The Mathematics of Referendum Elections and Separable Preferences

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Sam, age 17, is a bad driver. In fact, due to his recent vehicular mishaps, Sam's family is now in need of not just one, but two new cars. Mom, Dad, and Sam have narrowed the list of potential options down to three choices, and Dad, being committed to democracy in all of its various forms, has decided that a referendum election is the most appropriate mechanism for making the final decision. And so they vote, simultaneously casting ballots on each of three separate questions:

- **Question 1:** Should we purchase the BMW?
- **Question 2:** Should we purchase the Ford?
- **Question 3:** Should we purchase the Kia?

Eager to see democracy at work once again, Dad counts the ballots and reveals the outcome:

- On Question 1: **YES**, we should purchase the BMW!
- On Question 2: **YES**, we should purchase the Ford!
- On Question 3: **YES**, we should purchase the Kia!

Needless to say, this result leaves much to be desired. After all, the family was looking to purchase *two* cars, not three. Clearly, their election has failed to bring them any closer to a decision on which two cars to purchase. In fact, for financial reasons, Mom, Dad, and Sam all agree that buying three cars (the course of action suggested by the election) would be one of the worst decisions, if not *the* worst decision, that the family could make.

So what happened here? Are dimpled chads to blame? Were the ballots somehow corrupted or counted incorrectly? Or is strategic voting the real culprit? Perhaps. But there is another equally plausible explanation that merits consideration. What if each voter cast a yes vote on exactly two of the three questions: Mom voted yes on the BMW and the Ford, Dad voted yes on the BMW and the Kia, and Sam voted yes on the Ford and the Kia? Each question would then pass by a 2:1 margin, without even a hint of fraud, manipulation, or error.

Admittedly, the problem is easy to solve in this simple example. The family is not compelled to abide by the undesirable outcome of their election, and there are plenty of more reasonable ways to approach the decision they face. But what happens when the stakes are higher?

Consider, for instance, the 1990 California general election, where voters were faced with 28 statewide ballot initiatives, three of which pertained to environmental issues and requested between $340 million and $742 million in taxpayer funding via bonds. In their analysis of this election, Brams, Kilgour, and Zwicker [2] argue that "because all three propositions were pro-environement and involved the expenditure of substantial funds, there is good reason to believe that many voters saw them as related." They support this position with empirical data and go on to argue that the overall outcome, which resulted in two of the three related issues passing, was a "dubious 'compromise' choice," and that all three issues passing would have more accurately reflected the wishes of the electorate.

In these and many other other examples, we encounter a fundamental problem with referendum elections. Lacy and Niou [11] provide a nice summary of this problem, stating that in spite of the fact that "the resurrection of direct democracy through referendums is one of the clear trends of democratic politics ... referendums as currently practiced force people to separate their votes on issues that may be linked in their minds." This so-called *sep-*

arability problem is interesting from a practical perspective, but also leads to a number of important theoretical questions, many of which are highly mathematical in nature.

The purpose of this article is to summarize the most recent mathematical contributions to the separability problem and to suggest several directions for further research. We begin by formally defining the notion of *separability* as it pertains to voter preferences in referendum elections. We then show that separable preferences are structurally complex, rare, highly sensitive to small changes, and crucial to the problem of obtaining desirable outcomes in referendum elections.

Separable Preferences

The notion of separability formalizes the idea that a voter's preferences on one or more questions in a referendum election may depend on the known or perceived outcomes of other questions in the election. Preferences that are free from such interdependence are said to be *separable*.

To illustrate, consider again our first example. If Sam, who voted yes on both the Ford and the Kia, had known that the BMW was also going to be selected, then he may have chosen to vote no on one of the other two cars. In fact, doing so would have led to a more desirable outcome, as exactly two cars would have then been chosen. The problem, however, is that Sam had to vote on all three questions at the same time. He was forced by the election mechanism to register three separate votes on three issues that were undoubtedly linked in his mind. Sam's preferences, as well as those of his parents, were *nonseparable*, a fact that directly contributed to the undesirability of the resulting election outcome.

In order to formalize the notion of separability, we must first introduce a way to model voter preferences within a referendum election. For the purposes of this article, we will restrict our attention to elections that consist of a finite number, say n, of yes/no decisions. In this context, an *outcome* is defined to be any point in binary n-space (*i.e.*, the set of all n-tuples of zeros and ones, commonly denoted $\{0,1\}^n$). We may also refer to the outcome of a particular set of j questions (with $1 \leq j \leq n$), in which case we mean a point in a corresponding j-dimensional subspace of $\{0,1\}^n$. We typically think of a 1 in the k^{th} component of an outcome as representing an outcome of YES on question k, whereas a 0 represents an outcome of NO. We then use a linear order on $\{0,1\}^n$ to represent a voter's preferences over all possible outcomes.

Binary preference matrices (see [1]) provide a convenient way of visualizing the sorts of preferences that are of interest to us. In lieu of a formal definition, we consider the following example:

Example 1. Let $Q = \{1,2\}$. The 4×2 matrix below represents a voter's preferences over the four possible outcomes of an election on the two questions in Q:

$$P_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

To interpret these preferences, we adopt the convention that outcome A is preferred to outcome B if and only if A appears above B in P_1. We then note the following:

- The voter's most preferred outcome is YES on the first question and NO on the second question.
- The voter's least preferred outcome is NO on both questions.
- YES is always preferred to NO on the first question (regardless of the outcome on the second question).
- NO is preferred to YES on the second question, provided that the outcome on the first question is YES. If the outcome of the first question is NO, then this preference is reversed.

In the previous example, we noted that the voter's preference on the first question does not depend on the outcome of the second question. Because of this independence, we say that question 1 is *separable*. More precisely, we say that the set $\{1\}$ is separable with respect to P_1. On the other hand, since the voter's preference on the second question does depend on the outcome of the first question, we say that the set $\{2\}$ is *not* separable with respect to P_1. We adopt the convention that the entire question set, Q, and the empty set, $\emptyset$, are separable with respect to any preference matrix. Thus, we can associate with P_1 the following collection of separable sets, called the *character* of P_1 and denoted by $\text{char}(P_1)$:

$$\text{char}(P_1) = \{\emptyset, \{1\}, \{1,2\}\}$$

The next example further demonstrates the idea of separability and also motivates the precise definition we will adopt shortly.

EXAMPLE 2. Consider the following preference matrix for an election on 3 questions:

$$P_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Suppose we fix an outcome of 1 on the third question. This choice induces the following order on the possible outcomes of the first and second questions:

$$11 \succ 01 \succ 10 \succ 00$$

If we instead fix 0 on the third question, a different order is induced:

$$11 \succ 10 \succ 01 \succ 00$$

Since the ordering of the outcomes on the first and second questions depends on which choice we fix for the third question, we say that $\{1, 2\}$ is not separable (with respect to P_2).

The two induced orders from Example 2 correspond uniquely to the following submatrices of P_2:

$$P_2^{[\{3\},1]} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2^{[\{3\},0]} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The notation we use to describe these submatrices reflects the fact that we are fixing an outcome on a certain subset of Q. In general, if P is a preference matrix for an election on Q, S is a subset of Q, and x is an outcome on S, then we use the notation $P^{[S,x]}$ to denote the *submatrix of* P *induced by fixing* x *on* S. The formal definition of separability now follows naturally.

Definition 1. Let P be a preference matrix for an election on Q, and let S be a proper, nonempty subset of Q. We say that S is *separable with respect to* P, or that P is *separable on* S, if for any two outcomes x and y on $Q - S$,

$$P^{[Q-S,x]} = P^{[Q-S,y]}.$$

As noted above, we consider Q and $\emptyset$ to be separable with respect to any preference matrix.

Now that we have defined separability, we can also formally define the character of a preference matrix:

Definition 2. Let P be a preference matrix for an election on Q. The *character of* P, denoted $\text{char}(P)$, is the collection of all subsets of Q that are separable with respect to P; that is,

$$\text{char}(P) = \{S \subseteq Q : P \text{ is separable on } S\}.$$

When $\text{char}(P) = \mathcal{P}(Q)$ (the power set of Q), we say that P is *completely separable*. When $\text{char}(P) = \{\emptyset, Q\}$, we say that P is *completely nonseparable*.

Structural Properties

The notion of separability is important in a variety of disciplines, including economics, political science, and operations research. In these contexts, however, the sets of possible outcomes for each question are typically assumed to be arc-connected subsets of the real numbers. Gorman's Theorem [3] states that under this and a few other fairly innocuous assumptions, the property of separability is preserved by most set operations. In particular, if S and T are separable with respect to a given preference matrix, then $S \cup T$, $S \cap T$, $S - T$, $T - S$, and $S \triangle T$ are all separable as well.

Somewhat surprisingly, the same cannot be said when working within the context of referendum elections. Of course, the sets of binary outcomes inherent to such elections are far from being arc-connected. Nevertheless, it is somewhat counterintuitive to think, for instance, that a voter's preferences on two individual questions may not be influenced by the outcome of other questions in the election, yet when the two questions are viewed together, the voter's preferences on the combination may in fact exhibit such dependence.

The preference matrix P_2 from Example 2 illustrates exactly this phenomenon. In that example, we argued that the set $\{1, 2\}$ is not separable with respect to P_2. However, it is easy to verify that both $\{1\}$ and $\{2\}$ are separable with respect to P_2, each with induced order $1 \succ 0$. Note that this same matrix demonstrates the failure of separability to be preserved by relative complements (since $\{1, 2\} = \{1, 2, 3\} - \{3\}$ and both $\{1, 2, 3\}$ and $\{3\}$ are separable) and symmetric differences, since ($\{1, 2\} = \{1, 3\} \triangle \{2, 3\}$ and both $\{1, 3\}$ and $\{2, 3\}$ are separable).

Interestingly enough, separability is preserved by intersections, not only in this example, but in general [1]. This fact implies that the character of any binary preference matrix must be closed under intersections. The corresponding inverse problem is even more interesting:

Question 1. Let Q be a finite set of questions, and let $\mathcal{C}$ be any collection of subsets of Q that contains both $\emptyset$ and Q and is closed under intersections. Does there exist a preference matrix P such that $\mathrm{char}(P) = \mathcal{C}$?

Question 1 has been investigated for elections with 2, 3, and 4 questions [8]. In both the 2 and 3 question cases, the answer is a definitive *yes*. However, in the 4 question case, an interesting anomaly arises. Up to a naturally defined notion of isomorphism, there are 165 distinct collections of subsets of $Q = \{1, 2, 3, 4\}$ that satisfy the conditions specified in Question 1. Of these, 164 can be viewed as the character of one or more binary preference matrices. This one, however, cannot:

$$\mathcal{C} = \{\emptyset, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{3, 4\}, \{1, 2, 3, 4\}\}$$

It is of course difficult to generalize from one example, and to date, little progress has been made on the problem of classifying all possible characters for elections with 5 or more questions. This lack of progress is due partially to the large number of closed characters (14480 in the 5 question case) and the even larger number of preference matrices ($2^n!$ for an n question election) that must be considered.

Combinatorial Questions

Although preference separability is a fairly standard assumption throughout much of the literature in economics and social choice, the validity of this assumption in the context of referendum elections has been brought into question not only by empirical data from recent elections, but also by a body of mathematical research that shows separability to be exceedingly rare.

The task of counting separable preference matrices is a difficult one, and exact results are known only for elections with 2, 3, and 4 questions [1]. Other related combinatorial questions, however, have been more definitively answered. For instance, it has been shown [1] that every separable preference matrix must be *symmetric*, meaning that the k^{th} row from the top is the bitwise complement of the k^{th} row from the bottom. It is easy to show that the number of symmetric preference matrices for an election with n questions is

$$2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2 \quad = \quad 2^{2^{n-1}} \cdot 2^{n-1}!$$

Thus, the probability of a randomly selected preference matrix being separable is bounded above by

$$\frac{2^n \cdot (2^n - 2) \cdot (2^n - 4) \cdots 4 \cdot 2}{2^n \cdot (2^n - 1) \cdot (2^n - 2) \cdots 2 \cdot 1} = \frac{1}{(2^n - 1) \cdot (2^n - 3) \cdot (2^n - 5) \cdots 3 \cdot 1},$$

which approaches 0 (very quickly!) as $n \to \infty$. Even stronger asymptotic results have been obtained by considering *preseparable* and *strongly preseparable* preferences, which can be counted using the Catalan numbers and central binomial coefficients [4]. Finally, recent results [8] establish that the probability of *complete nonseparability* approaches 1 as $n \to \infty$. In particular, the probability of a preference matrix being separable on at least one nontrivial subset of Q (that is, a subset other than $\emptyset$ or Q) is bounded above by

$$\frac{1}{2^{2^{n-1}-n-1}}.$$

From a numerical perspective, this result implies that for elections with at least 5 questions, more than 99.9% of all possible preference matrices exhibit no separability whatsoever; that is, the preferences represented are as interdependent as possible.

Permutations of Separable Preference Orders

In light of the rarity of separable preferences, it is natural to ask which permutations, when applied to the rows of a separable preference matrix, will necessarily produce another separable matrix. To formally answer this question, note that for an election with n questions, the symmetric group of degree 2^n, denoted S_{2^n}, acts in a natural way on the rows of any preference matrix. A permutation $\sigma \in S_{2^n}$ is said to *preserve separability* if the image of each separable preference matrix under σ is also separable. A analogous definition applies to permutations that *preserve symmetry.*

It has been shown [5] that the sets of symmetry-preserving and separability-preserving permutations are each subgroups of S_{2^n}. Furthermore, for an election with n questions, the group of symmetry-preserving permutations is isomorphic to the group of symmetries of a 2^{n-1} dimensional hypercube, which is exactly the wreath product

$$Z_2 \wr S_{2^{n-1}} = [Z_2]^{2^{n-1}} \rtimes S_{2^{n-1}}.$$

More interesting is the fact that for every $n \geq 4$, the group of separability preserving permutations is isomorphic to the Klein 4-group, generated by

$$\sigma_1 = (2^{n-1}, 2^{n-1} + 1)$$

and

$$\sigma_2 = (1, 2^n)(2, 2^n - 1) \cdots (2^{n-1}, 2^{n-1} + 1).$$

It is significant that, for elections with at least four questions, only four permutations preserve the separability of all separable preference matrices. From a practical perspective, this result indicates that separable preferences are highly sensitive to small changes. With only a few exceptions, changes as simple as the transposition of two adjacent rows have the potential to introduce significant levels of nonseparability into a voter's preferences. To illustrate, note that two such transpositions, applied consecutively, are sufficient to transform a completely separable matrix into one that is completely nonseparable:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(23)} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(14)} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Practical Implications of Separability

In our first example, we suggested that it is possible for a referendum election to result in an outcome that is the least preferred choice of every voter in the election. To explicitly construct such an example, one need only choose an electorate that consists of equal numbers of voters having each of the following preference matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Note that none of these preference matrices are separable. This fact is significant, although it should be noted that highly undesirable results can occur even when all but one of the voters (in an arbitrarily large electorate) have separable preferences [6, 11].

Some research indicates that election outcomes can be improved significantly by forcing all voters to have separable preferences [9, 11, 12]. But such a restriction seems unrealistic, especially in light of the rarity of separable preferences. It is perhaps more reasonable to consider whether any incremental gains can be realized by forcing some, but not all, voters to have separable preferences. In other words, is some separability better than no separability, and is there a correlation between the relative degree of separability present in an electorate and the desirability of the resulting election outcomes?

Recent computer simulations suggest an affirmative answer to this question [7]. In particular, the data obtained from these simulations reveal a linear relationship between the percentage of voters in an election that have separable preferences and the desirability of the resulting outcome, as quantified by a measure called the *aggregate score index* (ASI). Figure 1 illustrates this relationship for the 3 question case. Note that, in this example, 0 is the lowest possible score (ASI) and 7 is the highest.

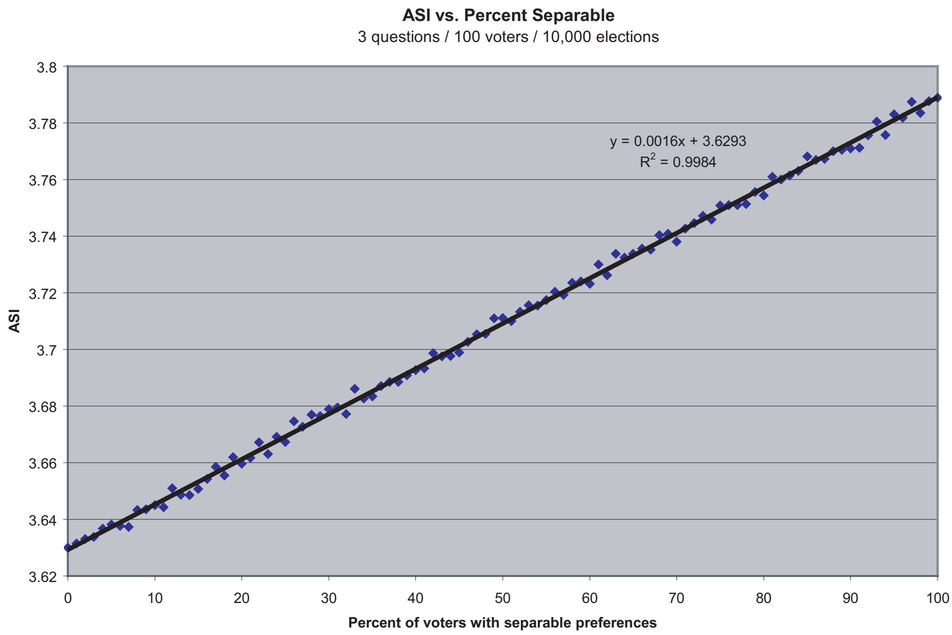

Figure 1: ASI vs. % Separable – Data from 10,000 simulated elections

The relatively small difference in score (3.63 to 3.79) between electorates with no separable preferences and those with all separable preferences might suggest that the effect of separability is minimal. However, if one considers not only the numerical score of each election result, but also how that result compares to other potential outcomes, a much stronger effect emerges, as shown in Table 1. (The data in this table is for 10,000 simulated elections, each with 3 questions and 100 voters.)

Note that when all voters have separable preferences, the first or second best outcome is selected in over 88% of the simulated elections. In contrast, when no voters have separable preferences, the first or second best outcome is selected less than 43% of the time. Equally striking is the fact that the worst or second worst outcome is selected in over 11% of all elections in

% Sep	Percent of elections resulting in the i^{th} best outcome, where $i = \ldots$							
↓	1	2	3	4	5	6	7	8
0	25.33	17.53	13.94	12.72	10.26	8.55	7.08	4.59
10	27.78	19.23	14.33	11.84	9.19	8.02	5.49	4.12
20	29.43	19.95	14.99	12.17	8.22	7.2	4.72	3.32
30	31.61	21.88	13.95	11.02	8.57	6.46	4.16	2.35
40	33.64	21.73	14.25	11.49	7.43	5.71	3.69	2.06
50	37.84	23.6	13.25	10.63	5.78	4.05	2.98	1.87
60	40.43	23.83	13.93	9.52	5.31	3.46	2.13	1.39
70	44.52	24.8	13.1	8.59	4.09	2.63	1.35	0.92
80	48.71	26.11	11.67	7.22	2.77	1.72	1.15	0.65
90	55.07	25.35	9.57	5.67	1.86	1.46	0.56	0.46
100	63.72	24.39	5.7	4.28	0.76	0.7	0.26	0.19

Table 1: The effect of separability on selecting near optimal outcomes

which voters have nonseparable preferences, as compared to less than 0.5% of elections in which voters' preferences are separable.

Finally, while one might expect highly paradoxical election outcomes (for instance, the aforementioned example in which the winning outcome was the last choice of every voter) to be quite rare, they are in fact common enough to appear randomly in simulations with as few as 100 elections. Such outcomes become increasingly rare as more and more voters are forced to have separable preferences.

Questions for Further Research

In the last decade, researchers' understanding of the separability problem has been significantly enhanced by the insights of mathematicians and mathematical social scientists. In particular, we have learned that voter preferences in referendum elections often contain structurally complex interdependencies, and that these interdependencies can substantially impact the desirability of election outcomes. Separable preferences, which are the most desirable from an interdependence standpoint, are rare and highly sensitive to small changes.

Of course, all of these results depend on the particular model used to represent voter preferences, and on the system used to aggregate these preferences and arrive at a collective decision. Other models of voter preference may produce different insights into the structure, prevalence, and importance of separable and nonseparable preferences. Furthermore, there are indications that alternative voting and aggregation methods (such as sequential voting and setwise aggregation) may hold some promise for obtaining practical solutions to the separability problem. Several recent discoveries suggest this (see [7], [10], [11]), although more work in this area is certainly needed.

From a theoretical perspective, many interesting combinatorial questions still remain open. For instance, no serious attempts have been made to enumerate separable preference matrices for elections with 5 or more questions, nor has any closed formula for counting such matrices been discovered. A good first step toward solving this and other related combinatorial problems would be to characterize the set of permutations that preserve the separability of one *particular* separable preference matrix (as compared to those that preserve the separability of *all* separable matrices). Such sets are never groups (see [5]), however, and so they may be more difficult to describe.

Finally, some preliminary investigations suggest that *complete* separability may not be necessary or even desirable for ensuring optimal or near optimal election outcomes. There is reason to believe that a class of *monoseparable* preferences (those for which each individual question is separable but for which larger sets of questions may not be) may be of most use in this regard. Thus, further research into monoseparable preferences seems likely to shed even more light on the separability problem.

All of these areas of investigation have the potential to yield rich mathematical results, some of which may have connections or applications to other fields of mathematics. They also provide opportunities for mathematicians to contribute to the ongoing debate about how to best implement the ideals of democracy. There may not be a simple solution to the separability problem, but further mathematical research holds the potential to help minimize its negative effects on the outcomes of referendum elections.

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