Research Interests in Coalgebras and Hopf Algebras

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Let K be a commutative ring, L a commutative K-algebra. If H is a left K-module, we can form the L-module $L \otimes H$. A natural question to ask in this context is which K-modules H' satisfy $L \otimes H' \cong L \otimes H$ as L-modules.

We can ask the same question for algebras, coalgebras, and Hopf algebras. Specifically,

Question 1. Given K, L as above, and a K-object H, what are all the K-objects H' such that $L \otimes H \cong L \otimes H'$ as L-objects?

Such K-objects H' are called L-forms of H.

Another interesting question arises when we relax the assumption that L be fixed.

Question 2. Given a K-object H, what are the K-objects which are L-forms of H for some suitable commutative K-algebra L?

For instance, [HP86] defines a form of H to be an L-form of H' for some faithfully flat commutative K-algebra L. We can define forms in other contexts, as long as we specify what is meant by a "suitable commutative K-algebra".

Question 2 was addressed in [HP86]. The authors studied Hopf algebra forms of group rings KG. They found a correspondence between Galois extensions $K \subseteq L$ of the base ring with Galois group $F = \mathrm{Aut}(G)$ and Hopf algebra forms of KG in the case when G is finitely generated, and F is finite. The Hopf algebra form was derived from the invariants of the action of KF on LG.

Question 1 was addressed in [Par89] for group algebras. Given $K \subseteq L$ a KF^*-Galois extension, and given a group action of F on G, the author constructed a certain twisted group ring $K_\Gamma G$. He showed that $K_\Gamma G$ is an L-form of KG, and that in the case when L is connected, all L-forms of KG are twisted group rings for some action of F on G.

In my work, the latter point of view is used to construct forms of Hopf algebras. The results appear in [Par01]. As with Pareigis' construction, I took my extension fields to be "Galois" in a suitable sense.

Definition 3. Let W be a Hopf algebra, and suppose $B \subseteq A$ is an extension of right W-comodule algebras (i.e. there is an algebra map $\rho : A \to A \otimes W$ satisfying conditions dual to module actions). This extension is said to be right W-Galois if

(i) $B = A^{coW} = \{a \in A : \rho(a) = a \otimes 1\}$

(ii) The map $\beta : A \otimes_B A \to A \otimes_K W$ given by $\beta(a \otimes b) = (a \otimes 1)\rho(b) = \sum ab_0 \otimes b_1$ is bijective.

If W is finite-dimensional, we can define a W^*-Galois extension in terms of the action of W on A.

Theorem 4. [KT81, Ulb82] Let W be a finite-dimensional Hopf algebra, A a left W-module algebra. Let $A^W = \{a \in A : w \cdot a = \varepsilon(w)a \text{ for all } w \in W\}$. The following are equivalent:

(*i*) $A^W \subseteq A$ is right W^*-Galois.

(*ii*) The map $\pi : A \otimes W \to End(A_{A^W})$ given by $\pi(a \otimes w)(b) = a(w \cdot b)$ is bijective, and A is a finitely generated projective right A^W-module.

(*iii*) If $0 \neq t \in \int_W^l$, then the map $[,] : A \otimes_{A^W} A \to A\#W$ given by $[a, b] = atb$ is surjective.

In particular, if the above is a finite extension of fields $K \subseteq L$, then (*ii*) implies that $|L : K| = \dim_K(W)$.

My goal was to obtain L-forms of H from actions of W on $L \otimes H$. But only certain actions give rise to such forms.

Definition 5. Let W be a Hopf algebra, and suppose H is a W-module algebra that is also a Hopf algebra. We say that the W-module structure on H is a commuting action if it commutes with the comultiplication, counit, and antipode of H.

Theorem 6. [Par01, Thm. 4.1] Suppose that $K \subseteq L$ is a W^*-Galois extension for W a finite-dimensional, semisimple Hopf algebra. Let H be any K-Hopf algebra, and suppose that we have a commuting action of W on $L \otimes H$ extending a Galois action on L. Then

(*i*) $H' = [L \otimes H]^W$ is a K-Hopf algebra, with structure inherited from the L-Hopf algebra structure of $L \otimes H$.

(*ii*) $L \otimes H' \cong L \otimes H$ as L-Hopf algebras, with the isomorphism given by $l \otimes \alpha \mapsto l\alpha$.

(*iii*) If F is another Hopf algebra form of H, then there is a suitable commuting action of W on $L \otimes H$ such that $F = [L \otimes H]^W$

It is natural to ask whether or not all commuting actions of W on $L \otimes H$ need to be considered to obtain all the L-forms of H up to isomorphism. While some examples suggest that we can restrict our attention to commuting actions on $L \otimes H$ which restrict to actions on H, this question is still open.

I have also computed forms for coalgebras, and found a necessary and sufficient condition for a coalgebra C to be a form for a grouplike coalgebra with respect to fields (i.e. C is an L-form for KG for some field extension $K \subseteq L$).

Theorem 7. [Par01, Thm. 3.1] Let H be a K-coalgebra. Then the following are equivalent.

(*i*) H is a form of a grouplike coalgebra with respect to fields.

(*ii*) H is cocommutative and cosemisimple with separable coradical.

A coalgebra is said to have separable coradical if, for each simple subcoalgebra D, it's dual D^* is a separable K-algebra. In particular, if D is cocommutative, then D^* is a separable field extension.

Given a coalgebra C, we can construct the coradical filtration as follows. We let C_0 (the coradical) be the sum of all simple subcoalgebras of C. For each $n \geq 1$, we then define $C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$. See [Mon93, 5.2] for basic properties of this filtration. We get the following corollary of Theorem 7.

Corollary 8. [Par01, Cor. 3.4] Let H be a cocommutative coalgebra, and suppose that $K \subseteq L$ is such that $L \otimes H$ is pointed (e.g. $L = \bar{K}$).
(i) $[L \otimes H]_n \subseteq L \otimes H_n$ for all $n \geq 0$.
(ii) Equality holds for all $n \geq 0$ if and only if H has separable coradical.

This result led me to study the effect that extension of the base field has on the coradical filtration. As Corollary 8(i) shows, it generally "breaks up" the coradical filtration into smaller pieces. One would guess that at some point, it could be broken down no further. Theorem 7 suggests that this occurs when we extend the field to one which contains the normal closure over the base field of all simple subcoalgebras. We get the following analogue of Theorem 7 .

Theorem 9. [Par02] Let C be a cocommutative coalgebra over K, and let $K \subseteq E$ be a field extension. Then $E \otimes C$ is pointed if and only if, for each simple subcoalgebra $D \subseteq C$, E contains the normal closure of the field D^* over K.

In the process of gaining information about the first level of the coradical filtration, it is clear that we need a result of Taft and Wilson (see [TW74] or a strengthened version in [Mon93, 5.4.1]).

Theorem 10. Let C be a pointed coalgebra, with $G = G(C)$. For each $g, h \in G$, let $P'_{g,h}(C)$ be any vector space complement of $K(g - h)$ in $P_{g,h}(C)$. Then
(i) $C_1 = KG \oplus (\oplus_{g,h \in G} P'_{g,h}(C))$
(ii) for any $n \geq 1$ and $c \in C_n$,

$$c = \sum_{g,h \in G} c_{g,h}, \text{ where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w$$

for some $w \in C_{n-1} \otimes C_{n-1}$.

I was able to generalize (i) to C_n.

Theorem 11. [Par02] For each $g, h \in G$, let $\bar{P}^{(n)}_{g,h}(C)$ be a vector space complement of $P^{(n)}_{g,h}(C) \cap C_{n-1}$ in $P^{(n)}_{g,h}(C)$. Then $C_n = C_{n-1} \oplus (\oplus_{g,h \in G} \bar{P}^{(n)}_{g,h}(C))$.

Theorem 12. [Par02] Let C be a cocommutative coalgebra. For each $g \in G$, let $\bar{P}^{(n)}_{g,g}(C)$ be a vector space complement of C_{n-2} in $P^{(n)}_{g,g}(C)$. Then $C_n = C_{n-2} \oplus (\oplus_{g \in G} \bar{P}^{(n)}_{g,g}(C))$.

We put these results to our advantage to obtain

Theorem 13. [Par02] Let $K \subseteq L$ be a finite field extension, and let E be a field containing the normal closure of L over K. Then $(E \otimes L^*)_n = \sum_{g \in G(E \otimes L^*)} C_g$, where each $C_g \cong (E \otimes_{L_s} H)_n$.

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