On the Coradical Filtration of Pointed Coalgebras

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Abstract

We investigate the coradical filtration of pointed coalgebras. First, we generalize a theorem of Taft and Wilson using techniques developed by Radford in [Rad78] and [Rad82]. We then look at the coradical filtration of duals of inseparable field extensions L^* upon extension of the base field K, where $K \subseteq L$ is a field extension. We reduce the problem to the case that the field extension is purely inseparable. We use this to prove that if E is a field containing the normal closure of L over K, then $E \otimes L^* = (E \otimes L^*)_1$ if and only if L/K is separable or $\mathrm{char}(K) = |L : L_s| = 2$, where L_s is the separable closure of K in L.

1 Introduction

In this paper, we study the coradical filtration of pointed coalgebras. Our first main result is a generalization of a theorem of Taft and Wilson from [TW74]. The following is a slightly stronger version of this result as proved in [Mon93, 5.4.1], using methods from [Rad78] and [Rad82].

Theorem 1.1. *Let C be a pointed coalgebra, with $G = G(C)$. For each $g, h \in G$, let $P'_{g,h}(C)$ be any vector space complement of $K(g - h)$ in $P_{g,h}(C)$. Then*

(*i*) $C_1 = KG \oplus (\oplus_{g,h \in G} P'_{g,h}(C))$

(*ii*) for any $n \geq 1$ and $c \in C_n$,

$$c = \sum_{g,h \in G} c_{g,h}, \text{ where } \Delta(c_{g,h}) = c_{g,h} \otimes g + h \otimes c_{g,h} + w$$

for some $w \in C_{n-1} \otimes C_{n-1}$.

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This theorem was one of the results used to prove theorems concerning the order of the antipode for finite-dimensional pointed Hopf algebras. The generalization we prove (Theorem 2.7) is an extension of (i) to C_n. The proof closely follows Montgomery's proof. We get a slightly stronger result in the case of cocommutative coalgebras (Theorem 2.10). In the process, we define an analogue of primitive elements for C_n and consider their dual notion in Section 5, which gives us a generalization of derivations.

We then consider duals of finite field extensions. This is motivated by the results in [Par01], where we investigated the behavior of cocommutative coalgebras when the base field is extended. Given a cocommutative coalgebra C, we found equivalent conditions for $(E \otimes C)_0 = E \otimes C_0$ for any field extension $K \subseteq E$, where C_0 is the coradical. In addition to C being cosemisimple, all simple subcoalgebras of C must be duals of separable field extensions of K.

In this paper, we study the coradical filtration of duals of inseparable field extensions. These are natural coalgebras to investigate, since they are the most elementary examples of cosemisimple coalgebras which are not cosemisimple when the field is extended. In particular, given a field extension $K \subseteq L$, we determine equivalent conditions on a field E so that $E \otimes L^*$ is pointed, and this leads to Theorem 3.11, which reduces the structure of $(E \otimes L^*)_n$ to that of the E-coalgebra $E \otimes_{L_s} Hom_{L_s}(L, L_s)$, where L_s is the separable closure of K in L. We use this, along with some results on purely inseparable field extensions, to prove that if E is a field containing the normal closure of L over the base field K, then $E \otimes L^* = (E \otimes L^*)_1$ if and only if either $K \subseteq L$ is a separable field extension or $\mathrm{char}(K) = |L : L_s| = 2$ (Theorem 4.5).

We refer the reader to [Mon93] for general facts about coalgebras. In particular, we make heavy use of the results and notation in sections 5.1, 5.2, and 5.4. All vector spaces and tensor products will be over a field K unless otherwise indicated.

2 Generalization of Taft-Wilson

In this section we generalize Theorem 1.1. Our goal will be to extend part (i) to C_n. We will, to a great extent, follow Montgomery's proof, which borrows heavily from methods in [Rad78] and [Rad82]. In particular, the following theorem will be crucial. Recall that a coalgebra has separable coradical if every simple subcoalgebra is the dual of a separable K-algebra.

Theorem 2.1. [Abe80, 2.3.11] Let C be a coalgebra with separable coradical. Then there exists a coideal I of C such that $C = I \oplus C_0$.

In particular, this applies to pointed coalgebras. So let C be a pointed coalgebra, and fix a coideal I such that $C = KG \oplus I$, where $G = G(C)$. For each $x \in G$, define $e_x \in C^*$ by $e_x(I) = 0$, $e_x(g) = \delta_{x,g}$ for all $g \in G$. Then $\varepsilon = \sum_{x \in G} e_x$ and the e_x are orthogonal idempotents, where multiplication is given by the convolution product (see [Mon93, 1.4]). For each $c \in C$, $x, y \in G$, define ${}^x c = c \leftharpoonup e_x = \sum e_x(c_1)c_2$, $c^y = e_y \rightharpoonup c = \sum e_x(c_2)c_1$, ${}^x c^y = {}^x(c^y) = ({}^x c)^y$. Let ${}^x C^y = \{{}^x c^y : c \in C\}$.

Proposition 2.2. Given the notation above,

(*i*) For all $c \in C$, $c = \sum_{x,y \in G} {}^x c^y$.

(*ii*) $I = \cap_{x \in G} Ker(e_x) = \oplus_{x,y \in G} ({}^x C^y)^+$.

(*iii*) ${}^x C^x = Kx \oplus ({}^x C^x)^+$, and ${}^x C^y = ({}^x C^y)^+$ for all $x \neq y$.

(*iv*) For all $c \in C$, ${}^x({}^g c^h)^y = \delta_{x,g}\delta_{h,y} {}^g c^h$.

(*v*) For all $c \in C$, $\Delta({}^x c^y) = \sum_{z \in G} {}^x c_1^z \otimes^z c_2^y$

(*vi*) For all $c \in ({}^x C_n^y)^+$, $\Delta(c) - c \otimes y - x \otimes c \in \sum_{g \in G} ({}^x C_{n-1}^g)^+ \otimes ({}^g C_{n-1}^y)^+$.

Proof. Parts $(i) - (iv)$ are easy to verify. For (v), [Rad78, p. 285, eqn 1.2a] gives us $\Delta({}^x c^y) = \sum (c_1 \leftharpoonup e_x) \otimes (e^y \rightharpoonup c_2)$. We then use the facts that $e_z \cdot e_z = e_z$ and $\sum_{z \in G} e_z = \varepsilon$ to give us $\sum_{z \in G} {}^x c_1^z \otimes^z c_2^y = \sum (c_1 \leftharpoonup e_x) \otimes (e^y \rightharpoonup c_2)$, completing the proof of (v). For (vi), it is shown in the proof of [Mon93, 5.4.1] that $\Delta(c) - c \otimes y - x \otimes c \in C_{n-1} \otimes C_{n-1}$ (see also eqn. 1.5 in [Rad78]). Applying (v) and the fact that $c \in {}^x C^y$, we get $\Delta(c) - c \otimes y - x \otimes c \in \sum_{g \in G} {}^x C_{n-1}^g \otimes {}^g C_{n-1}^y$. If we write $\Delta(c) - c \otimes y - x \otimes c = \sum v_{i,g} \otimes w_{i,g}$ with $v_{i,g} \in {}^x C_{n-1}^g$ and $w_{i,g} \in {}^g C_{n-1}^y$ with the $w_{i,g}$ linearly independent, apply $\varepsilon \otimes id$ to both sides to get $0 = \sum_{i,g} \varepsilon(v_{i,g}) w_{i,g}$. We get $v_{i,g} \in ({}^x C^g)^+$. Similarly, $w_{i,g} \in ({}^g C^y)^+$, giving us (vi). □

If we let $I_n = I \cap C_n$ for each $n \geq 0$, then we have $C_n = KG \oplus I_n$, and I_n is a coideal of C_n. Thus $I_n = \oplus_{g, \in G} ({}^g C_n^h)^+$.

To generalize Theorem 1.1, we will need an analogue of primitive elements in C_n.

Definition 2.3. For each $n \geq 0$, $g, h \in G$, we define the subspace $P_{g,h}^{(n)}(C)$ as follows.

$$P_{g,h}^{(0)} = 0, \quad P_{g,h}^{(n)} = \{c \in C : \Delta(c) - c \otimes g - h \otimes c \in \sum_{x \in G} P_{x,h}^{(n-1)} \otimes P_{g,x}^{(n-1)}\}$$

We then define $P^{(n)}(C) = \sum_{g,h \in G} P_{g,h}^{(n)}(C)$

Note that $P_{g,h}^{(1)}(C) = P_{g,h}(C)$, so this is a generalization of g, h-primitive elements.

Proposition 2.4. Let $g, h \in G$. Then

(*i*) $P_{g,h}^{(n)}(C) \subseteq P_{g,h}^{(n+1)}(C)$.

(*ii*) $\varepsilon(P^{(n)}(C)) = 0$

(*iii*) $P_{g,h}^{(n)}(C) \cap KG = KG^+$ for $n \geq 2$.

(*iv*) $({}^h C_n^g)^+ \subseteq P_{g,h}^{(n)}(C)$.

(*v*) $({}^\sigma C_{n-1}^g)^+, ({}^h C_{n-1}^\tau)^+ \subseteq P_{g,h}^{(n)}(C)$ for all $\sigma, \tau \in G$.

(*vi*) $({}^\sigma C_{n-2}^\tau)^+ \subseteq P_{g,h}^{(n)}(C)$ for all $\sigma, \tau \in G$.

(*vii*) $C_n = P^{(n)}(C) + KG$ for all $n \geq 0$.

Proof. Parts (i) and (ii) follow from an easy induction. For (iii), by (i) and (ii), it suffices to prove that $KG^+ \subseteq P^{(2)}_{g,h}(C)$. Let $c = \sum_{x\in G} k_x x$, where $k_x \in K$ and $\sum_{x\in G} k_x = 0$. Then

$$\begin{aligned}\Delta(c) - c\otimes g - h\otimes c &= \sum_{x\in G} k_x(x\otimes x - x\otimes g - h\otimes x)\\ &= \sum_{x\in G} k_x(x\otimes x - x\otimes g - h\otimes x + h\otimes g)\\ &= \sum_{x\in G} k_x(x-h)\otimes(x-g) \in \sum_{x\in G} P^{(1)}_{x,h}(C)\otimes P^{(1)}_{g,x}(C)\end{aligned}$$

so $c \in P^{(2)}_{g,h}(C)$.

For (iv), the $n=0$ case is trivial. By Proposition 2.2(vi), if $c \in ({}^hC^g_n)^+$ then $\Delta(c) - c\otimes g - h\otimes c \in \sum_{x\in G}({}^hC^x_{n-1})^+ \otimes ({}^xC^g_{n-1})^+$. Applying induction to $({}^hC^x_{n-1})^+$ and $({}^xC^g_{n-1})^+$ gives us the result.

We prove (v) by induction. The case $n=1$ is trivial. For $n>1$, let $c \in ({}^hC^\tau_{n-1})^+$. By Proposition 2.2$(vi)$, it follows that $\Delta(c) = c\otimes\tau + h\otimes c + \sum_{x\in G} {}^hv^x_i \otimes {}^xw^\tau_i$, where ${}^hv^x_i \in ({}^hC^x_{n-2})^+$ and ${}^xw^\tau_i \in ({}^xC^\tau_{n-2})^+$. By induction, ${}^xw^\tau \in P^{(n-1)}_{g,x}(C)$, giving us

$$\begin{aligned}\Delta(c) - c\otimes g - h\otimes c &= c\otimes(\tau - g) + \sum_{x\in G} {}^hv^x_i \otimes {}^xw^\tau_i\\ &\in P^{(n-1)}_{\tau,h}(C)\otimes P^{(n-1)}_{g,\tau}(C) + \sum_{x\in G} P^{(n-1)}_{x,h}(C)\otimes P^{(n-1)}_{g,x}(C)\end{aligned}$$

and the result follows. The argument is similar for $({}^\sigma C^g_{n-1})^+$.

We proceed similarly for (vi). Let $c \in ({}^\sigma C^\tau_{n-2})^+$. We then have, as before, $\Delta(c) - c\otimes g - h\otimes c = c\otimes(\tau-g) + (\sigma-h)\otimes c + \sum_{x\in G} {}^\sigma v^x_i \otimes {}^xw^\tau_i$, where ${}^\sigma v^x_i \in ({}^\sigma C^x_{n-3})^+$ and ${}^xw^\tau_i \in ({}^xC^\tau_{n-3})^+$. By (v), we have $c\otimes(\tau-g) \in P^{(n-1)}_{\tau,h}(C)\otimes P^{(n-1)}_{g,\tau}(C)$ and $(\sigma-h)\otimes c \in P^{(n-1)}_{\sigma,h}(C)\otimes P^{(n-1)}_{g,\sigma}(C)$. Thus, (vi) follows from induction.

For (vii), we have $C_n = KG \oplus I_n$ and $I_n = \oplus_{g,h\in G}({}^gC^h_n)^+$, so the result follows from (iv).

□

In Montgomery's proof of Theorem 1.1, a crucial step was proving that $P^{(1)}_{g,h}(C) = K(g-h) \oplus ({}^hC^g_1)^+$. The analogue for $n \geq 2$ is

Lemma 2.5. For all $n \geq 2$, we have

$$\begin{aligned}P^{(n)}_{g,h}(C) =& KG^+ \oplus (\oplus_{\sigma\neq h}({}^\sigma C^g_{n-1})^+) \oplus (\oplus_{\tau\neq g}({}^hC^\tau_{n-1})^+)\\ &\oplus (\oplus_{\substack{\sigma\neq h\\ \tau\neq g}}({}^\sigma C^\tau_{n-2})^+) \oplus ({}^hC^g_n)^+.\end{aligned}$$

Proof. By Proposition 2.4$(iii)-(vi)$, it suffices to show that $P^{(n)}_{g,h}(C) \subseteq (KG^+ \oplus (\oplus_{\sigma\neq h}({}^\sigma C^g_{n-1})^+) \oplus (\oplus_{\tau\neq g}({}^hC^\tau_{n-1})^+) \oplus (\oplus_{\substack{\sigma\neq h\\ \tau\neq g}}({}^\sigma C^\tau_{n-2})^+)) \oplus {}^hC^g_n$. Let $c \in P^{(n)}_{g,h}(C)$. In particular, $c \in C_n$, so

$c = v_G + \sum_{x,y\in G} {}^x v^y$, where $v_G \in KG$ and ${}^x v^y \in ({}^x C_n^y)^+$. Since $\varepsilon(c) = 0$, Proposition 2.4(ii) gives us $v_G \in KG^+$, and so, without loss of generality, ${}^x c^y \in ({}^x C_n^y)^+$ for all $x, y \in G$.

We have $\Delta(c) - c\otimes g - h\otimes c \in \sum_{u\in G} P_{u,h}^{(n-1)}(C) \otimes P_{g,u}^{(n-1)}(C)$. For $x, y \in G$, we apply Proposition 2.2$(iv), (v)$ to get

$$\Delta({}^x c^y) - \delta_{g,y}{}^x c^g \otimes g - \delta_{h,x} h \otimes {}^h c^y \in \sum_{u,z\in G} {}^x P_{u,h}^{(n-1)}(C)^z \otimes {}^z P_{g,u}^{(n-1)}(C)^y \tag{1}$$

If $y \neq g$, we can apply $id\otimes\varepsilon$ to the above and use Proposition 2.4(vii) to get ${}^x c^y \in C_{n-1}$. Similarly, if $x \neq h$, we have ${}^x c^y \in C_{n-1}$. Thus, if $x = h$ or $y = g$ (but not both), we have ${}^x c^y \in C_{n-1} \cap ({}^x C_n^y)^+ = ({}^x C_{n-1}^y)^+$. It thus suffices to show that for all $x \neq h$, $y \neq g$, we have ${}^x c^y \in {}^x C_{n-2}^y = C_{n-2} \cap ({}^x C_n^y)^+$. As before, we need only prove that ${}^x c^y \in C_{n-2}$. By induction, we have $P_{u,h}^{(n-1)}(C) = ({}^h C_{n-1}^u)^+ + C_{n-2}^+$ and $P_{g,u}^{(n-1)}(C) = ({}^u C_{n-1}^g)^+ + C_{n-2}^+$. Applying (1), and using the fact that $x \neq h$, $y \neq g$, we get $\Delta({}^x c^y) \in \sum_{z\in G} C_{n-2} \otimes C_{n-2}$. We now apply $id\otimes\varepsilon$ to get ${}^x c^y \in C_{n-2}$.

$\square$

Corollary 2.6. For all $n \geq 2$,

(i) $P_{g,h}^{(n)}(C) \cap C_{n-1} = KG^+ \oplus (\delta_{\sigma,h} + \delta_{\tau,g})(\oplus_{\sigma,\tau\in G}({}^\sigma C_{n-1}^\tau)^+) \oplus (\oplus_{\substack{\sigma\neq h\\ \tau\neq g}}({}^\sigma C_{n-2}^\tau)^+)$.

(ii) $P_{g,h}^{(n)}(C) \cap C_{n-2} = KG^+ \oplus (\oplus_{\sigma,\tau\in G}({}^\sigma C_{n-2}^\tau)^+)$

This brings us to the main theorem.

Theorem 2.7. For each $g, h \in G$, let $\bar{P}_{g,h}^{(n)}(C)$ be a vector space complement of $P_{g,h}^{(n)}(C) \cap C_{n-1}$ in $P_{g,h}^{(n)}(C)$. Then $C_n = C_{n-1} \oplus (\oplus_{g,h\in G} \bar{P}_{g,h}^{(n)}(C))$.

Proof. For each $g, h \in G$, Proposition 2.4(iv) gives us $({}^h C_n^g)^+ \subseteq P_{g,h}^{(n)}(C)$. Also, $KG \subseteq C_{n-1}$. Thus, $C_n = C_{n-1} + \sum_{g,h\in G} \bar{P}_{g,h}^{(n)}(C)$. It then suffices to prove that the sum is direct.

Suppose that there exist $v \in C_{n-1}$ and $\bar{v}_{g,h} \in \bar{P}_{g,h}^{(n)}(C)$ for each $g, h \in G$ with $v + \sum_{g,h\in G} \bar{v}_{g,h} = 0$. By Lemma 2.5, we can write $\bar{v}_{g,h} = v_{g,h} + w_{g,h}$, where $v_{g,h} \in {}^h C_n^g$ and $w_{g,h} \in C_{n-1}$. We then have $\sum_{g,h\in G} v_{g,h} = -v - \sum_{g,h\in G} w_{g,h} \in C_{n-1}$. Now ${}^x C_{n-1}^y \subseteq C_{n-1}$ for all $x, y \in G$. Thus, $v_{x,y} = {}^y(\sum_{g,h} v_{g,h})^x \in C_{n-1}$, and so $\bar{v}_{x,y} \in \bar{P}_{g,h}^{(n)}(C) \cap C_{n-1} = 0$. Then also $v = 0$, completing the proof.

$\square$

In the case of C cocommutative, we get a stronger result. We begin with a lemma.

Lemma 2.8. Let C be a cocommutative coalgebra.

(i) If $g, h \in G(C)$ with $g \neq h$, then $({}^g C_n^h)^+ = 0$.

(ii) If $c \in ({}^g C_n^g)^+$, then $\Delta(c) - c\otimes g - g\otimes c \in ({}^g C_{n-1}^g)^+ \otimes ({}^g C_{n-1}^g)^+$.

Proof. We prove (i) by induction. The case $n = 0$ is trivial. For $n > 0$, let $c \in ({}^gC_n^h)^+$. Applying Proposition 2.2(vi), induction, and the fact that $g \neq h$, we have $\Delta(c) - c \otimes h - g \otimes c \in \sum_{z \in G} ({}^gC_{n-1}^z)^+ \otimes ({}^zC_{n-1}^h)^+ = 0$. Thus, $c \in P_{h,g}(C)$. Since C is cocommutative, this implies $c \in K(g-h)$. But $K(g-h) \cap ({}^gC_n^h)^+ = 0$, completing the proof. Part (ii) follows from (i) and Proposition 2.2(vi). $\square$

This gives us the following.

Corollary 2.9. $C_n = \oplus_{g \in G} {}^gC_n^g$.

Proof. Lemma 2.8 implies that $C_n = KG + \sum_{g \in G} ({}^gC_n^g)^+$. For each $g \in G$, $g \in {}^gC_n^g$. Thus, $C_n = \sum_{g \in G} {}^gC_n^g$. It then follows that the ${}^gC_n^g$ are the irreducible components of C_n. [Mon93, 5.6.3] implies that the sum is direct. $\square$

This gives us a strengthened version of Theorem 2.7 for cocommutative coalgebras.

Theorem 2.10. Let C be a cocommutative coalgebra. For each $g \in G$, let $\bar{P}_{g,g}^{(n)}(C)$ be a vector space complement of C_{n-2} in $P_{g,g}^{(n)}(C)$. Then $C_n = C_{n-2} \oplus (\oplus_{g \in G} \bar{P}_{g,g}^{(n)}(C))$.

Proof. By Lemma 2.5 and Lemma 2.8, we have $P_{g,g}^{(n)}(C) \cap C_{n-1} \subseteq C_{n-2}$. The result then follows from Theorem 2.7. $\square$

3 Extensions of the Base Field for Duals of Field Extensions

Given a K-coalgebra C, one can extend the base field to an extension $K \subseteq E$. The resulting E-coalgebra is $E \otimes_K C$ (see proof of [Mon93, 2.2.2]). In this section, we consider how the coalgebra structure of C is affected by extension of the base field.

This question was considered in [Par01]. In particular, we obtained the following result.

Theorem 3.1. Let H be a K-coalgebra, and suppose $K \subseteq L$ is an extension of fields. Then the following are equivalent.

(i) $L \otimes H$ is a grouplike coalgebra.

(ii) H is cocommutative, cosemisimple with separable coradical, and L contains the normal closure of D^* for each simple subcoalgebra $D \subseteq H$.

We will prove an analogous result in the case that the simple subcoalgebras are not separable. One of the consequences of Theorem 3.1 is the following.

Corollary 3.2. Let H be a cocommutative coalgebra, and suppose that $K \subseteq L$ is such that $L \otimes H$ is pointed (e.g. $L = \bar{K}$). Let $\{H_n\}_{n=0}^{\infty}$ be the coradical filtration of H.

(i) $[L \otimes H]_n \subseteq L \otimes H_n$ for all $n \geq 0$.

(ii) Equality holds for all $n \geq 0$ if and only if H has separable coradical.

As in the proof of Theorem 3.1, we take the view of finite-dimensional coalgebras as duals of associative algebras. In fact, we take this a step further. Let A be a finite-dimensional K-algebra, and let $K \subseteq E$ be a field extension. If we take a basis $\{a_1, \cdots, a_n\}$ of A with dual basis $\{a_1^*, \cdots, a_n^*\} \subseteq A^*$, then we have an E-vector space isomorphism $\phi : Hom_K(A, E) \to E \otimes A^*$ given by $\phi(f) = \sum_i f(a_i) \otimes a_i^*$. Note that this map is independent of the choice of basis.

We use ϕ to give $Hom_K(A, E)$ a comultiplication. We can identify $Hom_K(A, E) \otimes_E Hom_K(A, E)$ with $(E \otimes A \otimes A)^*$. Then $\Delta(f)(\alpha \otimes a \otimes b) = \alpha f(ab)$ gives a comultiplication which is coassociative when A is associative. If A is associative, then $\varepsilon(f) = f(1)$ makes $Hom_K(A, E)$ an E-coalgebra.

Lemma 3.3. ϕ is an E-coalgebra isomorphism.

Proof. We have $a_i a_j = \sum_k \alpha_{ijk} a_k$ for some $\alpha_{ijk} \in K$. Then $\Delta(a_k^*) = \sum_{i,j} \alpha_{ijk} a_i^* \otimes a_j^*$. We need only prove that ϕ is a coalgebra morphism. Suppose $\Delta(f) = \sum f_1 \otimes_E f_2$, so $f(ab) = \sum f_1(a) f_2(b)$ for all $a, b \in A$. Then

$$\begin{aligned}\sum \phi(f_1) \otimes_E \phi(f_2) &= \sum_{i,j} (f_1(a_i) \otimes a_i^*) \otimes_E (f_2(a_j) \otimes a_j^*) \\ &= \sum_{i,j} f_1(a_i) f_2(a_j) \otimes a_i^* \otimes a_j^* = \sum_{i,j} f(a_i a_j) \otimes a_i^* \otimes a_j^* \\ &= \sum_{i,j,k} \alpha_{ijk} f(a_k) \otimes a_i \otimes a_j = \Delta(\phi(f))\end{aligned}$$

We also have $\varepsilon(\phi(f)) = \sum_i \varepsilon(a_i^*) f(a_i) = f(\sum_i a_i^*(1) a_i) = f(1) = \varepsilon(f)$, which completes the proof.

□

Let $\mathcal{G} = Alg_K(A, E)$. Then $\mathcal{G} = G(Hom_K(A, E))$. This, along with Lemma 3.3 gives us

Corollary 3.4. If A is a finite-dimensional associative algebra, and E is a field, then

(*i*) $G(E \otimes A^*) = \{\sum_i \sigma(a_i) \otimes a_i^* : \sigma \in \mathcal{G}\}$. Thus, $|G(E \otimes A^*)| = |Alg(A, E)|$.

(*ii*) Let $\sigma, \tau \in \mathcal{G}$. Then $\phi(P_{\sigma,\tau}^{(n)}(Hom_K(A, E))) = P_{\phi(\sigma),\phi(\tau)}^{(n)}(E \otimes A^*)$.

We also have that $Hom_K(A, E)$ is well behaved when we restrict maps to subalgebras.

Lemma 3.5. Let A be a finite-dimensional associative algebra, E be a field. Suppose that $B \subseteq A$ is a subalgebra, and let $\sigma, \tau \in \mathcal{G}$. If $f \in P_{\sigma,\tau}^{(n)}(Hom_K(A, E))$, then $f|_B \in P_{\sigma|_B,\tau|_B}^{(n)}(Hom_K(B, E))$.

Proof. Using the definition of comultiplication, it follows that $f \in P_{\sigma,\tau}^{(n)}(Hom_K(A, E))$ if and only if $f(ab) = \tau(a) f(b) + f(a)\sigma(b) + w(a \otimes b)$ for all $a, b \in A$, where $w \in \sum_{\kappa \in Alg_K(A,E)} P_{\sigma,\kappa}^{(n-1)}(Hom_K(A, E)) \otimes P_{\kappa,\tau}^{(n-1)}(Hom_K(A, E))$ for all $a, b \in A$. But when we restrict to B, this will still hold for $a, b \in B$. In addition, when we restrict to B, we will have $w \in \sum_{\kappa \in Alg_K(A,E)} P_{\sigma|_B,\kappa|_B}^{(n-1)}(Hom_K(B, E)) \otimes P_{\kappa|_B,\tau|_B}^{(n-1)}(Hom_K(B, E))$ by induction, so the proof is complete. □

Let C be a cocommutative K-coalgebra, and again let $K \subseteq E$ be a field extension. If E is algebraically closed, then $E \otimes C$ is pointed by [Mon93, 5.6]. We can be more specific about how much we need to extend the field. We first need the following lemma.

Lemma 3.6. Let $K \subseteq L$ be a finite field extension, and let $L \subseteq E$ be any field extension. Then $G(E \otimes L^*) \neq \emptyset$.

Proof. Let $\{a_1, \cdots, a_n\}$ be a basis for L over K. Then $\sum_i a_i \otimes a_i^* \in G(E \otimes L^*)$ by Corollary 3.4(i) (letting σ be the inclusion map). $\square$

We then get the following analogue of Theorem 3.1.

Theorem 3.7. Let C be a cocommutative coalgebra over K, and let $K \subseteq E$ be a field extension. Then $E \otimes C$ is pointed if and only if, for each simple subcoalgebra $D \subseteq C$, E contains the normal closure of the field D^* over K.

Proof. We first reduce the theorem to the case of C being a simple coalgebra. By Corollary 3.2, we have $(E \otimes C)_0 \subseteq E \otimes C_0$, so $(E \otimes C)_0 \subseteq (E \otimes C_0)_0$. In addition, $E \otimes C_0 \subseteq E \otimes C$, so $(E \otimes C_0)_0 \subseteq (E \otimes C)_0$. Thus, $(E \otimes C)_0 = (E \otimes C_0)_0$, and so $E \otimes C$ is pointed if and only if $E \otimes C_0$ is pointed. Now $C_0 = \oplus_i D_i$, where each D_i is a simple subcoalgebra. This gives us $E \otimes C_0 = \oplus_i E \otimes D_i$, so $E \otimes C_0$ is pointed if and only if each $E \otimes D_i$ is pointed. Thus, without loss of generality, C is simple. In particular, [Mon93, 5.1.4(3)] implies that C is finite-dimensional.

Without loss of generality, $K \subseteq E$ is an algebraic extension. Suppose that E contains the normal closure of C^* (note that C^* is a field since C is simple and cocommutative). If $\bar{K}$ is an algebraic closure of K containing E, then $\bar{K} \otimes C$ is a pointed coalgebra. Let $\sigma \in Alg(C^*, \bar{K})$. Since C^* is a field, we have $\sigma(C^*) \cong C^*$. Thus, $\sigma(C^*) \subseteq E$, and so $Alg(C^*, \bar{K}) = Alg(C^*, E)$. By Lemma 3.4, $|G(\bar{K} \otimes C)| = |Alg(C^*, \bar{K})| = |Alg(C^*, E)| = |G(E \otimes C)|$.

Suppose that $E \otimes C$ is not pointed. Then there exists a simple coalgebra $D \subseteq E \otimes C$ with $D \cap E(G(E \otimes C)) = 0$. Since D is a simple cocommutative E-coalgebra, then $D \cong F^* = Hom_E(F, E)$ for some field extension $E \subseteq F$. By Lemma 3.6, there is some $g \in G(\bar{K} \otimes_E D)$. Since $D \cap E(G(E \otimes C)) = 0$, we have $g \notin E(G(E \otimes C))$. This implies that $|G(\bar{K} \otimes C)| > |G(E \otimes C)|$, which is a contradiction. Thus, $E \otimes C$ is pointed.

Conversely, suppose that E does not contain the normal closure of C^*. Then there is some $\sigma \in Alg_K(C^*, \bar{K})$ such that $\sigma(C^*) \nsubseteq E$. Then Corollary 3.4 implies that $|G(\bar{K} \otimes C)| > |G(E \otimes C)|$. But since $\bar{K} \otimes C \cong \bar{K} \otimes_E (E \otimes C)$ is pointed, Corollary 3.2(i) implies that $(\bar{K} \otimes C)_0 \subseteq \bar{K} \otimes_E (E \otimes C)_0$. Since $\bar{K} \otimes C$ has more grouplikes than $E \otimes C$, $E \otimes C$ must have a simple subcoalgebra that is not grouplike. $\square$

Let $K \subseteq L$ be a finite field extension, $K \subseteq E$ any field extension. We now study the coradical filtration of $E \otimes L^*$. Note that while L^* is a simple coalgebra, it is possible that $E \otimes L^*$ is not if $K \subseteq L$ is an inseparable field extension (see Corollary 3.2(ii)). Since $E \otimes L^* \cong Hom_K(L, E)$, we can study $Hom_K(L, E)$ when convenient.

Lemma 3.8. Let $K \subseteq L$ be a finite field extension, $K \subseteq E$ any field extension.

(i) If $K \subseteq L$ is a separable extension, then $Hom_K(L, E)$ is cosemisimple.

(ii) If E contains the normal closure of L, then $Hom_K(L, E)$ is pointed.

Proof. These follow directly from Theorem 3.1 and Theorem 3.7, respectively. $\square$

Now suppose that F is the normal closure of L over K, and let L_s be the separable closure of K in L. Let $\{\beta_1, \cdots, \beta_r\}$ be a basis for L over L_s, and let $\{\beta_1^*, \cdots, \beta_r^*\}$ be the dual basis. We have that $H = Hom_{L_s}(L, L_s)$ is an L_s-coalgebra. Note that $F \otimes_{L_s} H$ is connected by Corollary 3.4 and Theorem 3.7, since $L_s \subseteq L$ is purely inseparable. Also note that $G(F \otimes_{L_s} H) = Ke$, where $e = \sum_i \beta_i \otimes \beta_i^*$.

If $c \in F \otimes_{L_s} H$, then we can write $c = \sum_i f(\beta_i) \otimes \beta_i^*$ for some $f \in Hom_{L_s}(L, F) \subseteq Hom_K(L, F)$. For each $\sigma \in Alg_K(L, F)$, we define $\Psi_\sigma : F \otimes_{L_s} H \to Hom_K(L, F)$ as follows. By a well-known theorem in field theory and the fact that $K \subseteq F$ is a normal extension, we can extend σ to a K-automorphism $\bar{\sigma} : F \to F$. Fix such an automorphism for each $\sigma \in Alg_K(L, F)$. We then define $\Psi_\sigma(\sum_i f(\beta_i) \otimes \beta_i^*) = \bar{\sigma} \circ f$. One can check that Ψ_σ is an F-coalgebra monomorphism.

Lemma 3.9. Let e, σ, L, and F be as above.

(i) $\Psi_\sigma(e) = \sigma$.

(ii) $\Psi_\sigma(P_{e,e}^{(n)}(F \otimes_{L_s} H)) \subseteq P_{\sigma,\sigma}^{(n)}(Hom_K(L, F))$

(iii) $P_{\sigma,\sigma}^{(n)}(Hom_K(L, F)) = F\mathcal{G}^+ \oplus \Psi_\sigma(P_{e,e}^{(n)}(F \otimes_{L_s} H))$, for $n \geq 2$, and $P_{\sigma,\sigma}(Hom_K(L, F)) = \Psi_\sigma(P_{e,e}(F \otimes_{L_s} H))$.

Proof. Part (i) follows directly. Part (ii) follows from a simple induction argument using (i) and the fact that Ψ_σ is a coalgebra morphism. For (iii), let $f \in P_{\sigma,\sigma}^{(n)}(Hom_K(L, F))$. Then $f|_{L_s} \in P_{\sigma|_{L_s},\sigma|_{L_s}}^{(n)}(Hom_K(L_s, F))$. By Lemma 3.8($i$), $f|_{L_s} \in E(Alg(L_s, F))$. By extending each element of $Alg(L_s, F)$ to L, we can consider $f|_{L_s} = h \in F\mathcal{G}$. If we let $f' = f - h$, then $f'(L_s) = 0$, and so $\bar{\sigma}^{-1} \circ f' \in Hom_{L_s}(L, F)$. Note that $\Psi_\sigma(\sum_i \bar{\sigma}^{-1}(f'(\beta_i)) \otimes \beta_i^*) = f'$, so it suffices to show that $\sum_i \bar{\sigma}^{-1}(f'(\beta_i)) \otimes \beta_i^* \in P_{e,e}^{(n)}(F \otimes_{L_s} H)$. This follows from the fact that $\Psi_\sigma^{-1} : \Psi_\sigma(F \otimes_{L_s} H) \to F \otimes_{L_s} H$ is a coalgebra isomorphism. $\square$

Applying Theorem 2.10 and Lemma 3.9 gives us the following.

Lemma 3.10. $(Hom_K(L, F))_n = F\mathcal{G} \oplus (\oplus_{\sigma \in \mathcal{G}} \Psi_\sigma(P_{e,e}^{(n)}(F \otimes_{L_s} H)))$.

This tells us a great deal about the coradical filtration of $E \otimes L^*$ when E contains the normal closure of L over K.

Theorem 3.11. Let $K \subseteq L$ be a finite field extension, and let E be a field containing the normal closure of L over K. Then $(E \otimes L^*)_n = \sum_{g \in G(E \otimes L^*)} C_g$, where each $C_g \cong (E \otimes_{L_s} H)_n$.

Proof. First note that we can identify $E \otimes L^*$ with $Hom_K(L, E)$. The case of $E = F$ then follows from Lemma 3.10, letting $C_g = Eg + \Psi_g(P^{(n)}_{e,e}(E \otimes_{L_s} H))$. For the general case, we have, by Corollary 3.2(*ii*) and Theorem 3.7,

$$(E \otimes L^*)_n = (E \otimes_F [F \otimes L^*])_n = E \otimes_F (F \otimes L^*)_n$$

The result follows directly. □

Thus, the coalgebra structure of $E \otimes L^*$, including the coradical filtration, is completely determined by that of $E \otimes_{L_s} H$. The following is immediate.

Corollary 3.12. Let $n = |L : L_s|$. Then $E \otimes L^* = (E \otimes L^*)_n$.

4 Duals of Purely Inseparable Field Extensions

The thrust of Corollary 3.11 is that we can understand the coalgebra structure of duals of field extensions when their base fields are extended as long as we understand the coalgebra structure of duals of purely inseparable field extensions under an extension of the base field.

Let $\text{char}(K) = p > 0$, and suppose that $K \subseteq L$ is a purely inseparable finite field extension. Let $L \subseteq E$ be a field extension. Then $E \otimes L^*$ is connected, with grouplike element g. We then have $(E \otimes L^*)_n = Eg \oplus P_n$, where $P_n = P^{(n)}_{g,g}(E \otimes L^*)$.

Corollary 3.12 gives an n such that $(E \otimes L^*)_n = E \otimes L^*$. One may ask what the minimum such n is. We shall address this question in this section. We begin with the following lemma.

Lemma 4.1. For all $n \geq 1$, $\dim_E(P_{n+1}/P_n) \leq \dim_E(P_n/P_{n-1})^2$.

Proof. Since P_{n-1} is a coideal for each $n \geq 1$, we have that $(E \otimes L^*)/P_{n-1}$ is a coalgebra. We define the linear map $f : P_{n+1}/P_{n-1} \to (P_n/P_{n-1}) \otimes (P_n/P_{n-1})$ by

$$f(c) = \Delta(c) - c \otimes (g + P_{n-1}) - (g + P_{n-1}) \otimes c$$

Since $\ker(f) = P_n/P_{n-1}$ and $im(f) \subseteq (P_n/P_{n-1}) \otimes (P_n/P_{n-1})$, we have $\dim_E(P_{n+1}/P_{n-1}) \leq \dim_E(P_n/P_{n-1}) + \dim_E(P_n/P_{n-1})^2$. Then $\dim_E(P_{n+1}/P_{n-1}) = \dim_E(P_{n+1}/P_n) + \dim_E(P_n/P_{n-1})$ gives us the result. □

By [Jac64, p. 182], there exists a p-basis of L over K. That is, there exist $\alpha_1, \cdots, \alpha_k \in L$ such that, for some $f_i > 0$, $\{\alpha_1^{e_1} \cdots \alpha_k^{e_k} : 0 \leq e_i < p^{f_i}\}$ is a basis for L over K. We can then define derivations $d_i \in Der_K(L, E)$ such that $d_i(\alpha_j) = \delta_{i,j}$. Let $\mathfrak{g} = K$-span$\{d_1, \cdots d_k\}$. Then $Der_K(L, E) = E \otimes \mathfrak{g}$. Since $\phi(Der_K(L, E)) = P_1$, we have that $\dim_K(P_1) = k$.

Corollary 4.2. Let k be the size of a p-basis of L over K.
(*i*) For all $n \geq 1$, $\dim_E(P_{n+1}/P_n) \leq k^{2^n}$.
(*ii*) For all $n \geq 0$, $\dim_E([E \otimes L^*]_n) \leq 1 + \sum_{i=0}^{n-1} k^{2^i}$.

Proof. For (i), we have $\dim_K(P_1/P_0) = \dim_K(P_1) = k$ by the above remarks, which gives us $n = 1$. For $n > 1$, we have $\dim_E(P_{n+1}/P_n) \leq \dim_E(P_n/P_{n-1})^2$ by Lemma 4.1. By induction, $\dim_E(P_n/P_{n-1})^2 \leq (k^{2^{n-1}})^2 = k^{2^n}$, and the result follows. Part (ii) follows directly from (i). $\square$

Since $\dim_E(E \otimes L^*) \geq p^k$, the following is immediate.

Corollary 4.3. If $E \otimes L^* = (E \otimes L^*)_n$ and k is the size of a p-basis of L over K, then $p^k \leq 1 + \sum_{i=0}^{n-1} k^{2^i}$.

This leads to some very restrictive conditions for $E \otimes L^* = (E \otimes L^*)_1$.

Corollary 4.4. Let $K \subseteq L$ be a purely inseparable extension. Then $E \otimes L^* = (E \otimes L^*)_1$ if and only if either $K = L$ or $\mathrm{char}(K) = |L : K| = 2$.

Proof. If $K = L$, then $E \otimes L^* = (E \otimes L^*)_0 \subseteq (E \otimes L^*)_1$. In the case that $K \neq L$, we have $k > 0$. By Corollary 4.3(ii), we have $p^k \leq 1 + k$. But this can only happen if $p = 2$ and $k = 1$, in which case we have equality. This forces $|L : K| = 2$. Since $\mathrm{char}(K) = p = 2$, this completes the proof. $\square$

We get corresponding results for the case $K \subseteq L$ is not purely inseparable, using Corollary 3.11.

Theorem 4.5. Let $K \subseteq L$ be a field extension with $\mathrm{char}(K) = p > 0$, and let L_s be the separable closure of K in L. Suppose E contains the normal closure of L over K. For each $g \in G(E \otimes L^*)$ and $n \geq 0$, let $\bar{P}_{g,g}^{(n)}$ be a vector space complement of KG^+ in $P_{g,g}^{(n)}(E \otimes L^*)$ such that $\bar{P}_{g,g}^{(n)} \subseteq \bar{P}_{g,g}^{(n+1)}$. Finally, let k be the size of a p-basis for L/L_s. Then

(i) For all $n \geq 1$, $\dim_E(\bar{P}_{g,g}^{(n+1)}/\bar{P}_{g,g}^{(n)}) \leq \dim_E(\bar{P}_{g,g}^{(n)}/\bar{P}_{g,g}^{(n-1)})^2$.
(ii) For all $n \geq 1$, $\dim_E(\bar{P}_{g,g}^{(n+1)}/\bar{P}_{g,g}^{(n)}) \leq k^{2^n}$.
(iii) For all $n \geq 0$, $\dim_E([E \otimes L^*]_n) \leq |G|(1 + \sum_{i=0}^{n-1} k^{2^i})$.
(iv) If $E \otimes L^* = (E \otimes L^*)_n$, then $p^k \leq 1 + \sum_{i=0}^{n-1} k^{2^i}$.
(v) $E \otimes L^* = (E \otimes L^*)_1$ if and only if either L is separable or $\mathrm{char}(K) = |L : L_s| = 2$.

Proof. These follow directly from Corollary 3.11, Lemma 4.1, Corollary 4.2, Corollary 4.3, and Corollary 4.4. $\square$

5 A Generalization of (σ, τ)-Derivations

Recall that if A and B are K-algebras (not necessarily associative), and $\sigma, \tau : A \to B$ are K-algebra homomorphisms, a (σ, τ)-derivation is a map $d : A \to B$ such that, for all $a, b \in A$, we have $d(ab) = \sigma(a)d(b) + d(a)\tau(b)$. There is a duality between derivations and primitive elements. In fact, if A is an associative algebra and B is a field, then Corollary 3.4 implies that $\sigma, \tau \in G(Hom_K(A, B))$ and $P_{\sigma,\tau}(Hom_K(A, B))$ consists of the (τ, σ)-derivations between A and B.

Since $P_{g,h}(C) = P^{(1)}_{g,h}(C)$, it makes sense to generalize (σ,τ)-derivations to the n^{th} level of the coradical filtration. Indeed, since the definition of $P^{(n)}_{g,h}(C)$ depends only on comultiplication in $Hom_K(A,B)$, we can define such maps when B is not a field, or even when A and B are not associative.

Definition 5.1. Let A and B be K-algebras, let $\mathcal{G}$ be the set of algebra homomorphisms $A \to B$, and let $\sigma, \tau \in \mathcal{G}$. The set of (σ,τ)-derivations of degree n (denoted $D^{(n)}_{\sigma,\tau}(A,B)$) can be defined inductively as follows.

$$\begin{aligned} &D^{(0)}_{\sigma,\tau}(A,B) = 0,\\ &D^{(n)}_{\sigma,\tau}(A,B) = \{f : A \to B : \forall a, b \in A, f(ab) = \sigma(a)f(b) + f(a)\tau(b)\\ &\qquad\qquad + w(a \otimes b),\ \text{where } w \in \sum_{\kappa \in \mathcal{G}} D^{(n-1)}_{\sigma,\kappa}(A,B) \otimes D^{(n-1)}_{\kappa,\tau}(A,B)\} \end{aligned}$$

Note that we do not require A to be finite-dimensional. An easy induction gives us the following.

Proposition 5.2. If A is an associative finite-dimensional K-algebra, and E is a field, then $D^{(n)}_{\sigma,\tau}(A,E) = P^{(n)}_{\tau,\sigma}(Hom_K(A,E))$.

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