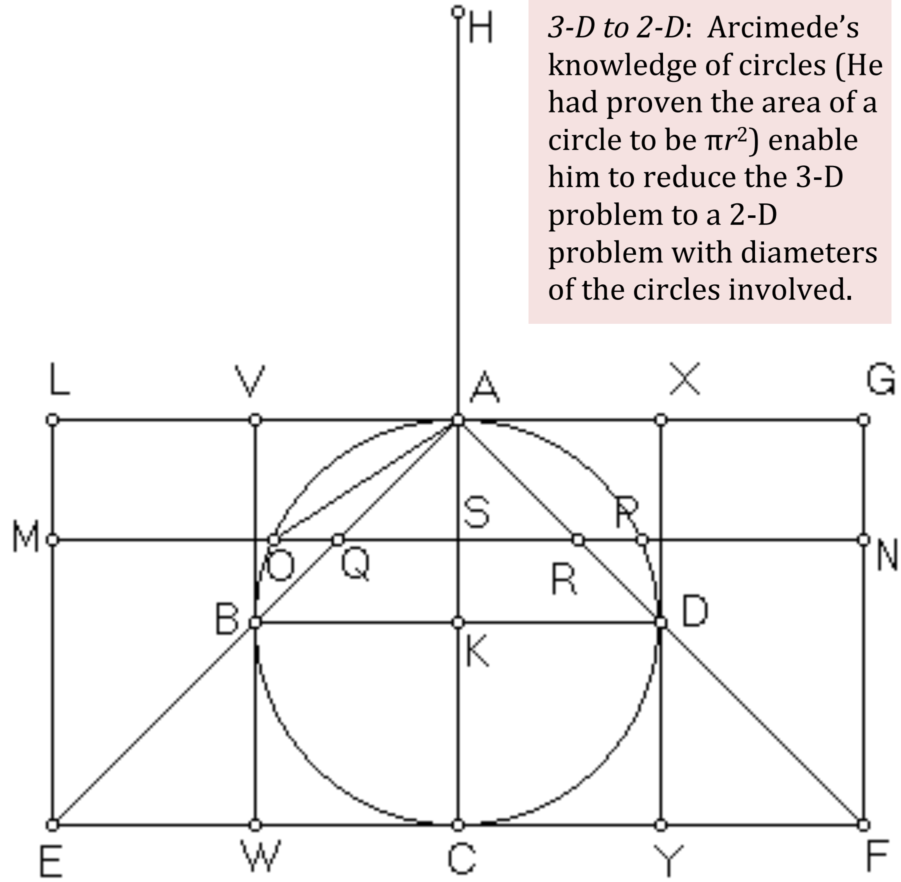
**Archimedes: Ball in a Can**
Adapted from: http://www.cut-the-knot.org/pythagoras/Archimedes.shtml

Archimedes was clearly one of the greatest geniuses our species has produced. And out of all his work, what he requested be put on his tombstone was the relationship he discovered and proved between the volume of a sphere and a circumscribed cylinder.

As an instructive example, I have reproduced below the entire text of Proposition 2, which determines the volume of a sphere. (Proposition 1 determines the area of a segment of a parabola; I have skipped ahead to Proposition 2 because the geometric prerequisites are much more familiar.) Archimedes considered this theorem his greatest achievement, and directed that a schematic representation of it be placed on his tombstone.

*3-D to 2-D*: Arcimede's knowledge of circles (He had proven the area of a circle to be $\pi r^2$) enable him to reduce the 3-D problem to a 2-D problem with diameters of the circles involved.




From The Method of Archimedes (the translation is that of T.L. Heath):

**Proposition 2**.

We can investigate by the [mechanical] method the propositions that

1. Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and
2. the cylinder with base equal to a great circle of the sphere and height equal to the diameter is 1½ times the sphere.

*How to draw it*: with no ability to reproduce figures, a common feature of ancient proofs is a description of how to draw your own!

1. Let ABCD be a great circle of a sphere, and AC, BD diameters at right angles to one another. Let a circle be drawn about BD as diameter and in a plane perpendicular to AC, and on this circle as base let a cone be described with A as vertex. Let the surface of this cone be produced and then cut by a plane through C parallel to its base; the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC, and produce CA to H, making AH equal to CA. Let CH be regarded as the bar of a balance, A being its middle point.

Draw any straight line MN in the plane of the circle ABCD and parallel to BD. Let MN meet the circle in O,P, the diameter AC in S, and the straight lines AE, AF in Q, R respectively. Join AO. Through MN draw a plane at right angles to AC; this plane will cut the cylinder in a circle with diameter MN, the sphere in a circle with diameter OP, and the cone in a circle with diameter QR.

*Notice*: MN could be at any height, so Archimedes is talking about just taking a cross section of his sphere, cylinder and cone.

*Notation*: Where we would write proportions (a\b =c\d), Archimedes writes products (ad=bc).

Now, since MS = AC, and QS = AS,

$$MS \cdot SQ = CA \cdot AS = AO^2 = OS^2 + SQ^2$$

*FYI*: Triangle CAO is a right triangle, similar to triangle OAS.

*Notation*: A:B denotes the ratio of A/B. The underlying idea here is that he knows the volume of a cylinder, so that if he can find how the volume of a sphere compares to a cylinder…

And, since HA = AC,

$$\begin{aligned} HA : AS &= CA : AS \\ &= MS : SQ \\ &= MS^2 : MS \cdot SQ \\ &= MS^2 : (OS^2 + SQ^2), \\ &= MN^2 : (OP^2 + QR^2) \\ &= \text{(circle, diam. MN) : (circle, diam. OP + circle, diam. QR).} \end{aligned}$$

*Notation*: By (circle, diam *x*), Archimedes is referring to the area of the circle. The ancients thought of circles in terms of diameters instead of radii.

That is, HA : AS = (circle in cylinder) : (circle in sphere + circle in cone). Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both latter circles are placed with their centres of gravity at H.

*Notation*: We'd call parallelogram LF by rectangle LGFE.

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF.

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder, the sphere, and the cone, and which make up those solids respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about A with the sphere and the cone together, when both are placed with their centres of gravity at H. Therefore, since K is the centre of gravity of the cylinder,

HA : AK = (cylinder) : (sphere + cone AEF). But HA = 2 ·AK; Therefore cylinder = 2 (sphere + cone AEF).

*FYI*: Now he's considering the special case where S is at K.

Now cylinder = 3 (cone AEF); [Eucl. XII 10] Therefore
Cone AEF = 2 (sphere). But, since EF = 2 ·BD,
Cone AEF = 8 (Cone ABD); Therefore,
sphere = 4 (Cone ABD).

*Euclid XII, Proposition 10*: The volume of a cone is one-third of the cylinder's volume with congruent base and height. He also seems to be using similarity of 3-D figures.

**2.** Through B, D draw VBW, XDY parallel to AC;
and imagine a cylinder which has AC for axis and the circles on VX, WY as diameters for bases.

Then cylinder VY = 2 (cylinder VD)
= 6 (cone ABD)[Eucl XII 10]
= 3/2 (sphere), from above.

Q.E.D.

"From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius."