjoint sets of real numbers, then
$$m\left(\bigcup_{i=1}^{i=\infty} E_i\right) = \sum_{i=1}^{i=\infty} m(E_i).$$
In words this means that for disjoint sets, the measure of the union of the sets equals the sum of their measures. If you think of measure as generalizing the notion of length, this property makes sense.
- IV. The measure m is “translational invariant.” What this means is that if we take a set E and move every element of the set y units to the left or right, we expect the

measure of the new set, call it $E + y$, to be the same as the measure of the original set E . In other words, $m(E + y) = m(E)$.

Keeping these four desired properties in mind, let us now attempt to construct this function m . We begin by making the following definition of what is known as “outer measure.”

definition: If E is a set of real numbers, the outer measure of E is defined as follows:

$$m^*(E) = \inf_{E \subseteq \bigcup I_n; I_n \text{ open}} \sum \ell(I_n).$$

Here’s what this definition is saying. Look at any collection of open intervals $\{I_n\}$ whose union contains the set E . Calculate the sum of the lengths of these intervals, and you get a nonnegative real number. Now take the infimum over all such collections of open intervals to get the outer measure of E . Note that this definition does not define the term “measure” that we’re looking for. It’s simply a first step toward doing so.

In class, we will prove three facts about outer measure:

- $m^*(\{\}) = 0$
- If $A \subseteq B$, then $m^*(A) \leq m^*(B)$
- If I is an interval, then $m^*(I) = \ell(I)$.

Now here are three theorems for you to prove.

1. If x is a real number, then $m^*(\{x\}) = 0$.
2. If $\{E_n\}$ is an infinite sequence of sets of real numbers, then $m^*\left(\bigcup_{n=1}^{n=\infty} E_n\right) \leq \sum_{n=1}^{n=\infty} m^*(E_n)$.

Note that the difference between this theorem, and the property given in (4) above is that we are not assuming here that the sets are pairwise disjoint. Some sets may overlap, which accounts for the inequality sign. This property is sometimes referred to as “countable subadditivity.” To prove this theorem, consider the following steps.

- a. Let $\varepsilon > 0$ be arbitrary. Explain why for each n there exists a sequence of open intervals $\{I_{n,i}\}_{i=1}^{i=\infty}$ with the property that $\sum_i \ell(I_{n,i}) < m^*(E_n) + \frac{\varepsilon}{2^n}$.
- b. Now consider the union of the above sequence of intervals, where the union is taken over all values of n . In other words, consider $\{I_{n,i}\}_{i,n=1}^{\infty}$. Prove that

$$\bigcup_{n=1}^{n=\infty} E_n \subseteq \{I_{n,i}\}_{i,n=1}^{\infty}, \text{ and then apply the definition of outer measure.}$$

3. If A is a countable set of real numbers, then $m^*(A) = 0$.

In class we will use the result of (3) to prove that the interval $[0,1]$ is uncountable.

Sadly, at this point in the discussion, I must inform you that it is impossible to construct the measure m that satisfies all four of our desired properties. So you say, “the project is over—let’s go celebrate!” Not so fast! It ends up that if one is willing to weaken property (I), then one can construct a measure m that satisfies the remaining three properties.

You will recall that property (I) stipulated that $m : P(\mathbb{R}) \rightarrow \mathbb{R}^+$ was a function. This means that the domain of m consists of all subsets of real numbers. What we are going to do is restrict the domain to include not all subsets of real numbers, but only those that satisfy a certain condition given in terms of their outer measures. Those sets that satisfy this certain condition are known as “measurable sets.”

definition: The set E of real numbers is measurable if and only if for every set A of real numbers,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where E^c denotes the complement of the set E in the real numbers. If the set E is measurable, we may then speak of its Lebesgue measure, $m(E)$, which is simply defined as the outer measure. In other words, if E is measurable, then $m(E) = m^*(E)$. The collection of all subsets of real numbers that are measurable we will denote by \mathcal{M} .

Our process now is to restrict the domain of m from all subsets of the real numbers $P(\mathbb{R})$ to the class of measurable sets \mathcal{M} . Obviously, we’d like to know which sets are measurable. For example, we want it to work out so that every interval $[a,b]$ is measurable. To classify some of the measurable sets, we will need to proceed through several steps.

4. Explain why it is always the case that $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$. This tells us that when proving a set E is measurable, we need only prove that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$ for every set of real numbers A .
5. Prove that if $m^*(E) = 0$, then the set E is measurable. In particular, the empty set is measurable.
6. Prove that if a set is measurable, so is its complement. This fact, plus (5), tells us that the set of real numbers is measurable.
7. Prove that every countable set of real numbers is measurable.

These results give us a few examples of measurable sets. We will now prove a few theorems about unions of measurable sets.

8. If E_1 and E_2 are measurable sets, then the union $E_1 \cup E_2$ is measurable as well. To prove this theorem you must start with an arbitrary set of real numbers A and prove

that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$, with the set E replaced by $E_1 \cup E_2$. To do this, consider proving and combining the following steps.

- a. $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$
- b. $m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$
- c. $m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$

9. If $\{E_n\}_{n=1}^N$ is finite sequence of disjoint measurable sets and if A is any set of real numbers, then $m^*\left(A \cap \left(\bigcup_{n=1}^{n=N} E_n\right)\right) = \sum_{n=1}^{n=N} m^*(A \cap E_n)$. Prove this theorem by induction on N . In the proof, consider proving and combining the following steps.

- a. $m^*\left(A \cap \left(\bigcup_{n=1}^{n=N} E_n\right)\right) = m^*\left(A \cap \left(\bigcup_{n=1}^{n=N} E_n\right) \cap E_N\right) + m^*\left(A \cap \left(\bigcup_{n=1}^{n=N} E_n\right) \cap E_N^c\right)$
- b. $A \cap \left(\bigcup_{n=1}^{n=N} E_n\right) \cap E_N = A \cap E_N$
- c. $A \cap \left(\bigcup_{n=1}^{n=N} E_n\right) \cap E_N^c = A \cap \left(\bigcup_{n=1}^{n=N-1} E_n\right)$

In class, we will use the above result to prove the following theorem.

Theorem: If $\{E_n\}$ is an infinite sequence of measurable sets, then the union $\bigcup_{n=1}^{\infty} E_n$ is measurable as well.

Thus, if we combine this theorem with other results, we see that we have established the following three properties about the class of measurable sets \mathcal{M} :

- i) If two sets belong to \mathcal{M} , their union belongs to \mathcal{M} as well.
- ii) If a set belongs to \mathcal{M} , its complement belongs to \mathcal{M} as well.
- iii) If an infinite collection of sets belongs to \mathcal{M} , the union of the sets belongs to \mathcal{M} as well.

Any collection of sets that is closed under finite unions, complements, and infinite unions is called a σ -algebra of sets.

10. Prove that the σ -algebra \mathcal{M} is also closed under finite and infinite intersections.

In class, we will then prove that for any real number a , the interval (a, ∞) is measurable.

11. Use this fact to prove that the σ -algebra \mathcal{M} includes all open and closed intervals as well as all open and closed sets. Consequently, since each open or closed interval is measurable, its Lebesgue measure is precisely its outer measure, which is its length!

The following theorem will prove indispensable for calculating the measure of many sets constructed as unions of simpler measurable sets.

12. Suppose $\{E_n\}$ is any sequence of measurable sets. Prove that

$$m\left(\bigcup_{n=1}^{n=\infty} E_n\right) \leq \sum_{n=1}^{n=\infty} m(E_n). \quad (1)$$

Furthermore, prove that if the sets in the sequence are pairwise disjoint, then

$$m\left(\bigcup_{n=1}^{n=\infty} E_n\right) = \sum_{n=1}^{n=\infty} m(E_n). \quad (2)$$

(Hint: For part (1) simply use the countable subadditivity of outer measure. To prove (2), first consider the case of the finite union, and apply the result of problem 10 with $A = IR$.)

This theorem establishes property (III) of the four properties outlined at the start of this project. In class, we will discuss the issue of translational invariance, property (IV). We will also prove a theorem regarding decreasing sequences of measurable sets and then construct an example of a non-measurable set of real numbers, i.e. an element of $P(IR)$ that is not an element of \mathcal{M} . Such an example illustrates why we had to weaken property (I).

To put some of these ideas together, let us consider a very interesting type of set known as the Cantor middle-thirds set. The set is constructed in the following manner. Take the unit interval $[0,1]$ and remove from it the middle open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Now take the resulting set, which consists of two closed intervals, and remove from each of them the middle open interval of length $\frac{1}{9}$. The remaining set then consists of four intervals of length $\frac{1}{9}$. Repeating this process inductively, we see at the n -th stage the remaining set consists of 2^n disjoint closed intervals of length $\frac{1}{3^n}$ each. The $(n+1)$ -stage consists of removing open intervals of length $\frac{1}{3^{n+1}}$ from the middle of each of the remaining closed intervals at the n -th stage. If this process continues forever, the resulting set K is known as the Cantor middle-thirds set. Note that this set has many points in it, for example, $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \dots$ all belong. In fact the Cantor set is uncountable, just like the interval $[0,1]$. It is also closed, nowhere dense, and perfect. To say that it is nowhere dense means that it contains no intervals. To say that it is perfect is to say that every point in the set is a limit point of other points in the set, i.e. there are no isolated points. Finally, the set is self-similar under magnification. It is a simple example of a fractal! WOW!!!

13. Find the Lebesgue measure of the Cantor middle-thirds set K .

