Two-Path Convexity in Clone-Free Regular Multipartite Tournaments

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Abstract

We present some results on two-path convexity in clone-free regular multipartite tournaments. After proving a structural result for regular multipartite tournaments with convexly independent sets of a given size, we determine tight upper bounds for their size (called the rank) in clone-free regular bipartite and tripartite tournaments. We use this to determine tight upper bounds for the Helly and Radon number in the bipartite case. We also derive an upper bound for the rank of a general clone-free regular multipartite tournament.

1 Introduction

Convexity has been studied in many contexts. These contexts have been generalized to the concept of a *convexity space*, which is a pair $\mathcal{C} = (V, C)$, where V is a set and C is a collection of subsets of V such that $\emptyset, V \in C$ and such that C is closed under arbitrary intersections and nested unions. The set C is called the set of *convex subsets of* $\mathcal{C}$. Given a subset $S \subseteq V$, the *convex hull of* S, denoted $C(S)$, is defined to be the smallest convex subset containing S.

In the case of graphs and digraphs, V is usually taken to be the vertex set and C to be a collection of vertex subsets that are determined by paths within the graph. For a (directed) graph $T = (V, E)$ and a set $\mathcal{P}$ of (directed) paths in T, a subset $A \subseteq V$ is called $\mathcal{P}$*-convex* if, whenever $v, w \in A$, any (directed) path in $\mathcal{P}$ that originates at v and ends at w can involve only vertices in A. We denote the collection of convex subsets of T by $\mathcal{C}(T)$.

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In the case $\mathcal{P}$ is the set of geodesics in T, we get *geodesic convexity*, which was introduced in undirected graphs by F. Harary and J. Nieminen in [HN81]. Geodesic convexity was also studied in [CFZ02] and [CCZ01]. When $\mathcal{P}$ is the set of all chordless paths, we get *induced path convexity* (see [Duc88]). Other types of convexity include *path convexity* (see [Pfa71] and [Nie81]), *two-path convexity* (see [Var76], [EFHM72], [EHM72], and [Moo72]) and *triangle path convexity* (see [CM99]).

All work in tournaments has been in *two-path convexity*, where $\mathcal{P}$ is the set of all 2-paths. This is natural, as J. Varlet noted in [Var76], since if all directed paths are allowed, then the only convex subsets of strong tournaments are V and $\emptyset$. Indeed, this is true even when all paths of length three or less are allowed. Our interest is in two-path convexity in multipartite tournaments, and henceforth all references to convexity will be two-path convexity.

A *convexly independent set* is a set $F \subseteq V$ such that $x \notin C(F - \{x\})$ for all $x \in F$. The *rank* $d(T)$ is the maximum size of a convexly independent set. The rank is an upper bound for the Helly number $h(T)$, the Radon number $r(T)$, and Caratheodory number $c(T)$, which are the most important convex invariants in convexity theory (see [vdV93, Ch. 2]). In [PWWa], we proved that $h(T) = r(T) = d(T)$ when T is a clone-free bipartite tournament. These convexity invariants were also studied in [PWWb]

In [PWWb], it was shown that a convexly independent set can have a nonempty intersection with at most two partite sets. Thus, we say that vertex sets A and B *form a convexly independent set* if $A \cup B$ is convexly independent and A and B are subsets of distinct partite sets. If this is the case, we must have either $A \to B$ or $B \to A$.

In this paper, we look at convexity in clone-free regular multipartite tournaments. In Section 2, we consider the structure of clone-free regular multipartite tournaments. The main result of this section is Theorem 2.8, which describes the orientation of the arcs between vertices in convexly independent sets and their distinguishing vertices.

Our other results center around determining upper bounds for the rank of clone-free regular multipartite tournaments. We study the bipartite case in Section 3, obtaining the bounds given in Theorem 3.3. We prove that these bounds are tight in Theorem 4.4 and use this result to obtain tight upper bounds for the Helly and Radon numbers. We then derive tight bounds for rank in the tripartite case (Theorem 5.3) and obtain bounds for rank in the p-partite case with $p \geq 3$ (Theorem 5.4). The latter bound is not tight for $p = 3$, and it is unknown whether it is tight for any $p \geq 4$.

Let $T = (V, E)$ be a digraph with V the vertex set and E the arc set. We denote an arc $(v, w) \in E$ by $v \to w$ and say that v dominates w. If $U, W \subseteq V$, then we write $U \to W$ to indicate that every vertex in U dominates every vertex in W. We denote by T^* the digraph with the same vertex set as T, and where (v, w) is an arc of T^* if and only if (w, v) is an arc of T. Recall that T is a p-partite tournament if one can partition V into p partite sets such that every two vertices in different partite sets have precisely one arc between them and no arcs exist between vertices in the same partite set. If $p = 2$, then T is a bipartite tournament. If $A, B \in \mathcal{C}(T)$, we define $A \vee B$ to be the convex hull of $A \cup B$. We write $v \vee w$ instead of $\{v\} \vee \{w\}$ for convenience when $v, w \in V$.

For each $v \in V$, the *outset of* v is $N^+(v) = \{w \in T : v \to w\}$ and the *inset of* v is

$N^-(v) = \{w \in T : w \to v\}$. T is *regular* if there is some integer r such that, for every $v \in T$, $|N^+(v)| = |N^-(v)| = r$. An immediate consequence of regularity in the case of multipartite tournaments is that each partite set must have the same number of vertices. Moreover, if there are an even number of partite sets, then each partite set must have an even number of vertices. This holds, in particular, for bipartite tournaments.

Two vertices u and v are said to be *clones* if $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$. A vertex w is said to *distinguish* the vertices x and y if $x \to w \to y$ or $y \to w \to x$. Thus, two vertices are clones if and only if there is no arc between them and no vertex distinguishes them. In particular, in a multipartite tournament, any two vertices that are clones must be in the same partite set. A digraph is said to be *clone-free* if it has no clones.

To facilitate our study of the rank of bipartite tournaments, it will be helpful to study their adjacency matrices. In the case of a bipartite tournament, however, the adjacency matrix can be represented more compactly. Let $\{v_1, \cdots, v_k\}$ and $\{w_1, \cdots, w_\ell\}$ be the partite sets of T for T a bipartite tournament. For each i and j with $1 \le i \le k$ and $1 \le j \le \ell$, let $m_{i,j} = 1$ if $v_i \to w_j$ and let $m_{i,j} = 0$ otherwise. We will call $M = (m_{i,j})$ the *matrix of* T, and we say that T is the *bipartite tournament induced by* M. If $S \subseteq V$ and N is the matrix of the bipartite tournament induced by S, we say that S induces N. Notice that v_i distinguishes w_j and w_k if and only if $m_{i,j} \ne m_{i,k}$ and w_i distinguishes v_j and v_k if and only if $m_{j,i} \ne m_{k,i}$. In addition, identical rows or columns of the matrix of T correspond to clones.

2 Convexly Independent Sets and Their Distinguishers

In clone-free multipartite tournaments, the vertices that distinguish vertices in convexly independent sets are very important. Let $C \subseteq V$. We define

$$D_C^{\rightarrow} = \{z \in V : z \to x \text{ for some } x \in C, y \to z \text{ for all } y \in C - \{x\}\}$$
$$D_C^{\leftarrow} = \{z \in V : z \leftarrow x \text{ for some } x \in C, z \to y \text{ for all } y \in C - \{x\}\}$$

In order to derive upper bounds for the rank of clone-free regular multipartite tournaments, we need to get a handle on the structure of such multipartite tournaments. We make extensive use of the following results, which also appear in [PWWb].

Theorem 2.1. Let T be a clone-free multipartite tournament. Let A and B form a convexly independent set, with $A \to B$ when both sets are nonempty.

1. If $A = \{x_1, \cdots, x_m\}$, $m \ge 2$, then one can order the vertices in A such that there exist $u_2, \cdots, u_m \in D_A^{\rightarrow}$ (resp., in $D_A^{\leftarrow}$ if $D_A^{\rightarrow} = \emptyset$) such that $u_i \to x_i$ (resp., $x_i \to u_i$).

2. If $|A| \ge 3$, then $D_A^{\rightarrow} \ne \emptyset$ if and only if $D_A^{\leftarrow} = \emptyset$, and $D_A^{\rightarrow}$ and $D_A^{\leftarrow}$ each lie in at most one partite set.

3. Suppose $A, B \neq \emptyset$. If $|A| \geq 2$, then $D_A^{\rightarrow}$ is in the same partite set as B, and if $|B| \geq 2$, then $D_B^{\leftarrow}$ is in the same partite set as A.

4. If $|A|, |B| \geq 2$, then $D_B^{\leftarrow} \rightarrow D_A^{\rightarrow}$.

5. Any vertex that distinguishes vertices in A must be in either $D_A^{\rightarrow}$ or $D_A^{\leftarrow}$ and any vertex that distinguishes vertices in B must be in $D_B^{\leftarrow}$ or $D_B^{\rightarrow}$. If $A, B \neq \emptyset$, then any vertex that distinguishes vertices in A must be in $D_A^{\rightarrow}$ and any vertex that distinguishes vertices in B must be in $D_B^{\leftarrow}$.

Proof. We begin with (1). If $m = 2$, let u_2 be any vertex distinguishing x_1 and x_2. By relabelling x_1 and x_2, if necessary, we have $x_1 \rightarrow u_2 \rightarrow x_2$. If $m = 3$, let u_2 distinguish x_1 and x_2. By relabelling and considering T^*, if necessary, we may assume $x_1 \rightarrow u_2 \rightarrow x_2$, and that $x_3 \rightarrow u_2$. Since T is clone-free, there is some u_3 that distinguishes x_1 and x_3. By switching x_1 and x_3 if necessary, we may assume that $x_1 \rightarrow u_3 \rightarrow x_3$. It suffices to show that $x_2 \rightarrow u_3$. If $u_3 \rightarrow x_2$, then $x_1 \rightarrow y_2 \rightarrow x_2$ and $x_1 \rightarrow u_3 \rightarrow x_2$, so $u_2, u_3 \in x_1 \vee x_2$. But then $u_3 \rightarrow x_3 \rightarrow u_2$, so $x_3 \in x_1 \vee x_2$, a contradiction. Thus, $x_2 \rightarrow y_3$.

Now assume the result for $r = m \geq 3$. For $r = m + 1$, we know there exist $y_2, \cdots, y_m$ such that $y_i \rightarrow x_i$ for all $2 \leq i \leq m$ and $x_i \rightarrow y_j$ for all $i \neq j$. It is easy to see that $x_i \vee x_j = y_i \vee y_j$ for all $2 \leq i \neq j \leq m$.

For the inductive step, we need to find $y_{m+1} \in D_A^{\rightarrow}$ with $y_{m+1} \rightarrow x_{m+1}$. To this end, we first show that $x_{m+1} \rightarrow y_i$ for all $i \leq m$. Suppose that $y_i \rightarrow x_{m+1}$ for some $i \leq m$. In this case, we find that $y_i \rightarrow x_{m+1}$ for all $i \leq m$. For if there is some j for which $x_{m+1} \rightarrow y_j$, then $x_{m+1} \in y_i \vee y_j = x_i \vee x_j$, contradicting convex independence. Since $m \geq 3$, there exist $y_i, y_j \rightarrow x_{m+1}, i \neq j$. We have $x_1 \rightarrow \{y_i, y_j\} \rightarrow x_m$, and so $x_i \vee x_j = y_i \vee y_j \subseteq x_1 \vee x_m$, a contradiction. Thus, $x_{m+1} \rightarrow y_i$ for all $i \leq m$. Now we just take y_{m+1} to be a vertex distinguishing x_1 and x_{m+1}. By switching x_1 and x_{m+1}, if necessary, we can assume that $x_1 \rightarrow y_{m+1} \rightarrow x_{m+1}$.

Finally, we have to show that $x_i \rightarrow y_{m+1}$ for all $2 \leq i \leq m$. If $y_{m+1} \rightarrow x_i$, then $x_{m+1} \in x_1 \vee x_i$ by arguments similar to the $r = 3$ case.

For the first part of (2), it follows from (1) that at least one of $D_A^{\rightarrow}$ and $D_A^{\leftarrow}$ is nonempty. For contradiction, let $u \in D_A^{\rightarrow}$, $v \in D_A^{\leftarrow}$. Let $x_1, x_2 \in A$ with $u \rightarrow x_1$ and $x_2 \rightarrow v$. Then $A - \{x_1\} \rightarrow u$ and $v \rightarrow A - \{x_2\}$. We have the cases $x_1 = x_2$ and $x_1 \neq x_2$. In the case $x_1 = x_2$, ignore the x_2 and then let $x_2, x_3 \in A - \{x_1\}$. In the case $x_1 \neq x_2$, let $x_3 \in A - \{x_1, x_2\}$. In either case, $u, v \in x_1 \vee x_2$. Then $v \rightarrow x_3 \rightarrow u$ implies $x_3 \in x_1 \vee x_2$, a contradiction.

For the second part of (2), suppose that $z_1, z_2 \in D_A^{\rightarrow}$ with $z_1 \rightarrow z_2$. Then there exist $x_1, x_2 \in A$ with $z_1 \rightarrow x_1$ and $z_2 \rightarrow x_2$. In the case $x_1 \neq x_2$, $|A| \geq 3$ implies that there exists some $x_3 \in A$ distinct from x_1 and x_2. By the definition of $D_A^{\rightarrow}$, we have $x_3 \rightarrow z_2$, so $x_3 \rightarrow z_2 \rightarrow x_2$, giving us $z_2 \in x_2 \vee x_3$. Similarly, we have $x_3 \rightarrow z_1 \rightarrow z_2$, and so $z_1 \in x_2 \vee x_3$. But $z_1 \rightarrow x_1 \rightarrow z_2$, so $x_1 \in x_2 \vee x_3$. This contradicts the convex independence of A. In the case of $x_1 = x_2$, again ignore the x_2 and let $x_2, x_3 \in A - \{x_1\}$. By (1), there exists, without loss of generality, $u \in D_A^{\rightarrow}$ with $u \rightarrow x_2$. Since z_1 and z_2 are in different partite sets, u must be in a partite set distinct from either z_1 or z_2, contradicting the $x_1 \neq x_2$ case.

For (3), suppose that $z \in D_A^{\rightarrow}$ with z not in the same partite set as B. Clearly, z is also not in the same partite set as A. Since $|A| \geq 2$, there exist $x_1, x_2 \in A$ such that $x_1 \to z \to x_2$. Let $y \in B$. If $z \to y$, then $x_1 \to z \to y$ and $z \to x_2 \to y$ imply $x_2 \in x_1 \vee y$, which contradicts convex independence. If instead $y \to z$, we have $z \in x_1 \vee x_2$, and so $x_2 \to y \to z$ implies $y \in x_1 \vee x_2$, which contradicts convex independence. This implies that z and y are incomparable and are thus in the same partite set. The argument for $D_B^{\leftarrow}$ is similar.

For (4), suppose that we have $z_1 \in D_A^{\rightarrow}$, $z_2 \in D_B^{\leftarrow}$ with $z_1 \to z_2$. Since $|A|, |B| \geq 2$, then there exist $x_1, x_2 \in A$, $y_1, y_2 \in B$ such that $x_1 \to z_1 \to x_2$ and $y_1 \to z_2 \to y_2$. It follows that $z_2 \in y_1 \vee y_2$. Then $x_1 \to z_1 \to z_2$ and $z_1 \to x_2 \to y_1$ imply $x_2 \in y_1 \vee y_2 \vee x_1$, a contradiction.

For the first part of (5), suppose there exists a vertex u that distinguishes two vertices in A but is neither in $D_A^{\leftarrow}$ nor $D_A^{\rightarrow}$. Then there exist $x_1, x_2, x_3, x_4 \in A$ with $\{x_1, x_2\} \to u \to \{x_3, x_4\}$. By (1), there exists, without loss of generality, $v \in D_A^{\rightarrow}$ with $v \to x_3$ (just choose v to be any vertex distinguishing x_3 and x_4 and consider T^* if necessary). We then have $x_1 \to \{u, v\} \to x_3$ and $u \to x_4 \to v$, and so $x_4 \in x_1 \vee x_3$, a contradiction.

For the second part of (5), let $y \in B$, and suppose $x_1 \to u \to x_2$ for $x_1, x_2 \in A$. If $u \to x_3$ for $x_3 \in A$, then $x_1 \to u \to x_2$ and $u \to x_3 \to y$ imply that $x_3 \in x_1 \vee x_2 \vee y$, a contradiction. Thus, $(A - \{x_2\}) \to u \to x_2$, and so $u \in D_A^{\rightarrow}$.

□

For the rest of the section, let T be a clone-free regular p-partite tournament, $p \geq 2$, with partite sets $P_1, \cdots, P_p$, each of size k. Also, let $A \subseteq P_1$ and $B \subseteq P_2$ form a convexly independent set with $|A| \geq |B|$ and $|A| \geq 2$. Choose T or T^* (which does not affect $\mathcal{C}(T)$) so that $A \to B$ when $A, B \neq \emptyset$ and $D_A^{\rightarrow} \neq \emptyset$ when $B = \emptyset$. We write $A = \{x_1, \cdots, x_m\}$, $B = \{y_1, \cdots, y_n\}$, and let $U = A \cup B$. We will primarily be interested in the bipartite tournament induced by $P_1 \cup P_2$ and by the matrix M induced by $P_1 \cup P_2$. We will always let P_1 represent the columns of M and P_2 represent the rows of M.

If we take the vertices $u_2, \cdots, u_m$ in Theorem 2.1(1) and line them up in order along the rows of M, they form all but the first row of the identity matrix I_m. To balance the insets and outsets of x_1 and x_2, there must be some vertex u_1 with $x_2 \to u_1 \to x_1$. By Theorem 2.1(5), we have $u_1 \in D_A^{\rightarrow} \subseteq P_2$, and so $x_i \to u_1$ for all $i \neq 1$, which gives us a full identity submatrix in M. If there are any additional vertices in $D_A^{\rightarrow}$, then it is easy to see that regularity and Theorem 2.1(5) demand that they form (possibly several) full identity matrices as well. A similar phenomenon occurs with B and $D_B^{\leftarrow}$ when $|B| \geq 2$. We get the following.

Lemma 2.2. If $|A| \geq 2$ (resp. $|B| \geq 2$), then the vertices in $A \cup D_A^{\rightarrow}$ (resp. $B \cup D_B^{\leftarrow}$) can be ordered so as to form a vertical (resp. horizontal) sequence of identity matrices in M.

We thus define an *identity block* of $D_A^{\rightarrow}$ as a subset $S \subseteq D_A^{\rightarrow}$ such that the matrix induced by $S \cup A$ is an identity matrix in M. An identity block of $D_B^{\leftarrow}$ is defined similarly. We get the following as a corollary.

Corollary 2.3. If $|A| \geq 2$ (resp. $|B| \geq 2$), then $D_A^{\rightarrow}$ (resp. $D_B^{\leftarrow}$) is a disjoint union of identity blocks.

Identity blocks play an important role in the construction of the matrices of clone-free regular bipartite tournaments. In fact, our main result states that if $|U| \geq 4$ or $|A| \geq 3$, then the matrix M will have the form

$$\begin{bmatrix} I_d & \cdots & I_d & 0 & 1 \\ \vdots & * & * & * & * \\ I_d & * & * & * & * \\ 0 & * & * & * & * \\ 1 & * & * & * & * \end{bmatrix} \tag{1}$$

Here, $d = |U|$, I_d is the $d \times d$ identity matrix, and the 0's and 1's are (possibly empty) blocks of 0's and 1's of appropriate sizes. The following lemma shows us how to form identity matrices in the case $B = \emptyset$.

Lemma 2.4. Let $S = \{u_1, \cdots, u_m\}$ be an identity block of $D_A^{\rightarrow}$ with $u_i \rightarrow x_i$. If $v \in P_1 - A$ with $u_i \rightarrow v \rightarrow u_j$ for some $1 \leq i, j \leq m$, then $v \rightarrow u_k$ for all $k \neq i$.

Proof. Suppose there exists some $k \neq i, j$ with $u_k \rightarrow v$. Clearly, $u_i, u_j \in x_i \vee x_j$. Since $u_i \rightarrow v \rightarrow u_j$ and $x_j \rightarrow u_k \rightarrow v$, we get $u_k \in x_i \vee x_j$. But $x_k \in u_j \vee u_k$, and so $x_k \in x_i \vee x_j$, a contradiction. □

Thus, when it is possible to distinguish vertices within an identity block of $D_A^{\rightarrow}$, we can form blocks of vertices that induce identity matrices with an identity block of $D_A^{\rightarrow}$. We get the following structure when we consider A and B individually.

Lemma 2.5. Let A and B form a convexly independent set. If $|A| \geq 2$, then the vertices of T can be ordered so that the matrix M of T has the form (1) with $d = |A|$. Moreover, this can be done so that A is represented by the columns of the upper left identity matrix. We get the same form when $|B| \geq 2$, except that B is represented by the rows of the upper left identity matrix and $d = |B|$.

Proof. Let $S = \{u_1, \cdots, u_m\} \subseteq D_A^{\rightarrow}$ such that $u_i \rightarrow x_i$. Let $x_1, \cdots, x_m$ represent the first m columns of M, and let $u_1, \cdots, u_m$ represent the first m rows of M. The identity matrices going across the top of M are formed from the vertices that distinguish vertices in S as in Lemma 2.4. The matrices going down on the left are formed from the vertices of $D_A^{\rightarrow}$ as in Lemma 2.2. The remaining blocks of 0's and 1's exist because Theorem 2.1(5) and Lemma 2.4 demand that the remaining vertices can distinguish neither vertices in A nor vertices in $D_A^{\rightarrow}$. The result for B follows similarly. □

Note that this gives us our desired result in the case $B = \emptyset$. For the case $A, B \neq \emptyset$, we first look at the arc relationship of vertices in $P_1 \cup P_2$ outside of $D_A^{\rightarrow}$ and $D_B^{\leftarrow}$ with the identity blocks of $D_A^{\rightarrow}$ and $D_B^{\leftarrow}$.

Lemma 2.6. Suppose $B \neq \emptyset$, $u \in P_1 - (A \cup D_B^{\leftarrow})$ and $v \in P_2 - (B \cup D_A^{\rightarrow})$.

1. If $|A| \geq 2$ and $v \to x_i$ for some i, then $v \to (A \cup D_{\overrightarrow{B}})$.

2. If $|B| \geq 2$ and $y_i \to u$ for some i, then $(B \cup D_{\overleftarrow{A}}) \to u$.

3. If $|A| \geq 3$, $|B| = 1$, S is an identity block of $D_{\overrightarrow{A}}$, and $y_1 \to u$, then u cannot distinguish any two vertices in S. In addition, if $u \to S$, then any vertex dominating u cannot distinguish any two vertices in A.

4. If $|A| \geq 3$ or $|A|, |B| \geq 2$, and if $u \to y_\ell$ for some ℓ, then u can be dominated by at most one vertex in each identity block of $D_{\overrightarrow{A}}$. Moreover, if $z, z' \in D_{\overrightarrow{A}}$ with $\{z, z'\} \to u$, then, for each j, $z \to x_j$ if and only if $z' \to x_j$.

Proof. For (1), the fact that $v \to A$ follows from $v \notin D_{\overrightarrow{A}}$. Suppose there exists $w \in D_{\overleftarrow{B}}$ with $w \to v$. Without loss of generality, $y_k \to w \to y_\ell$ for some k and ℓ. Consider x_j with $j \neq i$, which exists since $|A| \geq 2$. Clearly $w \in x_i \vee y_k \vee y_\ell$. Then $w \to v \to x_i$ and $v \to x_j \to y_k$ imply $x_j \in x_i \vee y_k \vee y_\ell$, a contradiction. Part (2) follows similarly.

For (3), suppose that $u_i \to u \to u_j$ for some $u_i, u_j \in S$ with $u_i \to x_i$ and $u_j \to x_j$. Since S is an identity block, $i \neq j$. Let $x_k \in A$ with $k \neq i, j$, which exists since $|A| \geq 3$. Since $x_k \to u_j \to x_j$, $y_1 \to u \to u_j$, and $x_j \to u_i \to u$, we get $u_i \in x_j \vee x_k \vee y_1$. Thus, $u_i \to x_i \to y_1$ implies $x_i \in x_j \vee x_k \vee y_1$, a contradiction. For the rest of (3), suppose that $u \to S$, that $z \to u$, and that $x_i \to z \to x_j$. Let u_i and u_j be as before. Clearly, $u_j, z \in x_i \vee x_j$. Moreover, $z \to u \to u_j$ and $x_i \to y_1 \to u$ imply $y_1 \in x_i \vee x_j$, a contradiction.

For (4), Let $u_i, u_j \in D_{\overrightarrow{A}}$ with $u_i \to x_i$, $u_j \to x_j$ and $i \neq j$. For contradiction, suppose that $\{u_i, u_j\} \to u$. In the case $|A| \geq 3$, let x_k be a third vertex in A. Then there exists some $u_k \in D_{\overrightarrow{A}}$ with $u_k \to x_k$. Clearly, we have $u_i \in x_i \vee x_k \vee y_\ell$. Then $u_i \to u \to y_\ell$, $x_i \to u_j \to u$, and $u_j \to x_j \to u_i$ imply $x_j \in x_i \vee x_k \vee y_\ell$, a contradiction. In the case $|A|, |B| \geq 2$, let $y_k \in B$ with $k \neq \ell$, and let $z_k, z_\ell \in D_{\overleftarrow{B}}$ with $y_k \to z_k$ and $y_\ell \to z_\ell$. As before, suppose $\{u_i, u_j\} \to u$. Clearly, $z_k, z_\ell \in x_i \vee y_k \vee y_\ell$. Since $z_k \to u_i \to x_i$ by Theorem 2.1(4), we get $u_i \in x_i \vee y_k \vee y_\ell$. Then $u_i \to u \to y_\ell$, $x_i \to u_j \to u$, and $u_j \to x_j \to y_k$ imply $x_j \in x_i \vee y_k \vee y_\ell$, a contradiction. For the rest of (4), if $z, z' \in D_{\overrightarrow{A}}$ with $z \to x_j$, $z' \to x_k$, $j \neq k$, then z and z' can be made a part of the same identity block. The result then follows from the first part of (4). $\square$

To simplify M, we would like to reduce our problem to one resembling the case where $B = \emptyset$. The following does just that.

Lemma 2.7. Suppose that $B \neq \emptyset$.

1. If $|A|, |B| \geq 2$, let S be any identity block of $D_{\overrightarrow{A}}$ and S' be any identity block of $D_{\overleftarrow{B}}$. Then there exist subsets $B_1, \cdots, B_r \subseteq P_2$ and $A_1, \cdots, A_s \subseteq P_1$ such that the matrices induced by $A \cup S' \cup B_i$ and $S \cup B \cup A_i$ are identity matrices and such that any vertex in $P_1 - A - S' - (\bigcup_{i=1}^{s} A_i)$ (resp. $P_2 - B - S - (\bigcup_{i=1}^{r} B_i)$) cannot distinguish any vertices in $B \cup S$ (resp. $A \cup S'$).

2. If $|A| \geq 3$ and $|B| = 1$, let S be any identity block of $D_{\overrightarrow{A}}$. Then there exists a vertex $u \in P_1$ with $B \to u \to S$, and there exist subsets $B_1, \cdots, B_r \subseteq P_2$ and

$A_1, \cdots, A_s \subseteq P_1$ such that the matrices induced by $A \cup \{u\} \cup B_i$ and $S \cup B \cup A_i$ are identity matrices and such that any vertex in $P_1 - A - \{u\} - (\bigcup_{i=1}^{s} A_i)$ (resp. $P_2 - B - S - (\bigcup_{i=1}^{r} B_i)$) cannot distinguish any vertices in $B \cup S$ (resp. $A \cup \{u\}$).

Proof. First note that, as we construct the matrix of T, we order vertices as follows. In P_1, we begin with $x_1, \cdots, x_m, w_1, \cdots, w_n$ where $w_i \in D_B^{\leftarrow}$ with $y_i \to w_i$. In P_2, we begin with $z_1, \cdots, z_m, y_1, \cdots, y_n$, where $z_i \in D_A^{\rightarrow}$ with $z_i \to x_i$. This makes the matrix induced by $A \cup S' \cup S \cup B$ an identity matrix.

For (1), let $u \in P_1$ distinguish any two vertices in $B \cup S$. It follows from Lemma 2.6(2) and (4) that v is dominated by a unique vertex in $B \cup S$. Thus, we can break up vertices in P_1 into blocks as we did in the construction of identity blocks. These are the A_i's. The remaining vertices cannot distinguish vertices in $B \cup S$ and we are done. The construction of the B_i are similar, which completes the proof of (1).

For (2), let $S = \{z_1, \cdots, z_m\}$, with $z_i \to x_i$. We have $A \to B$, so, by regularity, there exists $u \in T$ such that $y_1 \to u \to z_1$. If $u \notin P_1$, then it is easy to see that u dominates both A and D_A (otherwise y_1 makes $A \cup B$ convexly dependent). This then forces x_3 to make $A \cup B$ convexly dependent, a contradiction. Thus, $u \in P_1$. By Lemma 2.6(3), we must have $u \to S$. If we place u as the first column of M, we have, as before, $A \cup \{u\} \cup D_A^{\rightarrow} \cup B$ inducing an identity matrix. The construction of the A_i's then follows as before, using Lemma 2.4 and Lemma 2.6(3). The construction of the B_i's begins with an identity block of $D_A^{\rightarrow}$. By regularity, there must be some vertex z that dominates u and is dominated by some vertex in A. By Lemma 2.6(3), $A \to z$, which completes the B_i. The rest follows using Theorem 2.1(5) and Lemma 2.6(4). □

This gives us the main theorem of the section.

Theorem 2.8. Let $U = A \cup B$ be convexly independent with A and B in distinct partite sets and $|A| \geq |B|$. If $|U| \geq 4$ or $|A| \geq 3$ then the vertices of $P_1 \cup P_2$ can be ordered so that the matrix of T has the form given in (1). Furthermore, this can be done in such a way that A is represented by the last $|A|$ columns and that B is represented by the first $|B|$ rows of the upper left identity matrix.

Proof. Follows from Lemma 2.5 and Lemma 2.7. □

3 Upper Bounds on Rank in the Bipartite Case

We now consider rank in clone-free regular bipartite tournaments. Note that, by regularity, both partite sets have to be the same size. Moreover, each partite set must have an even number of vertices.

We will make use of Theorem 2.8, letting $d = d(T)$. Let T be a clone-free regular bipartite tournament with $2k$ vertices in each partite set. Note that, to preserve regularity, if there are b_1 identity matrices going across and b_2 identity matrices going down, the blocks of 1's are $d \times (k - b_1)$ and $(k - b_2) \times d$ and the blocks of 0's are $d \times (k - b_1(d-1))$ and $(k - b_2(d-1)) \times b$. The requirement $k - b_i(d-1) \geq 0$ gives us the following bound on $d(T)$.

Lemma 3.1. For $i = 1, 2$, $d(T) \leq \dfrac{k}{b_i} + 1$.

In the case $b_1 = b_2 = 1$, the next lemma identifies when vertices in one partite set are convexly independent.

Lemma 3.2. If $b_1 = b_2 = 1$, then the set of vertices represented by the rows of the identity matrix (resp. the columns of the identity matrix) is a convexly independent set.

Proof. Let $x_1, \cdots, x_b$ be the vertices represented by the rows of the identity matrix, and let $y_1, \cdots, y_b$ be represented by the columns of the identity matrix. It suffices to show that $x_i \notin x_1 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_b$ for each i. But it is easy to see that $x_1 \vee \cdots \vee x_{i-1} \vee x_{i+1} \vee \cdots \vee x_b = \{x_j, y_k : k, j \neq i\}$, which completes the proof. $\square$

The following theorem gives upper bounds for the rank of clone-free regular bipartite tournaments. We will show these bounds are tight in Section 4.

Theorem 3.3. Let T be a clone-free regular bipartite tournament with $2k$ vertices in each partite set, $k \geq 1$.

1. If $k \leq 2$, then $d(T) = 2$.
2. If $k \leq 6$, then $d(T) \leq 3$
3. If $k \geq 6$, then $d(T) \leq k + 1 - \sqrt{2k - 2}$.

Proof. For (1), the case of $k = 1$ is trivial. For $k = 2$, suppose that $d(T) \geq 3$. If $|A| = 3$, then Lemma 2.2 implies that the vertices of T can be ordered to form a 3×3 identity matrix. Thus, there are no more 0's in the first three rows, so the first three entries of the fourth row are 1's. This violates regularity. Otherwise, $A = \{x_1, x_2\}$ and $B = \{y_1\}$. Let $u_1, u_2 \in D_A^{\rightarrow}$ with $u_i \rightarrow x_i$, and let z be the remaining vertex in P_2. Let v_1 and v_2 be the remaining vertices in P_1. We order the vertices in P_1 by x_1, x_2, v_1, v_2 or x_1, x_2, v_2, v_1, and the vertices in P_2 by y_1, u_1, u_2, z. Then $A \rightarrow B$, the definition of $D_A^{\rightarrow}$, and regularity gives us the following matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Clearly, $u_1, u_2 \in x_1 \vee x_2$. Also, $v_1, v_2 \in x_1 \vee x_2$, since $u_1 \rightarrow v_1 \rightarrow u_2$ and $u_2 \rightarrow v_2 \rightarrow u_1$. But then $x_1 \rightarrow y_1 \rightarrow v_1$, and so $y_1 \in x_1 \vee x_2$, violating convex independence. This proves (1).

For (2) and (3), recall $d = d(T)$ and consider the case $b_i \geq 2$ for some $i = 1, 2$. By Lemma 3.1, we have $d \leq \frac{k}{b_i} + 1 \leq \frac{k}{2} + 1$, which implies $d \leq \frac{k+2}{2}$. If also $d > k + 1 - \sqrt{2k - 2}$, then we have $k + 1 - \sqrt{2k - 2} < \frac{k+2}{2}$. This only occurs when $k < 4 + 2\sqrt{2} < 7$, or $k \leq 6$. If $k \leq 5$, then $d \leq \frac{7}{2}$, so $d \leq 3$. If $k = 6$, then $d \leq 4$. We therefore need only show that there cannot be a clone-free regular bipartite tournament of rank 4 and with 12 vertices in each partite set.

Suppose, for contradiction, that there exists such a bipartite tournament. Without loss of generality, $b_1 \geq 2$. Then the matrix must have the following form, where each block is 4×4.

I_4	I_4	C'
E	$*$	D'
C	D	$*$

In each of the first four rows, there are six 0's taken up by the two I_4 matrices. Thus, each entry of C' is 1 by regularity. Also, we have $E = I_4$ or each row of E contains all 0's or all 1's by Theorem 2.8. In either case, we must have at least four rows of 1's in the first four columns, so we can arrange the rows of M so that every entry of C is 1. Let x_i and y_j be the vertices representing the ith column and jth row for $1 \leq i, j \leq 4$. If $d(T) = 4$, then we have a convexly independent set $\{u_1, u_2, u_3, u_4\}$, where $u_i = x_i$ or y_i for each i. In any case, we have $x_i, y_i \in u_1 \vee u_2 \vee u_3$ for each $1 \leq i \leq 3$. If x'_i represents the $(i+4)$th column of M for $1 \leq i \leq 4$, then $x'_i \in u_1 \vee u_2 \vee u_3$ for each $1 \leq i \leq 3$.

For any z representing a row in C, $z \notin u_1 \vee u_2 \vee u_3$, for otherwise we have $z \to x_4 \to y_1$, making $x_4 \in u_1 \vee u_2 \vee u_3$. It quickly follows that $y_4 \in u_1 \vee u_2 \vee u_3$, implying $u_4 \in u_1 \vee u_2 \vee u_3$, a contradiction. Thus, each such z must either dominate or be dominated by all x'_i for $1 \leq i \leq 3$. If z dominates all such x'_i, then $|N^+(z)| \geq 7$, violating regularity. Thus, $x'_i \to z$ for all $1 \leq i \leq 3$ and all z representing the last four rows of M. But now $|N^+(x'_i)| \geq 7$, violating regularity. Thus, $\{u_1, u_2, u_3, u_4\}$ is convexly dependent, implying $d(T) \leq 3$.

Thus, it suffices to dispose of the case $b_1 = b_2 = 1$. The matrix M of T has the following form.

	d	$k-d+1$	$k-1$
d	I_d	0	1
$k-d+1$	0	$*$	C
$k-1$	1	$*$	D

We focus our attention on C and D. First note that C and D contain all the 0's in the columns they occupy. Since D has $k-1$ rows, each column of C must have at least one

0. Columns of C that have exactly one 0 and whose unique 0's are in the same row must be identical (they have that 0 in the same entry, and the rest of the 0's in each column are in the last $k-1$ rows). Any two such columns of C represent vertices that are clones, so there are at most $k-d+1$ columns in C with precisely one 0. It follows that there are at least $d-2$ columns with at least two 0's. If q is the number of 0's in C, we must then have $q \geq k-d+1+2(d-2) = k+d-3$. In each row that M shares with C, we have accounted for d of the 0's (in the first d columns). Thus, each row of C has at most $k-d$ 0's. Since there are $k-d+1$ rows in C, this implies $q \leq (k-d)(k-d+1)$. Thus, $k+d-3 \leq (k-d)(k-d+1)$. Using $d \leq k+1$ by Lemma 3.1, this simplifies to $k \geq d+\sqrt{2d-3}$ or $d \leq k+1-\sqrt{2k-2}$. □

4 Clone-Free Regular Bipartite Tournaments of Maximum Rank

The object of this section is to show that the bounds in Theorem 3.3 are tight. It is not difficult to produce clone-free regular bipartite tournaments of rank 3 for $3 \leq k \leq 6$. Thus, it suffices to construct matrices of clone-free regular bipartite tournaments with $2k$ vertices in each partite set with rank d for each $k \geq 7$, where $d = \lfloor k+1-\sqrt{2k-2} \rfloor$. We begin with a matrix M of the following form.

$$
\begin{array}{c|c|c|c|}
 & d & k-d+1 & k-1 \\
\hline
d & I_d & 0 & 1 \\
\hline
k-d+1 & 0 & 1 & C \\
\hline
k-1 & 1 & C^T & D \\
\hline
\end{array}
$$

Note that Lemma 3.2 implies that the rank of the bipartite tournament T induced by the matrix is at least d. It suffices to construct the submatrices C and D so that no two rows and no two columns are identical and so that the number of 0's and 1's in each row and column of M are equal. Notice that the first d rows and columns are already distinct.

Our efforts begin with C. To maintain regularity, every row of C must have $k-d$ 0's. There are $k-d+1$ rows, which gives us a total of $(k-d)(k-d+1)$ 0's in C. Since there are $k-1$ columns, there are, on average $s = \frac{(k-d)(k-d+1)}{k-1}$ 0's in each column. This number is important in the construction of C, and the following shows that s can only take on a small range of values.

Lemma 4.1. $2 - \dfrac{\sqrt{2k-2}}{k-1} \leq s < 2 + \dfrac{\sqrt{2k-2}}{k-1}$.

Proof. Since $d = \lfloor k+1-\sqrt{2k-2} \rfloor$, we have $k-\sqrt{2k-2} < d \le k+1-\sqrt{2k-2}$, and thus $-1+\sqrt{2k-2} \le k-d < \sqrt{2k-2}$ and $\sqrt{2k-2} \le k-d+1 < 1+\sqrt{2k-2}$. The result follows from multiplying these two expressions together and dividing all sides by $k-1$. $\square$

We construct C so that two properties hold. First, no two rows of C will be identical. This ensures that the vertices represented by these rows (and, consequently, the vertices represented by the corresponding columns of C^T) are not clones. Second, the 0's in C are distributed as evenly as possible among the columns. Since Lemma 4.1 implies that s is quite close to 2, this forces most columns of C to have two 0's in them.

Define $r = (k-d)(k-d+1) - 2(k-1)$. By Lemma 4.1, $|r| \le \sqrt{2k-2}$. From another perspective, $(k-d)(k-d+1) = 2(k-1)+r$ is the number of 0's in C. Therefore, if we require the 0's of C to be distributed as evenly as possible among its columns, then there are precisely $|r|$ columns with three 0's each if $s \ge 2$ and precisely $|r|$ columns with one 0 each if $s < 2$. In either case, the remaining $k-|r|-1$ columns have two 0's each. The following will be useful.

Lemma 4.2. $r \ne 1$, and $|r| \le k-d+1$.

Proof. Suppose, for contradiction, that $r = 1$. Then we have $(k-d)(k-d+1) = 2k-1$. Solving for k, we get $k = \frac{2d+1\pm\sqrt{8b-3}}{2}$. Since k is an integer, $8d-3$ must be a perfect square. But no perfect square can be congruent -3 mod 8, which gives us a contradiction.

For the second part, if $|r| > k-d+1$, then $|r| \le \sqrt{2k-2}$ gives us $k-d+1 < \sqrt{2k-2}$, which implies that $d > k+1-\sqrt{2k-2}$, a contradiction. The result follows. $\square$

We split the columns of C into three parts; we call them C_1, C_2, and C_3. The matrix C_1 has $k-d+1$ columns, C_3 has $|r|$ columns, and thus C_2 has $d-|r|-2$. We need C_2 to have a nonnegative number of columns, so we must prove the following.

Lemma 4.3. If $k \ge 7$, then $d-|r|-2 \ge 0$.

Proof. Suppose, for contradiction, that $d-|r|-2 < 0$, so $d < |r|+2$. Since $d > k-\sqrt{2k-2}$ and $|r| \le \sqrt{2k-2}$, we have $k-\sqrt{2k-2} < 2+\sqrt{2k-2}$. Solving this inequality for k, we get $k < 6+2\sqrt{6} < 11$, and so $k \le 10$. One can check that the result holds for $7 \le k \le 10$, and so the result follows. $\square$

We now begin our construction of C_1. If we denote the entry of C_1 in the ith row and jth column by $C_1(i,j)$, then we let $C_1(i,i) = 0$, $C_1(j+1,j) = 0$ for all $1 \le i \le k-d+1$, $1 \le j \le k-d$, and $C_1(1,k-d+1) = 0$. The remaining entries of C_1 are 1. No two rows of C_1 are identical as long as $k-d+1 \ge 3$ (which is true for all $k \ge 4$). Therefore, no two rows of C are identical. Also, there are two 0's in each row and column of C_1.

For C_2, distribute two 0's in each column so that the 0's are distributed as evenly as possible among the rows. For example, in the first column, one might place the 0's in the first two rows. In the second column, the 0's can be placed in the third and fourth row, and so on. The 0's can be wrapped around to the first row when the end of a column is reached.

Before we begin the construction of C_3 and D, note that, out of the $(k-d)(k-d+1)$ 0's that need to be distributed in C (with $k-d$ in each row), we have placed $2(k-|r|-1)$ among C_1 and C_2, leaving $r+2|r|$ to be placed in C_3.

In the case of $r<0$, we have $-r=|r|$ 0's to place in C_3. Since C_3 has $|r|$ columns, we can then place one 0 in each column of C_3. To determine which rows to place the remaining 0's, we use $(k-d)(k-d+1)=2k-2+r=2k-2-|r|$ to get

$$2(k-|r|-1)=(k-d-1)(k-d+1)+(k-d+1-|r|).$$

Since the 0's in C_1 and C_2 are distributed as evenly as possible, and since $2(k-|r|-1)$ is the number of 0's in C_1 and C_2 combined, we get that the first $k-|r|-1$ columns of C each have at least $k-d-1$ 0's in each row, and $k-d+1-|r|$ of the rows have $k-d$ 0's. Note that Lemma 4.2 implies that $k-d+1-|r|\geq 0$. This leaves $|r|$ rows that have only $k-d-1$ 0's. We then place one 0 in each column of C_3 in a row that, in C_1 and C_2, has $k-d-1$ 0's. We place 1's in the remaining entries.

This brings us to the construction of D. We define $D(i,i)=1$ for each $1\leq i\leq k-|r|-1$ and $D(i,j)=0$ otherwise, making T regular. In addition, the way we have constructed the first $k-|r|-1$ rows of D guarantees that no two of the vertices represented by these rows and columns are clones. The way that C_3 is constructed guarantees that no two of the remaining vertices are clones. Thus, T is clone-free, and this completes the case of $r<0$.

In the case of $r\geq 0$, we have $3|r|=3r$ 0's to distribute into C_3. We distribute three 0's per column in such a way that the 0's in the matrix C are evenly distributed among the rows. This makes it so that each row of C now has $k-d$ 0's, as desired.

We begin the construction of D as before. We make $D(i,i)=1$ and $D(j,i)=0$ for each $1\leq i\leq k-r-1$, $i\neq j$. This ensures regularity and guarantees that each of the first $k-r-1$ rows and columns are distinct. For the remainder of the matrix, note that each of the last r rows and columns of D must have two 1's for regularity. For $r\geq 3$, we do this in the same manner as the construction of C_1. We put 1's down the main diagonal and down the diagonal just below the main diagonal (of the last r rows and columns of D), and then we put a 1 in the last column of D in the $(k-r)$th row. In the remainder of the entries of D, we place 0's. This arrangement ensures that the last r rows and columns are distinct. This also works in the case of $r=0$; the construction of the last r rows will be null. Thus, our matrix represents a clone-free regular bipartite tournament of rank d except in the case of $r=2$ (recall that Lemma 4.2 eliminates the case of $r=1$).

In the case of $r=2$, if we construct the matrix as above, in the bottom right-hand corner of D we have

$$\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}.$$

This can be replaced by

$$\begin{bmatrix}0&1&0\\0&1&1\\1&0&1\end{bmatrix}.$$

The replacement matrix preserves the number of 0's and 1's in each row and column, thus maintaining regularity and preventing clones. All of this together gives us the following.

Theorem 4.4. For each $k \geq 6$, there is a clone-free regular bipartite tournament of rank $d = \lfloor k+1-\sqrt{2k-2} \rfloor$. Thus, d is a tight upper bound for the rank clone-free regular bipartite tournaments with $2k$ vertices in each partite set and $k \geq 6$. The tight upper bounds for rank in the case $k \leq 5$ are given by Theorem 3.3.

Since $h(T) = r(T) = d(T)$ in clone-free bipartite tournaments [PWWa, Thm. 4.2], we have the following.

Corollary 4.5. For each $k \geq 6$, there is a clone-free regular bipartite tournament of Radon and Helly number $d = \lfloor k+1-\sqrt{2k-2} \rfloor$. Thus, d is a tight upper bound for the Radon and Helly numbers clone-free regular bipartite tournaments with $2k$ vertices in each partite set and $k \geq 6$. The tight upper bounds for the Radon and Helly number in the case $k \leq 5$ are the same as those in Theorem 3.3.

5 Upper Bounds on Rank in the Multipartite Case

Let T be a regular p-partite tournament with k vertices in each partite set. We consider the arcs between $P_1 \cup P_2$ and the remaining partite sets of T. We will use these arcs to derive upper bounds for the rank of clone-free regular p-partite tournaments for $p \geq 3$.

First note that regularity demands that the inset and outset of each vertex contains $\frac{(p-1)k}{2}$ vertices. Let $A \subseteq P_1$ and $B \subseteq P_2$ be as in the previous section. If $B = \emptyset$, we again assume $D_A^{\rightarrow} \neq \emptyset$ and choose P_2 so that $D_A^{\rightarrow} \subseteq P_2$.

Note that, by Theorem 2.1(5), if $|A \cup B| \geq 3$, then there are no vertices in $T-(P_1 \cup P_2)$ that distinguish vertices in A. When this is the case, we define $W = \{w \in T-(P_1 \cup P_2) : w \to A\}$ and $R = \{z \in T-(P_1 \cup P_2) : A \to z\}$. These two sets partition $T-(P_1 \cup P_2)$.

Lemma 5.1. $W \to B$, and $|W| \geq d(T)-1$.

Proof. Suppose there exists some $y_i \in B$ and $w \in W$ with $y_i \to w$. We have $y_i \to w \to x_1$ and $w \to x_2 \to y_i$, so $x_2 \in x_1 \vee y_i$, a contradiction. For the second part, we have that x_1 dominates all of B and at least $|A|-1$ vertices in $D_A^{\rightarrow}$. Thus, $|N^+(x_1) \cap P_2| \geq |B|+|A|-1 = d(T)-1$. That leaves at most $k-d(T)+1$ vertices in $P_2 \cap N^-(x_1)$. Since $|N^-(x_1)| = k$, that implies that $|N^-(x_1) - P_2| \geq d(T)-1$. And since only vertices in P_2 can distinguish vertices in A, we have $N^-(x_1) - P_2 \subseteq W$, which proves the result. □

There are no such restrictions on the arcs between vertices in R and B, so we can partition R into the sets $R_1 = \{z \in R : B \to z\}$ and $R_2 = \{z \in R : z \to B\}$.

Lemma 5.2. Let A and B be as above with $|A \cup B| \geq 3$.

1. If $B \neq \emptyset$, then $D_A^{\rightarrow} \to R_1$ and $W \to D_B^{\leftarrow}$.

2. If $|A| \geq 3$ or $|A|, |B| \geq 2$, then $W \to D_A^{\rightarrow}$ and $D_B^{\leftarrow} \to R_1$.

3. If $|A| \geq 3$, then no vertex in R can distinguish vertices in $D_A^{\rightarrow}$.

4. If $|A| \geq 3$, $|B| = 1$, and u is as in Lemma 2.7(2) (i.e., for some identity block S of $D_A^{\rightarrow}$, $B \to u \to S$), then $W \to u$.

Proof. For (1), let $u \in D_A^{\rightarrow}$ with $u \to x_i$, let $v \in R_1$, and let $j \neq i$ with $x_j \in A$. For contradiction, assume that $v \to u$. We have $x_j \to u \to x_i$, $x_i \to v \to u$, and $x_i \to y_1 \to v$, so $y_1 \in x_i \vee x_j$, a contradiction. Thus, $D_A^{\rightarrow} \to R_1$, and the rest of (1) follows similarly.

For (2), let $u \in D_A^{\rightarrow}$ with $u \to x_i$ and $w \in W$. For contradiction, suppose $u \to w$. In the case $|A| \geq 3$, let $j, k \neq i$ with $x_j, x_k \in A$. Clearly, $u \in x_i \vee x_j$. Then $u \to w \to x_i$ and $w \to x_k \to u$ imply $x_k \in x_i \vee x_j$, a contradiction. In the case $|B| \geq 2$, let $j \neq i$ with $x_j \in A$, and let $v \in D_B^{\leftarrow}$ with $y_1 \to v$. Again, suppose that $u \to w$. Since $y_1 \to v \to y_2$, we have $v \in x_i \vee y_1 \vee y_2$. Then $v \to u \to x_i$ gives us $u \in x_i \vee y_1 \vee y_2$. As in the $|A| \geq 3$ case, we get $x_j \in x_i \vee y_1 \vee y_2$, a contradiction. This gives us the first part of (2), and the rest follows similarly.

For (3), suppose we have $u_i, u_j \in D_A^{\rightarrow}$ with $u_i \to x_i$, $u_j \to x_j$, and $z \in R$ with $u_i \to z \to u_j$. Let $k \neq i, j$ with $x_k \in A$. We have $u_j \in x_j \vee x_k$. Then $x_j \to z \to u_j$, $x_j \to u_i \to z$, and $u_i \to x_i \to u_j$ imply $x_i \in x_j \vee x_k$, a contradiction.

Finally, for (4), suppose that $u \to w$ for some $w \in W$. Let $v \in S$ with $v \to x_1$. Then $v \in x_1 \vee x_2 \vee y_1$. We have $y_1 \to u \to v$, $u \to w \to x_1$, and $w \to x_3 \to y_1$, so $x_3 \in x_1 \vee x_2 \vee y_1$, a contradiction. ∎

If $|A| \geq 3$, and q is the number of vertices in R that are dominated by the vertices in $D_A^{\rightarrow}$, then $|R| - q$ is the number of vertices in R that dominate the vertices in $D_A^{\rightarrow}$. We can then consider the inset of $D_A^{\rightarrow}$. We get $|A| - 1$ vertices from A, $|W|$ vertices from W, and $|R| - q$ vertices from R. We then get

$$\frac{(p-1)k}{2} \geq |A| - 1 + |W| + |R| - q$$

Putting this together with $|R| + |W| = (p-2)k$ gives us $q \geq |A| - 1 + \frac{(p-3)k}{2}$. Since T has at least three partite sets, we get $q \geq |A| - 1$ and the following.

Theorem 5.3. Let T be a clone-free regular tripartite tournament with $k \geq 1$ vertices in each partite set.

1. If $k \leq 5$, then $d(T) = 2$.

2. If $k = 6$, then $d(T) \leq 3$.

3. If $k \geq 7$, then $d(T) < \frac{k}{2}$.

4. For each $k \geq 5$, there is a clone-free regular tripartite tournament with rank $\lfloor \frac{k-1}{2} \rfloor$, so the bound is tight.

Proof. Let T be a clone-free regular tripartite tournament with $d(T) \geq 3$, and let A and B form a maximum convexly independent subset of V. We claim that $d(T) < \frac{k}{2}$ unless $|A| = 2$ and $|B| = 1$. If $B = \emptyset$, then $q \geq |A| - 1 \geq 2$, so let $v, v' \in R$ with $D_A^{\rightarrow} \to \{v, v'\}$. By Theorem 2.8, $|D_A^{\rightarrow}| \geq |A|$. Thus, $k = |N^-(v)| \geq 2|A| = 2d(T)$, and so $d(T) \leq \frac{k}{2}$. If $d(T) = \frac{k}{2}$, then $N^-(v) = N^-(v') = A \cup D_A^{\rightarrow}$, and so v and v' are clones. This forces $d(T) < \frac{k}{2}$. The case $|B| \geq 2$ follows similarly, using W and its outset in place of $\{v, v'\}$ and its inset, and applying Lemma 5.1 and Lemma 5.2(1) and (2). If $|B| = 1$ and $|A| \geq 3$, then the result follows similarly using Lemma 5.2(4). This gives us (2) and (3).

For (1), assume that $k \leq 5$. If $d(T) \geq 3$, then, by the above argument, we must have $|A| = 2$ and $|B| = 1$. Let $u_1, u_2 \in D_A^{\rightarrow}$ with $u_i \to x_i$. Note that since $A \to y_1$, at most $k-2$ vertices in P_3 dominate y_1. Since $(W \cup R_2) \to y_1$, at least two vertices must be in R_1. Let $r_1, r_2 \in R_1$. We have $D_A^{\rightarrow} \to R_1$ by Lemma 5.2(1), and so $u_1, u_2 \in N^-(r_i)$, $i \in \{1, 2\}$. Combining this with $x_1, x_2, y_1, \in N^-(r_1)$ gives us $N^-(r_1) = N^-(r_2) = \{x_1, x_2, y_1, u_1, u_2\}$, making r_1 and r_2 are clones, a contradiction. Thus, $d(T) = 2$, which gives us (1).

We now prove the bounds are tight. The bound in (1) is trivially tight. The following is a clone-free regular tripartite tournament of rank 3 with 6 vertices in each partite set. The maximum convexly independent set is given by the first row and the second and third columns.

	P_1						P_3						P_2					
P_2	1	0	0	0	1	1	0	0	0	1	1	1						
	0	1	0	1	1	0	0	0	0	1	1	1						
	0	0	1	1	1	0	0	0	0	1	1	1						
	0	1	1	0	0	1	1	1	1	0	0	0						
	1	1	1	0	0	0	1	1	1	0	0	0						
	1	0	0	1	0	1	1	1	1	0	0	0						
P_3	0	1	1	1	0	0							1	1	1	0	0	0
	0	1	1	0	1	0							1	1	1	0	0	0
	0	1	1	0	0	1							1	1	1	0	0	0
	1	0	0	0	1	1							0	0	0	1	1	1
	1	0	0	1	0	1							0	0	0	1	1	1
	1	0	0	1	1	0							0	0	0	1	1	1

.

For the bound in (3), we must construct for each $k \geq 5$, a tripartite tournament T with rank $\frac{k-1}{2}$ when k is odd and $\frac{k}{2} - 1$ when k is even. In both cases, we partition $P_1 = A \cup P_1^d \cup P_1^r$, $P_2 = D_A^{\rightarrow} \cup P_2^d \cup P_2^r$, and $P_3 = W \cup R$.

If k is even and $d = \frac{k}{2} - 1$, then we let $|P_1^b| = |P_2^r| = 2$, $|P_1^r| = |P_2^b| = \frac{k}{2} - 1$, and $|W| = |R| = \frac{k}{2}$. We then construct the arcs of T according to the following matrix.

	A	P_1^b	P_1^r	W	R	$D_A^{\rightarrow}$	P_2^r	P_2^b
$D_A^{\rightarrow}$	I_d	0	1	0	1			
P_2^r	0	1	1	1	0			
P_2^b	1	1	0	$(I_d)_c$	0			
W	1	0	0			1	0	I_d
R	0	C'	D'			0	1	1

A blank block denotes an empty matrix (no arcs). If M is a binary matrix, we define M_c to be the matrix obtained from M by interchanging the 0's and 1's. Thus, we need only specify C and D. Let C be the $2 \times (d+1)$ matrix with entries $C(1,1) = C(2,2) = 0$ and 1's otherwise. Also, let D be the $d \times (d+1)$ matrix with entries $D(i,i) = 0$ for each $1 \leq i \leq d$, $D(i,i+2) = 0$ for each $1 \leq i \leq d-1$, $D(d,d+1) = 0$, and 1's otherwise. It is not difficult to see that T is regular and clone-free, and that A is a convexly independent set.

In the case k is odd, and $d = \frac{k-1}{2}$, we have $|P_1^d| = 2$, $|P_1^r| = d-1$, $|P_2^r| = 1$, $|P_2^d| = d$, $|W| = d$, and $|R| = d+1$. We then construct the arcs as follows.

	A	P_1^b	P_1^r	W	R	$D_A^{\rightarrow}$	P_2^r	P_2^b
$D_A^{\rightarrow}$	I_d	0	1	0	1			
P_2^r	0	1	1	1	0			
P_2^b	1	1	0	$(I_d)_c$	0			
W	1	0	0			1	0	I_d
R	0	C'	D'			0	1	1

In this case, let C' be the $(d+1) \times 2$ matrix with $C'(1,1) = C'(2,2) = 0$ and 1's otherwise, and let D' be the $(d+1) \times (d-1)$ matrix with $D'(i+2,i) = 0$ for $1 \leq i \leq d-1$ and 1's otherwise. $\square$

We achieve a bound almost as good in the general case.

Theorem 5.4. Let T be a clone-free regular p-partite tournament with $p \geq 3$ and $k \geq 2$ vertices in each partite set. Then $d(T) \leq \frac{k+2}{2}$.

Proof. Let A and B form a maximum convexly independent subset of V. For the case $B \neq \emptyset$, we get $\frac{(p-1)k}{2} = |N^+(x_i)| \geq |R_1| + |R_2| + d(T) - 1$. These correspond to vertices in R, B, and $D_A^{\rightarrow}$. Similarly, we get $\frac{(p-1)k}{2} = N^-(y_j) \geq |W| + |R_2| + d(T) - 1$. Combining these with $|W| + |R_1| + |R_2| = (p-2)k$ gives us $d(T) \leq \frac{k+2-|R_2|}{2} \leq \frac{k+2}{2}$.

We use a similar argument in the case $B = \emptyset$. In this case we may assume $|A| \geq 3$. We have $\frac{(p-1)k}{2} \geq |R| + d(T) - 1$, corresponding to vertices in R and $D_A^{\rightarrow}$. Similarly, we get $\frac{(p-1)}{2} \geq |W| + d(T) - 1$, since $W \rightarrow D_A^{\rightarrow}$ by Lemma 5.2(2). Combining these inequalities with $|R| + |W| = (p-2)k$ gives us the result. □

While this bound is tantalizingly close to the tight bound we derived in the tripartite case, it is unclear whether the above bound is tight. One might expect the bound to be asymptotically tight, but we have not yet found a proof for this.

6 Open Problems

We conclude with some open problems.

(1) *Determine the structure of clone-free regular multipartite tournaments.* Theorem 2.8 gives us significant insight into the rank of clone-free regular bipartite tournaments. Can anything more be said about the structure of clone-free regular bipartite tournaments? In particular, what form does the matrix have? What if there are three or more partite sets?

(2) *Determine tight upper bounds for the rank of clone-free regular multipartite tournaments.* We have taken care of the case $p = 2, 3$ with Theorem 4.4 and Theorem 5.3. One would guess that the bound in Theorem 5.4 is either tight or very nearly tight, but we have not verified this.

(3) *Classify all clone-free regular bipartite/tripartite tournaments of maximum rank.*

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