# Research Interests in Graph Theory

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My interests in graph theory began in studying two-path convexity in multipartite tournaments. Lately, my interests have broadened to multidesigns and a solitaire game on graphs called *Lights Out*.

## 1 Convexity in Multipartite Tournaments

Convexity has been studied in many contexts. Examples include Euclidean $n$-space, partially ordered sets, graphs, and directed graphs. These contexts have been generalized to the concept of a *convexity space*, which is also known as a *convex structure* or *aligned space*. Whatever it is called, it is a pair $\mathcal{C} = (V, C)$, where $V$ is a set and $C$ is a family of subsets of $V$ such that $\emptyset, V \in C$ and $C$ is closed under arbitrary intersections and nested unions. We call the sets in $C$ the *convex sets of* $\mathcal{C}$. Given $S \subseteq V$, the *convex hull* $C(S)$ is the smallest convex set containing $S$.

Given a graph or directed graph, one can impose a convex structure in a natural way.

**Definition.** Let $T = (V, E)$ be a (directed) graph with vertex set $V$ and edge (or arc) set $E$, and let $\mathcal{P}$ be a collection of (directed) paths in $T$. A subset $S \subseteq T$ of vertices is called $\mathcal{P}$-*convex* if, for every $v, w \in S$, any (directed) path in $\mathcal{P}$ that originates at $v$ and ends at $w$ can involve only vertices in $S$.

In the case where $\mathcal{P}$ is the set of *geodesics* (i.e. shortest paths between two vertices), we get *geodesic convexity*, which is the most commonly studied form of convexity (see [CZ99], [CFZ02], [ES85], and [HN81]). When $\mathcal{P}$ is the set of all *induced paths* (i.e. paths without chords), we get *induced path convexity* (see [Duc88]). When $\mathcal{P}$ consists of all 2-paths, we get *two-path convexity* (see [EFHM72], [Var76]), and [Moo72]). Other forms of convexity have been studied as well, such as path convexity, where $\mathcal{P}$ is the set of all paths (see [Nie81] and [Pfa71]) and triangle path convexity (see [CM99]), where $\mathcal{P}$ is the set of all paths that are chordless except perhaps for chords that form triangles.

Among the most studied convexity invariants are the Helly, Radon, and Caratheodory numbers (see [JN84], [Pol95], and [CM99]). These are based on notions of independence.

**Definition.** Let $\mathcal{C} = (V, C)$ be a convexity space, and let $F \subseteq V$.

1. We say $F$ is $H$-*independent* if $\bigcap_{p \in F} C(F - \{p\}) = \emptyset$. The *Helly number* $h(\mathcal{C})$ is the size of a largest $H$-independent set.
2. We say $F$ is $R$-*independent* if $F$ does not have a *Radon partition.* That is, there is no partition $F = A \cup B$ with $C(A) \cap C(B) \neq \emptyset$. The Radon number $r(\mathcal{C})$ is the size of a largest $R$-independent set.
3. We say $F$ is $C$-*independent* if $C(F) \nsubseteq \bigcup_{a \in F} C(F - \{a\})$. The *Caratheodory number* $c(\mathcal{C})$ is the size of a largest $C$-independent set.

4. We say $F$ is *convexly independent* if, for each $p \in F$, we have $p \notin C(F - \{p\})$. The rank $d(\mathcal{C})$ is the size of a largest convexly independent set.

5. We say $F$ is a hull set if $C(F) = V$. The *hull number* is the size of a smallest hull set.

The above definition of the Radon number is not universally accepted. Often, it is defined as the smallest number $r$ in which every set of size $r$ is $R$-dependent. This is one larger than in our definition. The Levi inequality is $h(\mathcal{C}) \leq r(\mathcal{C}) \leq d(\mathcal{C})$ (see, e.g. [vdV93, p. 169, p. 227]).

Our interest is in two-path convexity when $T$ is a multipartite tournament. There have been three basic problems we have been pursuing.

1. Given a class of multipartite tournaments with $n$ vertices, which have maximum rank/Helly number/Radon number/Caratheodory number/Hull number?

2. Under what circumstances do we have $h(T) = r(T) = d(T)$?

3. What is the relationship between rank and the cycle structure of the multipartite tournament?

In [PWW08], we determine, given $|V|$, the maximum rank, Helly number, Radon number, Caratheodory number, Hull number, and rank for general multipartite tournaments and clone-free multipartite tournaments (i.e. multipartite tournaments in which no two vertices have the same inset and outset). We then classify the multipartite tournaments and clone-free multipartite tournaments of maximum Helly number, Radon number, Hull number, and rank.

In [PWW09], we study convexly independent subsets of $V$ and prove a structural theorem for clone-free multipartite tournaments. We use this result to prove that $h(T) = r(T) = d(T)$ when $T$ is a clone-free bipartite tournament. Given a convexly independent set, we determine conditions equivalent to $H$-independence and $R$-independence. We use this to show that $h(T) = r(T)$ for $T$ a clone-free multipartite tournament, except perhaps when $h(T) = 2$ and $r(T) = 3$.

In [PWW06], we find tight upper bounds for the Helly number, Radon number, and rank for regular clone-free bipartite tournaments. We also find tight upper bounds for rank in regular clone-free tripartite tournaments and upper bounds for rank in clone-free regular $p$-partite tournaments, $p \geq 4$.

In [PWW], we study bipartite tournaments of rank 2 and the relationship between bipartite tournaments of minimum rank and bipartite tournaments with many 4-cycles.

In [ADEP05], we look at convex subsets from a lattice-theoretic point of view. The convex subsets of a convexity space form a lattice, where the meet and join are given by $A \wedge B = A \cap B$ and $A \vee B = C(A \cup B)$. We study the problem of recovering information about a multipartite tournament from its lattice of convex subsets. We determine, in most cases, when two vertices are clones (i.e. when they dominate and are dominated by the same vertices). We also determine, in most cases, when a multipartite tournament is bipartite, and what the possibilities are in the ambiguous case. Finally, we proved an existence result for lattices satisfying what we called the *anti-bipartite condition*.

## 2 Multidesigns of Complete Graphs

Another endeavor has been my work with Drs. Atif Abueida, Mike Daven, Wiebke Diestelkamp, and Stephanie Edwards on multidesigns. This work appears in [ADD$^+$].

Consider $K_6$, the complete (undirected) graph on six vertices (i.e. a graph with six vertices and with precisely one edge between each vertex). We decompose the edges of $K_6$ into three subgraphs such that no two sets of edges form isomorphic graphs and none leave a vertex without an edge. This is called a *graph-triple of order 6.* Given a graph-triple $T$ of order 6, we seek to find all complete graphs $K_n$ whose edges can be decomposed into (perhaps multiple) copies of the graphs in $T$. This is called a $T$*-multidecomposition* of $K_n$.

We also looked at $T$*-multipackings* and $T$*-multicoverings.* *Multipackings* differ from multidecompositions in that they may not contain all the edges, but get as close as possible. *Multicoverings* contain all edges, but have a minimum number of edges that are used twice. A *Multidesign* is a multidecomposition or an optimal multipacking or multicovering. There are 131 graph-triples of order 6, and we determined all multidesigns for 37 of these.

## 3 Lights Out and Domination Theory

More recently, I worked with undergraduates Alexander Giffen Trochelman on the mathematics of a solitaire game on graphs called *Lights Out*, which has deep connections to domination theory. Our work will appear in [GP]

Let $G$ be a graph with vertex set $V$ and edge set $E$. The classical problem in domination theory is to find a set $S \subseteq V$of minimum size such that every vertex in $V$ is either in $S$ or adjacent to some vertex in $S$. Such a set $S$ is called a *dominating set.* Another way of stating this is the following. For $v \in V$, let $N[v] = \{v\} \cup \{w \in V : vw \in E\}$. Then $S$ is a dominating set if $N[v] \cap S \neq \emptyset$ (equivalently $|N[v] \cap S| \neq 0$) for every $v \in V$.

Suppose we label each vertex in $V$ as either odd or even. We say that $S$ is a *parity dominating set* for this labeling if $|N[v] \cap S|$ is odd when $v$ is odd and even when $v$ is even. This problem was studied in [AS96], [ACS98], and [ASZ02]. In particular, they studied graphs that have a parity dominating set for all possible labelings of $V$. These are called *all-parity realizable (APR) graphs.*

Now consider the following "Lights Out" game on $G$. As above, we begin with a labeling of the vertices. This time the labeling of a vertex is either "on" or "off". We play the game by toggling a vertex. Each time a vertex is toggled, it reverses the labeling of both the vertex toggled and the adjacent vertices. We win the game when each vertex is labeled "off". Alex and I studyied graphs in which the game can be won regardless of the initial labeling. We call these *always winnable (AW) graphs.*

Both of these problems are equivalent. If we write $V = \{v_1, v_2, \cdots, v_n\}$, consider the $n \times n$ matrix $N = [n_{ij}]$, where

$$n_{ij} = \begin{cases} 1, & i = j \text{ or } v_i v_j \in E \\ 0, & \text{otherwise} \end{cases}$$

If $S \subseteq V$, let $\mathbf{x} \in \mathbb{Z}_2^n$ with $\mathbf{x}[i] = 0$ if $v_i \notin S$ and $\mathbf{x}[i] = 1$ if $v_i \in S$. With a little thought, one can see that $\sum_{j=1}^{n} n_{ij}\mathbf{x}[j]$ is the number of vertices (modulo 2) that are adjacent to $v_i$.

If we let $\mathbf{b} \in \mathbb{Z}_2^n$ with $\mathbf{b}[i] = 0$ when $v_i$ is even and $\mathbf{b}[i] = 1$ if $v_i$ is odd, then $S$ is a parity dominating set if and only if

$$\sum_{j=1}^{n} n_{ij}\mathbf{x}[j] = \mathbf{b}[i] \text{ for all } i \;\Leftrightarrow N\mathbf{x} = \mathbf{b}$$

Now suppose we are playing the Lights Out game, and let $S \subseteq V$ be the set of vertices that are toggled, and define $\mathbf{x}$ as before. Then $\sum_{j=1}^{n} n_{ij}\mathbf{x}[j]$ is the number of times either $v_i$ or a vertex adjacent to $v_i$ is toggled. Define $\mathbf{b} \in \mathbb{Z}_2^n$ by $\mathbf{b}[i] = 0$ if $v_i$ is initially "off" and $\mathbf{b}[i] = 1$ if $v_i$ is initially "on". Then toggling the vertices in $S$ wins the Lights Out game if and only if $N\mathbf{x} = \mathbf{b}$. We get the following.

**Theorem.** Let $G$ be a graph whose vertices are labeled by $\mathbf{Z}_2$. There exists a parity dominating set for this labeling if and only if the Lights Out game can be won.

Alex and I generalized the game to consider vertices labeled with elements of $\mathbb{Z}_k$, $k \geq 2$. We classified all *AW* paths and cycles, and devised a constructive method for generating all caterpillar graphs that are not *AW* in the case $k = p^n$, $p$ prime.

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