# On Generalizing the "Lights Out" Game and a Generalization of Parity Domination

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May 25, 2009

MR Subject Classifications: 05C78, 05C15, 91A43,
Keywords: Lights Out, parity domination

**Abstract**

The Lights Out game on a graph $G$ is played as follows. Begin with a (not necessarily proper) coloring of $V(G)$ with elements of $\mathbb{Z}_2$. When a vertex is toggled, that vertex and all adjacent vertices change their colors from 0 to 1 or vice-versa. The game is won when all vertices have color 0. The winnability of this game is related to the existence of a parity dominating set. We generalize this game to $\mathbb{Z}_k$, $k \geq 2$, and use this to define a generalization of parity dominating sets. We determine all paths, cycles, and complete bipartite graphs in which the game over $\mathbb{Z}_k$ can be won regardless of the initial coloring, and we determine a constructive method for creating all caterpillar graphs in which the Lights Out game cannot always be won.

## 1 Introduction

The game Lights Out was originally a handheld game by Tiger Electronics. This game has since been generalized to graphs as follows. Let $G$ be a graph with a (not necessarily proper) vertex coloring by the set $\mathbb{Z}_2 = \{0, 1\}$. When a vertex is toggled, that vertex and all of its neighbors change colors (from 0 to 1 or vice-versa). The game is won when all vertices have the color 0.

Strategies for winning this game (when victory is possible) and some variations of the game have been studied in [AF98], [Aua00], [Pel87], [Sto89], and [Sut89]. The Lights Out game has connections with domination theory, specifically parity domination, which has been explored by Amin, Clark, Slater, and Zhang (see [AS92], [AS96], [ACS98], and [ASZ02]). J. Goldwasser and W. Klostermeyer were the first to discover the connection

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between Lights Out and parity domination (see [GK97] and [GKT97]). In particular, they proved that the existence of a parity dominating set is equivalent to whether a corresponding game of Lights Out can be won.

In Section 2, we generalize the game of Lights Out to an arbitrary set of colors $\mathcal{C}$, where the result of toggling a vertex is determined by a function $T : \mathcal{C} \to \mathcal{C}$. We focus on the case where $T$ is a permutation and reduce this problem to the case where $T$ is a cycle.

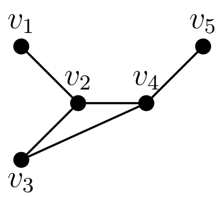
In Section 3, we recast the task of winning Lights Out on a graph whose vertices are labeled by $\mathbb{Z}_k$ as a solution to a linear system of equations over $\mathbb{Z}_k$. This will illustrate the connection between the Lights Out game over $\mathbb{Z}_2$ and parity domination and will allow us to use our generalized Lights Out game to generalize parity domination.

In Section 4, we characterize the labelings for paths, cycles, and complete bipartite graphs in which the Lights Out game over $\mathbb{Z}_k$ can be won. We use these results to determine the paths, cycles, and complete bipartite graphs in which the Lights Out game can be won regardless of the initial labeling. In Section 5, we generalize a result of A. Amin and P. Slater on the construction of caterpillar graphs in which the Lights Out game cannot always be won.

## 2 Generalized Lights Out

To play Lights Out, one needs to know the graph, the colors, and what the "off" color is. In addition, one needs to know the rule used to change the colors of each toggled vertex and its neighbors. Let $G$ be a graph with vertex coloring $\pi : V(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set, and let $0 \in \mathcal{C}$ be designated as the *off* color. We then define a *toggling function* $T : \mathcal{C} \to \mathcal{C}$ so that if $v \in V(G)$ is toggled, the resulting coloring is $\pi'$ with $\pi'(w) = T(\pi(w))$ if $w = v$ or $wv \in E(G)$, and $\pi'(w) = \pi(w)$ otherwise. We define the game to be won when the coloring is $\pi_0$, where $\pi_0(v) = 0$ for all $v \in V(G)$. In the standard Lights Out game, $\mathcal{C} = \mathbb{Z}_2$, the off color is 0, and the toggling function is $T(c) = c + 1$.

**Example 2.1.** Consider the following graph:




Let $\mathcal{C} = \mathbb{Z}_5$ with off color 0, and define the toggle function $T : \mathcal{C} \to \mathcal{C}$ by $T(0) = 2$, $T(1) = 0$, $T(2) = 3$, $T(3) = 1$, $T(4) = 3$. Let the initial coloring be $\pi(v_1) = 2$, $\pi(v_2) = 4$, $\pi(v_3) = 1$, $\pi(v_4) = 0$, and $\pi(v_5) = 2$. If we toggle $v_2$ once, we change the color of $v_1$ to $T(2) = 3$, $v_2$ to $T(4) = 3$, $v_3$ to $T(1) = 0$, and $v_4$ to $T(0) = 2$. We then toggle $v_1$ twice, giving $v_1$ and $v_2$ the color $T(T(3)) = 0$. After toggling $v_5$ thrice, all vertices have color 0, and the game is won.

Now suppose the toggling function $T$ is a permutation, so we can write $T$ as a product of disjoint cycles. If $T(0) = 0$, the game can be won if and only if all vertices are initially colored 0. Otherwise, let $\sigma$ be the cycle in $T$ with $\sigma(0) \neq 0$. If any vertex of the graph has a color that is fixed by $\sigma$, then the game cannot be won. If the vertices are colored only by colors that are not fixed by $\sigma$, then the other disjoint cycles have no effect on the game. This gives us the following.

**Proposition 2.2.** Let $G$ be a graph whose vertices are colored by $\mathcal{C}$, and whose toggling function is a permutation $T = \sigma_1\sigma_2\cdots\sigma_m$, where the $\sigma_i$'s are disjoint cycles and $\sigma_1(0) \neq 0$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the set of colors that are not fixed by $\sigma_1$. Then the Lights Out game can be won if and only if

1. All vertices are colored by elements of $\mathcal{C}'$ and
2. The Lights Out game can be won with toggling function $\sigma_1$.

Thus, the question of whether a Lights Out game can be won when the toggling function is a permutation can be reduced to the case where the toggling function is a cycle. If the cycle $T$ has order $k$, we identify $T^c(0)$ with $c \in \mathbb{Z}_k$, and we can thus let $\mathcal{C} = \mathbb{Z}_k$ with toggling function $T(c) = c + 1$. The traditional Lights Out game operates this way with $k = 2$. Note that we can consider $\pi : V(G) \to \mathbb{Z}_k$ a labeling of $V(G)$.

# 3 Matrix Methods and Parity Domination

As before, let $G$ be a graph with labeling $\pi : V(G) \to \mathbb{Z}_k$ and toggling function $T(c) = c + 1$. Let $V(G) = \{v_1, \ldots, v_n\}$, with $\pi(v_i) = b_i$. In this section, we address the question of whether, given this initial labeling, the Lights Out game can be won.

We proceed as in [AF98]. One can easily check that the order in which the vertices are toggled has no impact on the resulting labeling. All that matters is how many times each vertex is toggled. Let $x_i$ be the number of times that $v_i$ is toggled, and let $\mathbf{x}$ be the $n$-dimensional vector with $\mathbf{x}[i] = -x_i$. Similarly, let $\mathbf{b}$ be the $n$-dimensional vector with $\mathbf{b}[i] = b_i$.

Let $A$ be the adjacency matrix of $G$. Then $N = A + I_n$ is the *neighborhood matrix* or *augmented adjacency matrix* of $G$. Notice that the label of $v_i$ is increased by one each time either $v_i$ or a neighbor of $v_i$ is toggled. Thus, the label of $v_i$ after the toggling given by $\mathbf{x}$ is $b_i + \sum_{j=1}^{n} N_{ij}x_j$. This gives us the following.

**Lemma 3.1.** The toggling given by $\mathbf{x}$ can be used to win the Lights Out game if and only if $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_k$.

**Example 3.2.** Let $G = C_4$, and suppose that we have an initial labeling $\pi : V(G) \to \mathbb{Z}_8$ given by $\pi(v_1) = 4$, $\pi(v_2) = 1$, $\pi(v_3) = 5$, $\pi(v_4) = 3$. We then have

$$N = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

Suppose $k = 8$. We then solve the equation $N\mathbf{x} = \mathbf{b}$ by row reduction modulo 8 to get $\mathbf{x}[1] = 2$, $\mathbf{x}[2] = 4$, $\mathbf{x}[3] = 3$, and $\mathbf{x}[4] = 6$. Thus, the game can be won. Note that if we row reduce in $\mathbb{Z}_3$, there is no solution. In this case, the game cannot be won.

These methods are reminiscent of those used in domination theory (see, for example, [AS92] and [ASZ02]). The classical domination problem is to find a set $S \subseteq V(G)$ (called a *dominating set*) of minimum cardinality such that every vertex of $G$ is either in $S$ or adjacent to a vertex in $S$.

For each $v \in V(G)$, define $N[v] = \{w \in V : vw \in E(G) \text{ or } w = v\}$. Then $S \subseteq V(G)$ is a dominating set if and only if $|N[v] \cap S| \geq 1$ for all $v \in V(G)$. Other types of domination have been studied by placing various restrictions on $|N[v] \cap S|$. Note that if we let $\mathbf{x}$ be the $n$-dimensional vector with $x_i = 1$ if $v_i \in S$ and $x_i = 0$ otherwise, then

$$|N[v] \cap S| = \sum_{j=1}^{n} N_{ij} x_j \tag{1}$$

In *parity domination*, we begin with a labeling $\pi : V(G) \to \mathbb{Z}_2$. We call a set $S \subseteq V(G)$ a parity dominating set of $\pi$ if $|N[v] \cap S| \equiv \pi(v) \pmod 2$ for all $v \in V(G)$. Using (1), this is equivalent to $S$ satisfying the equation $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_2$, where $\mathbf{x}(i) = 1$ if $v_i \in S$ and $\mathbf{x}(i) = 0$ otherwise. Thus, we have a parity domination set $S$ for $\pi$ if and only if the Lights Out game with initial labeling $\pi$ can be won by toggling precisely the vertices in $S$.

To extend parity domination to labelings $\pi : V(G) \to \mathbb{Z}_k$, $k \geq 3$, we must address the possibility that a solution to $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_k$ may have an entry that is neither 0 nor 1. We resolve this issue by using multisets. Recall that a *multiset* is a pair $M = (S, m)$, where $S$ is a set (called the *underlying set*), and $m : S \to \mathbb{N}$ is a function. For $s \in S$, we call $m(s)$ the *multiplicity of $s$ in $M$*.

**Definition 3.3.** For each $v \in V(G)$, let $N_k[v]$ be the multiset with underlying set $N[v]$ and with each element having multiplicity $k-1$. Let $M$ be a multiset whose underlying set is a subset of $V(G)$ and whose elements each have multiplicity at most $k-1$. For a labeling $\pi : V(G) \to \mathbb{Z}_k$, we call $M$ a *$\mathbb{Z}_k$-dominating multiset* for $\pi$ if $|N_k[v] \cap M| \equiv \pi(v) \pmod k$ for all $v \in V(G)$.

Note that a $\mathbb{Z}_2$-dominating multiset is merely a parity dominating set. The following describes the relationship between $\mathbb{Z}_k$-domination and the Lights Out game.

**Theorem 3.4.** Let $G$ be a graph with labeling $\pi : V(G) \to \mathbb{Z}_k$. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, let $\mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{b}[i] = \pi(v_i)$. The following are equivalent.

1. There exists a $\mathbb{Z}_k$-dominating multiset for $\pi$.

2. There is a solution to the equation $N\mathbf{x} = \mathbf{b}$ over $\mathbb{Z}_k$.

3. The Lights Out game with initial labeling $\pi$ can be won.

*Proof.* By Lemma 3.1, 2 and 3 are equivalent. For 1 $\Rightarrow$ 2, let $S$ be a $\mathbb{Z}_k$-dominating multiset. Define $\mathbf{x}$ so that $\mathbf{x}[i]$ is the multiplicity of $v_i$ in $S$. It is not hard to show that $\sum_{j=1}^{n} N_{ij}x_j = |N_k[v] \cap S|$. Since also $S$ is a $\mathbb{Z}_k$-dominating multiset, we have, for each $v \in V$,

$$\sum_{j=1}^{n} N_{ij}x_j = |N_k[v] \cap S| \equiv \pi(v) \pmod{k}$$

and so $\mathbf{x}$ is a solution to $N\mathbf{x} = \mathbf{b}$. The proof of 2 $\Rightarrow$ 1 is similar. □

# 4 Winnable Labelings and AW Graphs

We use notation as in previous sections, with our graph $G$, labeling set $\mathbb{Z}_k$, and toggling function $T(c) = c + 1$. We say the labelings $\pi$ and $\pi'$ are *equivalent under $T$* (or merely *equivalent* if the context is clear) if, given the initial labeling $\pi$, there is a sequence of toggles such that the terminal labeling is $\pi'$. We denote this relation by $\mathcal{R}_G^k$. It is easy to see that $\mathcal{R}_G^k$ is an equivalence relation. We call a labeling $\pi$ *winnable* if $\pi$ is equivalent to $\pi_0$, where $\pi_0(v) = 0$ for all $v \in V(G)$. We say that $G$ is *always winnable over $\mathbb{Z}_k$* (or simply *always winnable* or *AW* if the context is clear) if all labelings $\pi : V(G) \to \mathbb{Z}_k$ are winnable.

**Example 4.1.** $P_3$ has neighborhood matrix $N = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Since $\det(N) = -1$, $N$ is invertible, and so the equation $N\mathbf{x} = \mathbf{b}$ can always be solved. By Lemma 3.1, $P_3$ is AW over all $\mathbb{Z}_k$, $k \geq 2$.

On the other hand, $K_n$ is non-AW for all $k \geq 2$. Every labeling $\pi \neq \pi_0$ is not winnable, since toggling any vertex has the effect of increasing the label of every vertex by one.

In this section, we study the winnable labelings of paths, cycles and complete bipartite graphs. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. We let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1 v_n, v_i v_{i+1} : 1 \leq i \leq n-1\}$. Finally, we let $V(K_{m,n}) = \{v_i, w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_{m,n}) = \{v_i w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. We begin by proving that every labeling of the vertex sets of each of these graphs is equivalent to a "nice" labeling.

**Lemma 4.2.**
1. Each labeling of $V(P_n)$ by $\mathbb{Z}_k$ is equivalent to some $\pi$ where $\pi(v_i) = 0$ for all $i \neq 1$.
2. Each labeling of $V(C_n)$ by $\mathbb{Z}_k$ is equivalent to some $\pi$, where $\pi(v_i) = 0$ for all $i \neq 1, 2$.
3. Each labeling of $V(K_{m,n})$ by $\mathbb{Z}_k$ is equivalent to some $\pi$, where $\pi(w_j) = 0$ for all $1 \leq j \leq n$.

*Proof.* For 1, if all vertices have label 0, then we are done. Otherwise, let $m$ be maximum such that the label of $v_m$ is $c \neq 0$. If $m = 1$, we are done. If $m \geq 2$, we toggle $v_{m-1}$ $k - c$

times (or $-c$ times modulo $k$). The resulting labeling has the label of 0 for $v_i$, $i \geq m$. By induction, this labeling is equivalent to one in which all vertices except perhaps $v_1$ have label 0. Part 2 follows similarly.

For 3, let the label of $w_j$ be $c_j$ for each $1 \leq i \leq n$. We merely toggle $w_j$ $k - c_j$ times to get the desired labeling. $\square$

Lemma 4.2 motivates the following labelings. For each $z \in \mathbb{Z}_k$, we define $\pi_z : V(P_n) \to \mathbb{Z}_k$ to be $\pi_z(v_i) = \delta_{i,1}z$. For each $y, z \in \mathbb{Z}_k$, we define $\pi_{y,z} : V(C_n) \to \mathbb{Z}_k$ to be $\pi_{y,z}(v_i) = \delta_{1,i}y + \delta_{2,i}z$. Finally, for each $z_1, \ldots, z_m \in \mathbb{Z}_k$, let $\mathbf{z} \in \mathbb{Z}_k^m$ with $\mathbf{z}(i) = z_i$. We define $\pi_{\mathbf{z}} : V(K_{m,n}) \to \mathbb{Z}_k$ by $\pi_{\mathbf{z}}(v_i) = z_i$ and $\pi_{\mathbf{z}}(w_j) = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$. Our main result tells precisely when $\pi_z$, $\pi_{y,z}$, and $\pi_{\mathbf{z}}$ are winnable.

**Theorem 4.3.** Let $\pi_z$, $\pi_{y,z}$, and $\pi_{\mathbf{z}}$ be as above

1. For $P_n$, $\pi_z$ is winnable if and only if either $n \equiv 0, 1 \pmod 3$ or $z = 0$.
2. For $C_n$, $\pi_{y,z}$ is winnable if and only if one of the following holds.
   (a) $n \equiv 1, 2 \pmod 3$ and $\gcd(3, k) = 1$
   (b) $n \equiv 1 \pmod 3$, $3|k$, and the equivalence $3x \equiv -y - z \pmod k$ has a solution.
   (c) $n \equiv 2 \pmod 3$, $3|k$, and the equivalence $3x \equiv y - 2z \pmod k$ has a solution.
   (d) $y = z = 0$.
3. For $K_{m,n}$, $\pi_{\mathbf{z}}$ is winnable if and only if $(mn-1)x \equiv \sum_{i=1}^{m} \mathbf{z}(i) \pmod k$ has a solution.

*Proof.* For 1, suppose we toggle the vertices in the order $v_1, v_2, \ldots, v_n$. Let $t_i$ be the number of times $v_i$ is toggled, and let $d_i$ be the label of $v_i$ after $v_i$ is toggled. Clearly $t_i = -d_{i-1}$ for all $i \geq 2$. We then have $d_i = t_i + t_{i-1} = -d_{i-1} - d_{i-2}$, and so $d_i + d_{i-1} + d_{i-2} = 0$. This, along with $d_1 = t_1 + z$ and $d_2 = -z$, gives us

$$d_i = \begin{cases} -t_1, & i \equiv 0 \pmod 3 \\ t_1 + z, & i \equiv 1 \pmod 3 \\ -z, & i \equiv 2 \pmod 3 \end{cases} \tag{2}$$

Note that $\pi_z$ is winnable if and only if there exists $t_1 \in \mathbb{Z}_k$ such that $d_n \equiv 0 \pmod k$. If $n \equiv 0 \pmod 3$, let $t_1 = 0$; if $n \equiv 1 \pmod 3$, let $t_1 = -z$. For $n \equiv 2 \pmod 3$, $\pi_z$ is winnable precisely when $z = 0$.

We proceed similarly for 2. Let $t_i$ be the number of times $v_i$ is toggled, and let $d_i$ be the label of $v_i$ after $v_i$ is toggled. Similarly as before, we have $t_i = -d_{i-1}$ for $3 \leq i \leq n$ and $d_i + d_{i-1} + d_{i-2} = 0$ for $4 \leq i \leq n-1$. This is not the case with $i = 3$ since $t_2$ is not necessarily $-d_1$ ($v_1$ is adjacent to both $v_2$ and $v_n$). We have $d_2 = t_1 + t_2 + z$ and $d_3 = -t_1 - z$, and so

$$d_i = \begin{cases} -t_1 - z, & i \equiv 0 \pmod 3 \\ -t_2, & i \equiv 1 \pmod 3 \\ t_1 + t_2 + z, & i \equiv 2 \pmod 3 \end{cases} \tag{3}$$

After $v_n$ is toggled, we have $d_n = t_{n-1} + t_n + t_1$ and $v_1$ has label $t_n + t_1 + t_2 + y$. If $n \equiv 0 \pmod 3$, we have $d_n = -z$ and $v_1$ has label $y - z$, so $\pi_{y,z}$ is winnable if and only if $y = z = 0$. If $n \equiv 1 \pmod 3$, we have $d_n = t_1 - t_2$ and $v_1$ has label $2t_1 + t_2 + y + z$, and so $\pi_{y,z}$ is winnable if and only if $t_2 = t_1$ and $3t_1 \equiv -y - z \pmod k$. Finally, if $n \equiv 2 \pmod 3$, we have $d_n = 2t_1 + t_2 + z$ and $v_1$ has label $t_1 + 2t_2 + y$. Eliminating $t_2$ gives us $3t_1 \equiv y - 2z \pmod k$.

For 3, let $z_i = \mathbf{z}(i)$, and let $x_i$ be the number of times $v_i$ is toggled. Once the $v_i$'s have been toggled, $v_i$ has label $x_i + z_i$ and each $w_j$ has label $\sum_{\ell=1}^m x_\ell$. In order to have a final label of 0, each $w_j$ must be toggled $-\sum_{\ell=1}^m x_\ell$ times. This leaves $v_i$ with the label $z_i + x_i - n\sum_{\ell=1}^m x_\ell = z_i + (1-n)x_i - n\sum_{\ell \neq i}^m x_\ell$. We must then have $(n-1)x_i + n\sum_{\ell \neq i}^m x_\ell = z_i$, a linear system over $\mathbb{Z}_k$. For each $p \in \mathbb{N}$, let $B(p) = [b_{ij}]$ be the $p \times p$ matrix with $b_{ii} = n-1$ for $1 \leq i \leq p$ and $b_{ij} = n$ otherwise. Then the augmented matrix for our linear system is $[B(m)|\mathbf{z}]$.

We now seek to put $[B(m)|\mathbf{z}]$ in row echelon form. If we subtract row 2 from row 1 and add $n$ times the new row 1 to every other row, we get the matrix

$$\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & z_1 - z_2 \\ 0 & 2n-1 & n & \cdots & n & nz_1 - nz_2 + z_2 \\ 0 & 2n & & & & nz_1 - nz_2 + z_3 \\ \vdots & \vdots & & B(m-2) & & \vdots \\ 0 & 2n & & & & nz_1 - nz_2 + z_m \end{bmatrix}$$

If we iterate this process $j-1$ times, $2 \leq j \leq m$, we get

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & \cdots & 0 & z_1 - z_2 \\ 0 & \ddots & \ddots & 0 & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & -1 & 1 & 0 & \cdots & 0 & z_{j-1} - z_j \\ 0 & \cdots & 0 & jn-1 & n & \cdots & n & \left(\sum_{\ell=1}^{j-1} nz_\ell\right) - (j-1)nz_j + z_j \\ 0 & \cdots & 0 & jn & & & & \left(\sum_{\ell=1}^{j-1} nz_\ell\right) - (j-1)nz_j + z_{j+1} \\ \vdots & \ddots & \vdots & \vdots & & B(m-j) & & \vdots \\ 0 & \cdots & 0 & jn & & & & \left(\sum_{\ell=1}^{j-1} nz_\ell\right) - (j-1)nz_j + z_m \end{bmatrix}$$

We get our echelon form by setting $j = m$. Since all row operations used to obtain the echelon form are invertible, this echelon form is equivalent to the original system. Moreover, the first $m-1$ leading entries are $-1$ and the last leading entry is $mn-1$, so $\pi_{\mathbf{z}}$ is winnable if and only if

$$(mn-1)x = \left(\sum_{\ell=1}^{m-1} nz_\ell\right) - (m-1)nz_m + z_m = \left(\sum_{\ell=1}^{m} nz_\ell\right) - (mn-1)z_m$$

has a solution in $\mathbb{Z}_k$. This is equivalent to $(mn-1)x \equiv \sum_{\ell=1}^m nz_\ell$ having a solution. Since $\gcd(mn-1, n) = 1$, this is equivalent to $(mn-1)x \equiv \sum_{\ell=1}^m z_\ell$ having a solution, which completes the proof. $\square$

Let $G$ be any graph. Recall the equivalence relation $\mathcal{R}_G^k$ between labelings of $V(G)$ by $\mathbb{Z}_k$, and note that the number of winnable labelings is $|[\pi_0]|$. There is a natural group action of $\mathbb{Z}_k{}^{|V(G)|}$ on the labelings of $V(G)$ in which the equivalence classes of $\mathcal{R}_G^k$ are the orbits of the action. It follows that all equivalence classes of $\mathcal{R}_G^k$ have the same size. Furthermore, Lemma 4.2 implies that $\pi_z$, $\pi_{y,z}$, and $\pi_{\mathbf{z}}$ represent all equivalence classes of $\mathcal{R}_{P_n}^k$, $\mathcal{R}_{C_n}^k$, and $\mathcal{R}_{K_{m,n}}^k$, although not necessarily uniquely.

Armed with this information, we can now count the winnable labelings for $P_n$, $C_n$, and $K_{m,n}$.

**Theorem 4.4.** 1. The number of winnable labelings of $V(P_n)$ by $\mathbb{Z}_k$ is

(a) $k^n$ if $n \equiv 0, 1 \pmod 3$.

(b) $k^{n-1}$ if $n \equiv 2 \pmod 3$

2. The number of winnable labelings of $V(C_n)$ by $\mathbb{Z}_k$ is

(a) $k^n$ if $n \equiv 1, 2 \pmod 3$ and $\gcd(3, k) = 1$.

(b) $k^{n-2}$ if $n \equiv 0 \pmod 3$.

(c) $\frac{k^n}{3}$ if $3|k$ and $n \equiv 1, 2 \pmod 3$.

3. $K_{m,n}$ has $\dfrac{k^{m+n}}{\gcd(k, mn-1)}$ winnable labelings by $\mathbb{Z}_k$.

*Proof.* For 1, Lemma 4.2(1) implies that the collection of all $\pi_z$, $z \in \mathbb{Z}_k$ represents all equivalence classes. If $k \equiv 0, 1 \pmod 3$, all labelings are winnable, so we have $k^n$ winnable labelings. If $k \equiv 2 \pmod 3$, suppose that $\pi_y$ and $\pi_z$ are equivalent. Then the same toggling sequence that takes $\pi_z$ to $\pi_y$ will take $\pi_{z-y}$ to $\pi_0$. By Theorem 4.3(1), we have $z - y = 0$, so $z = y$. Thus, there are $k$ equivalence classes. Since each has the same cardinality, each equivalence class has $\frac{k^n}{k} = k^{n-1}$ labelings.

For 2, Lemma 4.2(2) implies that the $\pi_{y,z}$ represent all equivalence classes. Suppose $\pi_{y_1,z_1}$ and $\pi_{y_2,z_2}$ are equivalent. As before, if $y = y_2 - y_1$ and $z = z_2 - z_1$, then $\pi_{y,z}$ is winnable. Let $n \equiv 1$ or $2 \pmod 3$. If $\gcd(3, k) = 1$, then Theorem 4.3(2a) implies that all $k^n$ labelings are winnable. If $n \equiv 0 \pmod 3$, then Theorem 4.3(2) implies that $\pi_{y,z}$ is winnable if and only if $y = z = 0$. It follows that there are $k^2$ equivalence classes, and therefore $k^{n-2}$ winnable labelings. If $3|k$ and $n \equiv 1 \pmod 3$, then by Theorem 4.3(2b), $\pi_{y,z}$ is winnable if and only if $3x \equiv -y - z \pmod k$ has a solution. This occurs precisely when $3|y + z$, or, equivalently, $y_1 + z_1 \equiv y_2 + z_2 \pmod 3$. There are then 3 equivalence classes and therefore $\frac{k^n}{3}$ winnable labelings. The $n \equiv 2 \pmod 3$ case follows similarly, using Theorem 4.3(2c).

For 3, Lemma 4.2(3) implies that the $\pi_{\mathbf{z}}$ represent all equivalence classes. Let $\mathbf{y} \in \mathbb{Z}_k^m$ be fixed. As before, if $\pi_{\mathbf{y}}$ and $\pi_{\mathbf{z}}$ are equivalent, then $\pi_{\mathbf{z}-\mathbf{y}}$ is winnable. By Theorem 4.3(3), this occurs precisely when $(mn-1)x = \sum_{i=1}^m [\mathbf{z}(i) - \mathbf{y}(i)] = \sum_{i=1}^m \mathbf{z}(i) - \sum_{i=1}^m \mathbf{y}(i)$ has a solution in $\mathbb{Z}_k$. If $d = \gcd(k, mn-1)$, then there are $\frac{k}{d}$ values of $\sum_{i=1}^m \mathbf{z}(i)$ for which a solution to this congruence exists. For each $r$ of these $\frac{k}{d}$ values, there are $k^{m-1}$ different

$\mathbf{z} \in \mathbb{Z}_k^m$ whose entries add up to $r$. Thus, there are $\frac{k^m}{d}$ $\pi_{\mathbf{z}}$'s in each equivalence class, and so there are $\frac{k^m}{k^m/d} = d$ equivalence classes. The result follows. □

From this, we can easily determine the AW paths, cycles, and complete bipartite graphs.

**Corollary 4.5.**
1. $P_n$ is AW over $\mathbb{Z}_k$ if and only if $n \equiv 0$ or $1 \pmod 3$.
2. $C_n$ is AW over $\mathbb{Z}_k$ if and only if $n \equiv 1$ or $2 \pmod 3$ and $\gcd(3, k) = 1$.
3. $K_{m,n}$ is AW over $\mathbb{Z}_k$ if and only if $\gcd(mn - 1, k) = 1$

# 5 Non-AW Caterpillar Graphs

One of the results in [AS96] gives a constructive method for generating all caterpillar graphs $G$ for which there exists a labeling $\pi : V(G) \to \mathbb{Z}_2$ that does not admit a parity domination set. In this section, we derive a similar constructive method for generating non-AW caterpillar graphs that will give Amin and Slater's result as a special case.

Recall that a caterpillar graph is a graph in which the vertices that are not leaves (called the *spine*) induce a path. Let $v_1(G), v_2(G), \ldots, v_n(G)$ be the vertices of the spine with $v_i(G)v_{i+1}(G) \in E(G)$, and let $\ell_i(G)$ be the number of leaves adjacent to $v_i(G)$. We can leave out the argument $G$ if $G$ is known. We begin with a result similar to Lemma 4.2.

**Lemma 5.1.** If $G$ is a caterpillar graph, then every labeling of $G$ is equivalent to some labeling $\pi$ such that $\pi(v) = 0$ for all $v \in V(G) - \{v_1\}$.

*Proof.* Follows from an argument similar to Lemma 4.2(1) and (3). □

For each $z \in \mathbb{Z}_k$, let $\pi_z$ be the labeling of the caterpillar graph $G$ given by $\pi_z(v_1) = z$ and $\pi_z(v) = 0$ for all $v \neq v_1$. Let $C$ be the set of all equivalence classes of labelings of $V(G)$. If $y, z \in \mathbb{Z}_k$, one can easily verify that the binary operation $[\pi_y] + [\pi_z] = [\pi_{y+z}]$ is well-defined, making $C$ an additive group. Moreover, the map $\Psi : \mathbb{Z}_k \to C$ given by $\Psi(z) = [\pi_z]$ is an epimorphism whose kernel is the set of all $z \in \mathbb{Z}_k$ such that $\pi_z$ is winnable. We then use standard group theory to get the following.

**Lemma 5.2.** Let $G$ be a caterpillar graph.

1. If $\pi_d$ is winnable, then $\pi_{md}$ is winnable for all $m \in \mathbb{Z}$.
2. $\pi_y$ and $\pi_z$ are winnable if and only if $\pi_{\gcd(y,z)}$ is winnable.
3. $G$ is AW if and only if $\pi_1$ is winnable.

Thus, in our study of AW (and non-AW) graphs, we will begin with the labeling $\pi_1$. To make the computations more convenient, let $m_i(G) = \ell_i(G) - 1$. We proceed as with $P_n$, toggling the vertices of the spine in the order $v_1, v_2, \ldots, v_n$. After each $v_i$ is toggled, we toggle the leaves adjacent to $v_i$ so that they each have label 0. Let $t_i$ be the number of times $v_i$ is toggled with $t_1 = x$, and let $d_i^G(x)$ (or simply $d_i(x)$ when $G$ is known) be the label of $v_i$ after $v_i$ and all its adjacent leaves are toggled. After toggling $v_1$, we must toggle each adjacent leaf $-x$ times to get $d_1(x) = 1 + x - \ell_1 x = -m_1 x + 1$. We toggle $v_2$ and its adjacent leaves similarly to get $d_2(x) = (1 - m_1 m_2)x + m_2$. For the remaining vertices, we get

$$d_i(x) = t_i - \ell_i t_i + t_{i-1} = -d_{i-1}(x) + \ell_i d_{i-1}(x) - d_{i-2}(x) = m_i d_{i-1}(x) - d_{i-2}(x)$$

This gives us the following.

**Lemma 5.3.** For each $1 \leq i \leq n$, we have $d_i(x) = a_i x + b_i$, where the sequences $\{d_i(x)\}$, $\{a_i\}$, and $\{b_i\}$ satisfy the homogeneous linear difference equation $y_j = m_j y_{j-1} - y_{j-2}$ with initial values $a_1 = -m_1$, $a_2 = 1 - m_1 m_2$, $b_1 = 1$, and $b_2 = m_2$.

Note that if the spine of $G$ is $P_n$, then $G$ is AW if and only if the equivalence $a_n x + b_n \equiv 0 \pmod{k}$ can be solved, which occurs precisely when $\gcd(a_n, k) | b_n$. Using techniques similar to the proof of Lemma 5.3, we get a slight strengthening of Lemma 5.2(1).

**Lemma 5.4.** Suppose that $\pi_d$ can be won by toggling $v_1$ $y$ times. Then $\pi_{md}$ can be won by toggling $v_1$ $my$ times.

Amin and Slater use the following construction to generate non-AW caterpillar graphs in the case $k = 2$.

**Definition 5.5.** Let $G_1$ and $G_2$ be caterpillar graphs whose spines are $P_{n_1}$ and $P_{n_2}$, respectively. We define $G_1.w.G_2(r)$ to be the caterpillar graph with vertex set $V(G_1) \cup V(G_2) \cup \{w, x_1, \ldots, x_r\}$, where $r \geq 0$ ($r = 0$ denotes no $x_i$'s) and edge set $E(G_1) \cup E(G_2) \cup \{v_{n_1}(G_1)w, wv_1(G_2), wx_1, \ldots, wx_r\}$. We call this construction a *pasting* of $G_1$ and $G_2$.

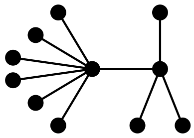
Note that the construction depends on which order the vertices of the spine are written. This can be made clear by defining the $\ell_i$'s. We call $K_{1,n}$ *type* $T_1$ if $n$ is odd, and we call a caterpillar graph *type* $T_{2,j}$, $j \geq 0$, if it has spine $P_{j+2}$, if $\ell_1$ and $\ell_{j+2}$ are even, and if $\ell_i$ are odd for $2 \leq i \leq j+1$. Note that $T_1$ and $T_{2,j}$ are unique if we consider each $\ell_i$ modulo 2. We can now state Amin and Slater's result (in Lights Out terminology) as follows.

**Theorem 5.6.** [AS96]

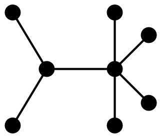
1. Let $G_1$ and $G_2$ be two non-AW caterpillar graphs over $\mathbb{Z}_2$. Then $G_1.w.G_2(r)$ is non-AW over $\mathbb{Z}_2$.
2. A caterpillar graph $G$ is non-AW over $\mathbb{Z}_2$ if and only if either
   (a) $G$ is of type $T_1$ or $T_{2,j}$.
   (b) $G$ can be obtained by repeated pastings of graphs of types $T_1$ and $T_{2,j}$.

The following example shows that this result does not hold for all $\mathbb{Z}_k$.

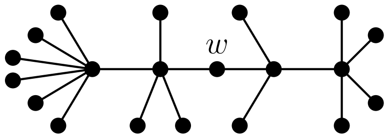
**Example 5.7.** Consider $G_1$, $G_2$, and $G_1.w.G_2(0)$ as follows.



$G_1$



$G_2$




$G_1.w.G_2(0)$

We have $m_1(G_1) = 5$, $m_2(G_1) = 2$, $m_1(G_2) = 1$, and $m_2(G_2) = 3$. If $k = 6$, then, $d_2^{G_1}(x) = 3x + 2$, and $d_2^{G_2}(x) = 4x + 3$. Since neither $d_2^{G_i}(x)$ can be 0 modulo 6, $G_1$ and $G_2$ are not AW. However, $G_1.w.G_2(0)$ is AW (using $x = 0$ for $\pi_1$). Thus, it is possible to paste two non-AW caterpillars together to get an AW caterpillar.

While we do not have an analogue of Theorem 5.6 for arbitrary $k$, our main result generalizes the theorem to $k = p^e$, where $p$ is prime. We begin with a lemma.

**Lemma 5.8.** Suppose $p$ is a prime such that $p|k$, and let $a_i$ and $b_i$ be as in Lemma 5.3. Then for each $1 \leq i \leq n$, $p$ cannot divide both $a_i$ and $b_i$.

*Proof.* For contradiction, let $i$ be minimal such that $p|a_i$ and $p|b_i$. By Lemma 5.3, we have $i \geq 3$, and (in $\mathbb{Z}_k$) $a_i = m_i a_{i-1} - a_{i-2}$ and $b_i = m_i b_{i-1} - b_{i-2}$. Since $p|k$, these equations also hold when we consider them over $\mathbb{Z}_p$. In this context, $a_i = b_i = 0$, and so $a_{i-2} = m_i a_{i-1}$ and $b_{i-2} = m_i b_{i-1}$.

We claim that $a_{j+1}b_j \equiv a_j b_{j+1} \pmod{p}$ for all $1 \leq j \leq i-2$. We induct on $i - j - 2$. For $i - j - 2 = 0$, by the minimality of $i$, either $a_{i-1}$ or $b_{i-1}$ is nonzero in $\mathbb{Z}_p$. Without loss of generality, $a_{i-1} \neq 0$ in $\mathbb{Z}_p$. Doing computations in $\mathbb{Z}_p$, we get $m_i = \frac{a_{i-2}}{a_{i-1}}$, and so $b_{i-2} = \frac{a_{i-2}b_{i-1}}{a_{i-1}}$. Thus, $a_{i-1}b_{i-2} \equiv a_{i-2}b_{i-1} \pmod{p}$. For the induction step, suppose $a_{j+1}b_j \equiv a_j b_{j+1} \pmod{p}$. We then have, in $\mathbb{Z}_p$,

$$a_{j+1}b_j = a_j b_{j+1} = a_j(m_{j+1}b_j - b_{j-1})$$

A little algebra gives us $a_j b_{j-1} = b_j(m_{j+1}a_j - a_{j+1}) = a_{j-1}b_j$, which proves the claim.

In particular, we have $a_1 b_2 \equiv a_2 b_1 \pmod{p}$, and so $-m_1 m_2 \equiv 1 - m_1 m_2 \pmod{p}$. This implies that $1 \equiv 0 \pmod{p}$, a contradiction, which completes the proof. $\square$

As a consequence, we get the following.

**Theorem 5.9.** Let $G_1$ and $G_2$ be non-AW caterpillar graphs over $\mathbb{Z}_k$, where $k = p^e$ with $p$ a prime. Then $G_1.w.G_2(r)$ is non-AW over $\mathbb{Z}_k$ for all $r \geq 0$.

*Proof.* Let the spine of $G_i$ be $P_{n_i}$ for $i = 1, 2$. Since $d_{n_1}^{G_1}(x) \equiv 0 \pmod{k}$ has no solution, we cannot have $\gcd(a_{n_1}, k) = 1$. Thus, $p|a_{n_1}$. By Lemma 5.8, $p$ does not divide $b_{n_1}$, and so $p$ does not divide $d_{n_1}$. Therefore $\gcd(d_{n_1}, k) = 1$.

After $v_{n_1}(G_1)$ is toggled, we toggle $w$ $-d_{n_1}$ times, and we toggle each leaf of $w$ $d_{n_1}$ times. This leaves $v_1(G_2)$ with label $-d_{n_1}$. Now the only vertices that remain to be toggled are $v_1(G_2), \ldots, v_{n_2}(G_2)$. If it were possible to toggle these vertices so that $v_1(G_2), \ldots, v_{n_2}(G_2)$ have label 0 (even if we ignore the label of $w$), then $\pi_{-d_{n_1}}$ would be a winnable labeling for $G_2$. This would imply, by Lemma 5.2(2), that $\pi_1$ is a winnable labeling, which implies that $G_2$ is AW by Lemma 5.2(3). This is a contradiction and completes the proof. $\square$

We now derive a set of non-AW caterpillar graphs that we paste together to generate all non-AW caterpillars. We call a non-AW caterpillar graph *irreducibly non-AW* if it cannot be written $G_1.w.G_2(r)$ for any non-AW $G_1$ and $G_2$, and any $r \geq 0$. The following is a useful characterization of irreducibly non-AW caterpillar graphs.

**Lemma 5.10.** Let $k = p^e$ with $p$ prime, and $G$ be a caterpillar graph over with spine $P_n$. Then $G$ is irreducibly non-AW over $\mathbb{Z}_k$ if and only if $\gcd(a_i, k) = 1$ for all $i \leq n-1$ and $\gcd(a_n, k) \neq 1$.

*Proof.* Suppose that $\gcd(a_i, k) = 1$ for all $i \leq n-1$ and $\gcd(a_n, k) \neq 1$. Since $\gcd(a_n, k) \neq 1$, $G$ is not AW by Lemma 5.8. Furthermore, if $G = G_1 v_i G_2$, then $\gcd(a_{i-1}, k) = 1$ implies that $G_1$ is AW. Thus, $G$ is irreducibly non-AW.

Conversely, suppose that $G$ is irreducibly non-AW. For contradiction, assume $\gcd(a_i, k) \neq 1$ for some $i \leq n-1$. Let $j$ be minimal such that $\gcd(a_j, k) \neq 1$. We claim that $\gcd(a_{j+1}, k) = 1$. If $j = 1$, then $a_2 = 1 - m_1 m_2$. Since $p$ divides $a_1 = -m_1$, then $a_2 \equiv 1 \pmod{p}$. If $j \geq 2$, then $a_{j+1} = m_{j+1} a_j - a_{j-1}$ and $p|a_j$ imply that $a_{j+1} \equiv a_{j-1} \pmod{p}$. In either case, $\gcd(a_{j+1}, k) = 1$. Since $G$ is not AW, we must have $j \leq n-2$. We have $G = G_1.v_{j+1}.G_2$, where $\ell_i(G_1) = \ell_i(G)$ for $1 \leq i \leq j$, and $\ell_i(G_2) = \ell_{i+j+1}(G)$ for $1 \leq i \leq n-j-1$. We claim that $G_1$ and $G_2$ are non-AW, which would imply that $G$ is not irreducibly AW, completing the proof.

Since $\gcd(a_j, k) \neq 1$, $G_1$ is non-AW. It suffices to prove that $G_2$ is non-AW. Suppose, for contradiction, that $G_2$ is AW, and let $y$ be the number of times $v_{j+2}$ is toggled to win $\pi_1$. As we toggle the vertices of $G$ in an attempt to win $\pi_1$, consider the situation after $v_{j+1}$ and its adjacent leaves are toggled. Then $v_{j+1}$ has label $d_{j+1}(x)$, $v_{j+2}$ has label $t_{j+1} = -d_j(x)$, and the remaining vertices have label 0. Note that only vertices in $G_2$ are toggled from here on out, and that $v_{j+2}$ will be toggled $-d_{j+1}(x)$ times. But Lemma 5.4 implies that if $v_{j+2}$ is toggled $-yd_j(x)$ times, then the game can be won. Thus, the game can be won if $-yd_j(x) \equiv -d_{j+1}(x) \pmod{k}$ can be solved for $x$. By substituting $d_j(x) = a_j x + b_j$ and $d_{j+1}(x) = a_{j+1}x + b_{j+1}$, this equivalence becomes

$$(a_{j+1} - a_j y)x \equiv b_j y - b_{j+1} \pmod{k}$$

We know $p$ divides $a_j$ but not $a_{j+1}$, so $\gcd(a_{j+1} - a_j y, k) = 1$. Therefore, the equivalence can be solved, which makes $G$ an AW graph. This is a contradiction, and so $G_2$ is non-AW. $\square$

This characterization of irreducibly non-AW caterpillar graphs makes them relatively straightforward to construct. Let $k = p^e$, let $G$ be irreducibly non-AW over $\mathbb{Z}_k$, and let $P_n$ be the spine of $G$. If $n = 1$, then $p$ divides $a_1 = -m_1$, giving us $p^{e-1}$ choices for $m_1$ (and thus $\ell_1$) modulo $k$. If $n = 2$, then since $p$ does not divide $a_1$, we have $p^{e-1}(p-1)$ choices for $m_1$ modulo $k$. We then need $p|a_2$, and so $m_1 m_2 \equiv 1 \pmod{p}$. If $a$ is the inverse of $m_1$ modulo $p$, then $a + rp$ are incongruent solutions for $0 \leq r \leq p^{e-1} - 1$, giving us $p^{e-1}$ choices for $m_2$, and $p^{2(e-1)}(p-1)$ possible irreducibly non-AW caterpillar graphs. For $n \geq 3$, we proceed similarly. For $j \geq 3$, suppose that we have arranged that $p$ does not divide $a_i$ for $1 \leq i \leq j-1$. We have $a_j = m_j a_{j-1} - a_{j-2}$, and so $p|a_j$ precisely when $m_j a_{j-1} \equiv a_{j-2} \pmod{p}$. This equivalence has a unique solution $m_j = a$, and, as in the $n = 2$ case, we get inequivalent solutions $a + rp$ modulo $k$, where $0 \leq r \leq p^{e-1} - 1$. If $j = n$, we can choose any of the $p^{e-1}$ solutions modulo $k$, and if $j < n$, we choose any of the other $p^{e-1}(p-1)$ equivalence classes to make $G$ irreducibly non-AW. Note that applying this process to the case $k = 2$ gives us the graphs of types $T_1$ and $T_{2,j}$ in Theorem 5.6(2). Thus, the task of generating all non-AW caterpillars can be reduced to inverting elements of $\mathbb{Z}_p$. We also get the following.

**Corollary 5.11.** Let $k = p^e$, where $p$ is prime. If we consider each $\ell_i$ modulo $k$, then the number of irreducibly non-AW caterpillar graphs whose spine is $P_n$ is $p^{n(e-1)}(p-1)^{n-1}$.

The following generalization of Amin and Slater's result follows directly from Lemma 5.10.

**Theorem 5.12.** Let $k = p^e$, where $p$ is prime. Then all non-AW caterpillar graphs over $\mathbb{Z}_k$ can be generated by repeated applications of pasting irreducibly non-AW caterpillar graphs.

# References


[ACS98] A.T. Amin, L.H. Clark, and P.J. Slater, *Parity Dimension for Graphs*, Discrete Math. **187** (1998), 1–17.

[AF98] M. Anderson and T. Feil, *Turning Lights Out with Linear Algebra*, Math. Mag. **71** (1998), 300–303.

[AS92] A.T. Amin and P.J. Slater, *Neighborhood domination with parity restrictions in graphs*, Congr. Numer. **91** (1992), 19–30.

[AS96] ______, *All Parity Realizable Trees*, JCMCC **20** (1996), 53–63.

[ASZ02] A.T. Amin, P.J. Slater, and G. Zhang, *Parity Dimension for Graphs - A Linear Algebraic Approach*, Linear and Multilinear Algebra **50** (2002), 327–342.

[Aua00] P.V. Auaujo, *How to turn all the lights out*, Elem. Math. **55** (2000), 135–141.

[GK97] J. Goldwasser and W. Klostermeyer, *Maximization versions of "Lights Out" games in grids and graphs*, Congr. Numer. **126** (1997), 99–111.

[GKT97] J. Goldwasser, W. Klostermeyer, and G. Trapp, *Characterizing switch-setting problems*, Linear Multilinear Algebra **43** (1997), 121–135.

[Pel87] D. Pelletier, *Merlin's Magic Square*, Amer. Math. Monthly **94** (1987), 143–150.

[Sto89] D.L. Stock, *Merlin's Magic Square Revisited*, Amer. Math. Monthly **96** (1989), 608–610.

[Sut89] K. Sutner, *Linear cellular automata and the Garden-of-Eden*, Math. Intelligencer **11** (1989), 49–53.