# Final Report

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## 1 Introduction

Felix Hausdorff developed the Hausdorff metric $h$ early in the $20^{\text{th}}$ century as a means of measuring the distance between certain sets in $n$-dimensional real space. We can talk about betweenness in the Hausdorff Metric Geometry in a similar manner as we would discuss it in Euclidean geometry, except for the fact that we can have multiple elements at a given location between two sets in our Hausdorff geometry. Although we can find infinitely many sets which have a different exact number of elements at each location between them, including sets with one through 18 and 20 through 36 elements at each location between them, we will prove the surprising result that no two sets have exactly 19 elements with this property, and we will also talk about the implications of this discovery.

## 2 Topology

To fully understand what is happening in the Hausdorff geometry, we must first internalize several important concepts from topology. Thus, we will devote this section to definitions, theorems, and examples that will lay the groundwork for the work that will follow.

**Definition 2.1.** *A* ***topological space*** $(X,\mathcal{U})$ *consists of a set* $X$ *and a family* $\mathcal{U}$ *(known as a* ***topology****) of subsets of* $X$ *which fulfills the following:*

1. $X$ *and* $\emptyset$ *are in* $\mathcal{U}$
2. $\bigcup_{\alpha\in I} A_\alpha \in \mathcal{U}$ *for any indexing set* $I$*, where* $A_\alpha \in U$ *for all* $\alpha \in I$*.*
3. $A \bigcap B \in \mathcal{U}$ *for any* $A, B \in \mathcal{U}$

*A set* $A \subset (X,\mathcal{U})$ *is called an* ***open set*** *if and only if* $A \in \mathcal{U}$*. If* $X - A \in \mathcal{U}$*, then we say that* $A$ *is a* ***closed set****.*

To better elucidate the concept of a topological space, we have included the following examples:

**Example 2.1.** *If* $X = \{a, b, c, d\}$ *then* $\mathcal{U}_1 = \{\emptyset, \{c\}, \{c, d\}, X\}$ *is a topology on* $X$*. However,* $\mathcal{U}_2 = \{\emptyset, \{a, b\}, \{b, c\}, X\}$ *is not a topology on* $X$ *since* $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{U}_2$*.*

The following definitions will be necessary when we eventually get into Hausdorff Metric Geometry, the underlying focus of all our research.

**Definition 2.2.** *Let* $A \subset (X,\mathcal{U})$*. An* ***open covering*** *of* $A$ *is of a collection of open sets, the union of which contains* $A$*. A* ***subcovering*** *of a covering of* $A$ *is a subcollection of that covering whose union contains* $A$*. A* ***finite covering*** *of* $A$ *is an open covering of* $A$ *containing a finite number of sets.*

**Definition 2.3.** *Let* $B \subset (X,\mathcal{U})$*. We say* $B$ *is* ***compact*** *if every open covering of* $B$ *contains a finite subcovering of* $B$*.*

The sets on which we perform our measurements must be compact; without compactness the Hausdorff distance could be undefined, as will be explained later in this paper.

**Definition 2.4.** *A set $C \subset \mathbb{R}^n$ is **bounded** if for some $r \in \mathbb{R}$, every point in $C$ is within a distance $r$ from the origin.*

The concepts of bounds and compactness are interrelated, as we can see in the following theorem:

**Heine-Borel Theorem.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded [6].*

This theorem allows us to characterize compactness in a slightly less abstract manner in $\mathbb{R}^n$, and we will use its stipulations interchangably with compactness since we are working in $\mathbb{R}^n$.

**Example 2.2.** *In the **usual topology** on $\mathbb{R}$, open sets are defined as open intervals or the union of open intervals. For any $a, b \in \mathbb{R}, [a, b]$ is a compact subset of $\mathbb{R}$. The sets $[a, \infty)$ and $(a, b)$ are not compact under the usual topology of $\mathbb{R}$, since $[a, \infty)$ is not bounded from above and in this topology, sets must be either open or closed, and $(a, b)$ is an open set. There exists no open set whose complement is an open interval in $\mathbb{R}$.*

All of our work will be within a specialized metric space, which we will define later. For now, let us generally define a metric space. The purpose of a metric in a metric space is to provide a way of measuring some sort of distance in a given topological space.

**Definition 2.5.** *A **metric space** is a set $X$ together with a function $d : X \times X \to \mathbb{R}$ (called a **metric**), which satisfies, for every $x, y, z \in X$,*

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ *with equality if and only if* $x = y$
3. $d(x, z) \leq d(x, y) + d(y, z)$

The Euclidean distance metric is one with which most readers are probably familiar:

**Definition 2.6.** *The **Euclidean distance** between two points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{R}^n$ (denoted $d_E(x, y)$) is equal to $\sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$.*

The Euclidean distance will be important to our definition of the Hausdorff metric. From this point onward, we will let $\mathcal{H}(\mathbb{R}^n)$ represent the set of all non-empty compact subsets of $\mathbb{R}^n$ - this is the collection of sets which, combined with the Hausdorff metric, will form our metric space. One special sequence within a metric space that will help make sure that the Hausdorff metric is defined is known as a Cauchy sequence.

**Definition 2.7.** *A sequence $\{a_n\}_{n=1}^{\infty}$ in a metric space with metric $d$ is **Cauchy** if*

$$\lim_{\min(m,n)\to\infty} d(a_m, a_n) = 0.$$

Material from [6] deomstrates that every compact subset $X$ of a topological space is also complete, which means that every Cauchy sequence in $X$ converges to an element of $X$.

**Definition 2.8.** *Let $A$ be a subset of a topological space $X$.*

- *The **closure** of $A$ (denoted $\overline{A}$) is the intersection of all closed sets containing $A$.*
- *The **interior** of $A$ (denoted int($A$)) is the union of all open sets contained in $A$.*
- *The **boundary** of $A$ (denoted $\partial A$) is equal to $\overline{A} \cap \overline{X - A}$.*

Thus it is also true that $\overline{A} = int(A) \cup \partial A$, and $\overline{A}$ is the smallest closed set in $X$ containing A, whereas int($A$) is the largest open subset of $A$ in $X$ [6].

**Example 2.3.** *Let $A = \{(x, y) \in \mathbb{R}^2\text{: } x^2 + y^2 < 1\}$. The set $A$ is equal to its interior, its boundary is the unit circle, and its closure is the unit disk.*

## 3 The Hausdorff Metric

Now we have the information necessary to define the Hausdorff distance. The Hausdorff metric $h$ gives us a method of measuring the distance between two compact subsets of $\mathbb{R}^n$:

**Definition 3.1.** *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$.*

- *For any $x \in \mathbb{R}^n$, the distance from $x$ to $B$ is*

$$d(x, B) = \min_{b \in B}\{d_E(x, b)\}.$$

  *A picture of this distance is depicted in Figure 1.*

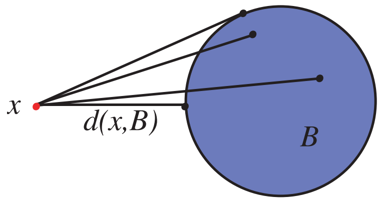



Figure 1: Distance from a point to a compact set.

- *The distance from $A$ to $B$ is*

$$d(A, B) = \max_{x \in A}\{d(x, B)\}.$$

  *Notice that this distance is not necessarily symmetric as shown in Figure 2.*

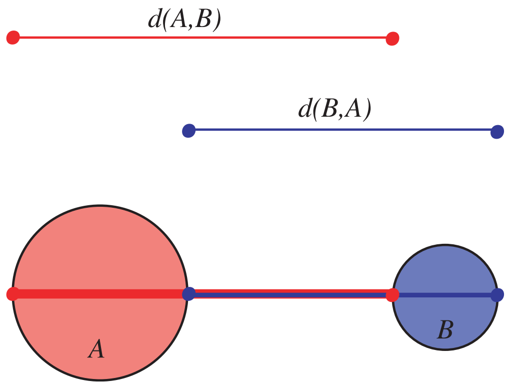



Figure 2: Distance from a compact set to a compact set.

- *The Hausdorff distance, $h(A, B)$, between $A$ and $B$ is*

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

  *In Figure 2, $h(A, B) = d(A, B) > d(B, A)$.*

The definition of the Hausdorff distance uses the function $d : \mathbb{R}^n \times \mathcal{H}(\mathbb{R}^n) \to \mathbb{R}$ defined by $d(x, B) = \min_{b \in B}\{d_E(x, B)\}$ for any $x \in \mathbb{R}^n$. A question arises: Is this function always defined? That is, will this distance actually achieve a minimum value for some $b \in B$?

**Definition 3.2.** *Given a point $x$ in $\mathbb{R}^n$, a subset $N$ of $\mathbb{R}^n$ is a* ***neighborhood*** *of $x$ (denoted $N(x)$) if we can find an open set $O$ in $\mathbb{R}^n$ satisfying $x \in O \subset N$.*

We will concern ourselves with a specific type of neighborhood, which will be of use to us in the proof that will follow shortly.

**Definition 3.3.** *For any $r$ in $\mathbb{R}^n$, the* ***r-neighborhood*** *of a point $x$ in $\mathbb{R}^n$ is the set $N_r(x) = \{y \in \mathbb{R}^n : d_E(x, y) < r\}$.*

Another tool we need to show that the Hausdorff distance is defined is continuity, which we can efficiently connect to compactness.

**Definition 3.4.** *A function $f : X \to Y$ is* ***continuous*** *if for every $x \in X$ and every neighborhood $N(f(x)) \in Y$, the set $f^{-1}(N(f(x)))$ is a neighborhood of $x$ in $X$.*

**Lemma 3.1.** *The continuous image of a compact subset of $\mathbb{R}^n$ is compact [11].*

These definitions and the related lemma will be essential to the proof that follows:

**Theorem 3.1.** *For any $x \in \mathbb{R}^n$ and any $B \in \mathcal{H}(\mathbb{R}^n)$, $f : B \to \mathbb{R}$ defined by $f(b) = d_E(x, b)$ for all $b \in B$ will reach a minimum value.*

*Proof.* Let $B \in \mathcal{H}(\mathbb{R}^n)$, and let $x \in \mathbb{R}^n$. Define $f : B \to \mathbb{R}$ by $f(b) = d_E(x, b)$. Any neighborhood $N$ of $f(b)$ in $\mathbb{R}$ must contain an open interval centered around $f(b)$. Call this interval $Y$ and let $r$ equal its radius, so $Y = (f(b) - r, f(b) + r)$. Thus, in order to show that $f$ is continuous, it will be sufficient to show that $f^{-1}(Y)$ is a neighborhood of $b$ in $B$. We know that

$$\begin{aligned} f^{-1}(Y) &= \{b' \in B : |f(b) - f(b')| < r\} \\ &= \{b' \in B : |d_E(b, x) - d_E(b', x)| < r\} \end{aligned}$$

Let Z be the r-neighborhood in $B$ centered around $b$. For any $b'' \in Z$, the triangle inequality says that

$$\begin{aligned} d_E(b, x) &\leq d_E(b, b'') + d_E(b'', x) \\ d_E(b, x) - d_E(b'', x) &\leq d_E(b, b'') < r \end{aligned} \tag{3.1}$$

Similarly,

$$\begin{aligned} d_E(b, x) + d_E(b, b'') &\geq d_E(b'', x) \\ d_E(b, x) - d_E(b'', x) &\geq -d_E(b, b'') > -r. \end{aligned} \tag{3.2}$$

Combining the two inequalities (7.1) and (7.2) yields

$$-r < d_E(b, x) - d_E(b'', x) < r$$

which is the same as

$$|d_E(b, x) - d_E(b'', x)| < r.$$

Hence, $b'' \in f^{-1}(Y)$, which means $Z \subset f^{-1}(Y)$ and the function $f$ is continuous. Lemma 1 states that $f(B)$ must then be compact since $B$ itself is compact. The Heine-Borel Theorem says that $f(B)$ must be bounded, so there exists a $c \in \mathbb{R}$ such that $c = \inf_{b \in B} f(B)$. Consider the sequence $a_n = c + \frac{1}{n}$. Clearly $\{a_n\}$ is a Cauchy sequence converging to $c$, and $a_n > \inf_{b \in B} f(B)$ for each $n$. Thus, by the definition of infimum, we can find a $b_n \in f(B)$ such that $c \leq b_n \leq a_n$ for each $a_n$. The new sequence $\{b_n\}$ is also Cauchy, converges to $c$, and is contained in $f(B)$. Since $f(B)$ is compact and therefore complete, $c$ must necessarily be in $f(B)$, and thus $f$ is well-defined. □

Thus, $h(A, B)$ will always have a definite value. The next question we need to address is this: Is the Hausdorff distance actually a metric? Just as the definition of the Hausdorff distance depends on Euclidean distance, its metric properties as will rely on those of the Euclidean distance metric.

**Theorem 3.2.** *The Hausdorff distance $h$ satisfies the necessary conditions to be a metric on $\mathcal{H}(\mathbb{R}^n)$.*

*Proof.* Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. I first must show that $h(A, B) = h(B, A)$. By the definition $h(A, B)$ I have

$$\begin{aligned} h(A, B) &= \max\{d(A, B), d(B, A)\} \\ &= \max\{d(B, A), d(A, B)\} \\ &= h(B, A) \end{aligned}$$

Next, I will show that $h(A, B) \geq 0$ with equality if and only if $A = B$. Clearly, $h(A, B) = d_E(a, b)$ for some $a \in A$ and $b \in B$, $and d_E(a, b) \geq 0$.

Suppose that $h(A, B) = 0$, which means that $d(A, B) = d(B, A) = 0$. Thus, $d(a, B) = 0$ for all $a \in A$, so for every $a \in A$ there exists a $b \in B$ such that $d_E(a, b) = 0$. Hence each $a \in B$, so $A \subset B$. By a similar argument, $B \subset A$ and we have $A = B$.

Now suppose that $A = B$. This means that for every $a \in A$, there exists a $b \in B$ such that $a = b$, or $d_E(a, b) = 0$. Thus, $d(a, B) = 0$ for each $a \in A$, and $d(A, B) = 0$. Similarly, $d(B, A) = 0$; hence, $h(A, B) = 0$.

Finally, I must show that for any $A, B, C \in \mathcal{H}(\mathbb{R}^n)$, $h(A, C) \leq h(A, B) + h(B, C)$. Let $A, B, C \in \mathcal{H}(\mathbb{R}^n)$. Without loss of generality, we assume that $h(A, C) = d(A, C)$. There must exist a point, call it $a$, in $A$ such that $d(a, C) = d(A, C)$. There must also be a $b \in B$ such that $d_E(a, b) = d(a, B)$, and we can then find a $c \in C$ with $d_E(b, c) = d(b, C)$. By the triangle inequality,

$$\begin{aligned} d_E(a, c) &\leq d_E(a, b) + d_E(b, c) \\ &\leq d(a, B) + d(b, C) \end{aligned}$$

by substitution. Note that $d(A, C) = d(a, C)$ and $d(a, C) = \min_{c' \in C}\{d_E(a, c')\}$, which means that $d(a, C) \leq d_E(a, c)$. Also, $d(A, B) = \max_{a' \in A}\{d(a', B)\}$ and $d(B, C) = \max_{b' \in B}\{d(b', C)\}$, so $d(a, B) \leq d(A, B)$ and $d(b, C) \leq d(B, C)$. Then

$$\begin{aligned} d(A, C) = d(a, C) &\leq d_E(a, c) \\ &\leq d(a, B) + d(b, C) \\ &\leq d(A, B) + d(B, C) \end{aligned}$$

Finally, remember that $d(A, C) = h(A, C)$, and note that $d(A, B) \leq h(A, B)$ and $d(B, C) \leq h(B, C)$. Thus,

$$\begin{aligned} h(A, C) &\leq d(A, B) + d(B, C) \\ &\leq h(A, B) + h(B, C) \end{aligned}$$

Therefore, the Hausdorff distance is indeed a metric. ☐

**Example 3.1.** *Let $A$ be the unit circle and $B = \{(2, 0)\}$ in $\mathbb{R}^2$, and let $D$ be any closed subset of the disk radius one centered at $(2, 0)$ in $\mathbb{R}^2$. Then $C = \{(1, 0)\} \cup D$ in $\mathbb{R}^2$ satisfies $h(A, C) = 2$ and $h(B, C) = 1$. Notice that $d_E((1, 0), (2, 0)) = 1$, and any point in $D$ will be no further than one unit from $(2, 0)$, so $d(C, B) = 1$. By the same reasoning, the distance from $(2, 0)$ to any point in $C$ will be no greater than one unit, so $d(B, C) \leq 1$ and thus $h(B, C) = 1$. The distance between the point $(-1, 0)$ in $A$ and the point $(1, 0)$ in $C$ is two units, and $\overline{N_2(-1, 0)} \cap \overline{N_1(2, 0)}$ is the point $(1, 0)$, which means $(1, 0)$ is the closest point in $C$ to $(-1, 0)$. Each other point $a \in A$ will be closer than two units to $(1, 0)$: For any point $a \in A$, we know that $d_E(a, (0, 0)) \leq 1$ and $d_E((0, 0), (1, 0)) = 1$, so by the triangle inequality, $d_E(a, (1, 0)) \leq d_E(a, (0, 0)) + d_E((0, 0), (1, 0)) \leq 1 + 1 = 2$, so $d(A, C) = 2$. By a similar argument, the distance from any point in $C$ can be no greater than two units away from $A$, so $d(C, A) \leq 2$, and $h(A, C) = 2$.*

# 4 Applications of Subspace Topologies

In this section, we discuss betweenness in $\mathcal{H}(\mathbb{R}^n)$ and examine different ways to look at and count the exact number of elements between two sets. Specifically, we will develop a useful method of counting this number by employing subspace topologies. From this point forward, we set the convention that $A, B \in \mathcal{H}(\mathbb{R}^n)$. In

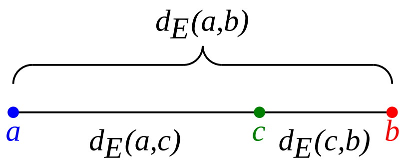



Figure 3: Betweenness in Euclidean Geometry

Euclidean geometry, a point $c \in \mathbb{R}^n$ is on the line segment between points $a$ and $b$ in $\mathbb{R}^n$ if and only if $d_E(a,b) = d_E(a,c) + d_E(c,b)$ (see Figure 3).

We will talk about betweenness in the Hausdorff Metric Geometry in a manner analagous to that in Euclidean Geometry, although we will see that Hausdorff betweenness does not imply uniqueness as does Euclidean Geometry.

**Definition 4.1.** *A set $C \in \mathcal{H}(\mathbb{R}^n)$ is **between** $A$ and $B$ at a location $t$ from $A$ if and only if $h(A,C) = t$ and $h(A,B) = h(A,C) + h(C,B)$.*

If $C$ is between $A$ and $B$ at a given location, we say that $C$ satisfies $ACB$. To understand that betweenness is not unique, we need to understand the term "extension".

**Definition 4.2.** *The **extension** of the set $A$ by the real number $t > 0$ is defined as the set*

$$A + t = \{x \in \mathbb{R}^n : d_E(x,a) \leq t \text{ for some } a \in A\}.$$

We can now set another convention: Let $r = h(A,B)$, and if $0 < s < r$, let $t = r - s$. Let $C = (A+t) \cap (B+s)$. Note that $C$ is the intersection of two closed and bounded sets, so it must be closed and bounded itself, and $C \in \mathcal{H}(\mathbb{R}^n)$.

In [2], the authors demonstrate that multiple elements can exist at a given location between $A$ and $B$ with the following theorem:

**Theorem 4.1.** *Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$, with $d(B,A) \leq d(A,B)$, then if $C'$ is a compact subset of $C$ containing $\partial((A+t) \cap (B+s))$, then $C'$ satisfies $AC'B$.*

It follows that $h(A,C) = t$ and $h(B,C) = s$. For infinitely many different integers $k$, we can finds sets $A$ and $B$ with exactly $k$ elements at each location between them. We would like to focus only on these finite scenarios; however, we can also find sets $A$ and $B$ with infinitely many elements at each location between them, so how do we tell which sets $A$ and $B$ will give us what we are looking for? The authors of [9] examine this phenomenon at a deeper level and demonstrate the conditions under which $A$ and $B$ have a finite number of elements at each location between them.

**Definition 4.3.** *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. We say that $A$ and $B$ satisfy the **PFAEL** (Possibly Finite At Each Location) conditions if*

$$h(A,B) = d(b,A) = d(a,B) \text{ for all } a \in A \text{ and } b \in B. \tag{4.1}$$

Note that these conditions are necessary but not sufficient to produce a finite number of elements at each location between $A$ and $B$. For example, consider the following sets $A$ and $B$ in $\mathbb{R}^3$:

**Example 4.1.** *Let $A_1 = \{(x,y,z) : x^2+y^2 = 1, z = 0\}$, $A_2 = \{(x,y,z) : x^2+y^2 = 1, z = 2\}$, $B_1 = \{(x,y,z) : x^2+y^2 = 1, z = 1\}$, $B_2 = \{(x,y,z) : x^2+y^2 = 1, z = 3\}$, and let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Notice that $h(A,B) = 1$, and each point in $A$ or $B$ is a distance one unit away from the other set. Thus, $A$ and $B$ satisfy the PFAEL conditions. In this case, however, if we have $C_1 = \{(x,y,z) : x^2+y^2 = 1, z = \frac{1}{2} \text{ or } \frac{5}{2}\}$ and $C_2$ is any compact subset of the set $\{(x,y,z) : x^2+y^2 = 1, z = \frac{3}{2}\}$, then there are infinitely many sets $C' = C_1 \cup C_2$ that satisfy $AC'B$ with $h(A,C') = \frac{1}{2}$.*

Another key concept, which is intertwined with the PFAEL conditions and will be of much use to us later in our investigation, is adjacency.

**Definition 4.4.** *Let $A$ and $B$ satisfy the PFAEL conditions and let $a \in A, b \in B$. Then $a$ is* ***adjacent*** *to $b$ (denoted $a \Bumpeq b$) if and only if $d_E(a, b) = h(A, B)$.*

**Example 4.2.** *In Figure 4, we have sets $A$ and $B = B_1 \cup \{b_0\}$, and a set $C = C_1 \cup C_2$ between $A$ and $B$.*

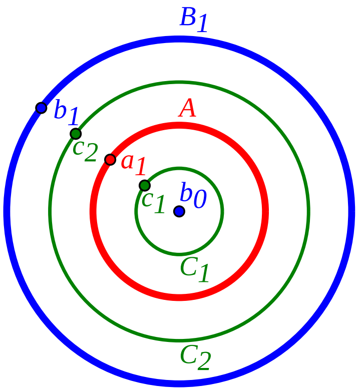



Figure 4: Sets $A$ and $B$ in $\mathcal{H}(\mathbb{R}^n)$

*In this figure, notice that $a_1 \Bumpeq b_1$ and $a_1 \Bumpeq b_0$. Also, $b_0 \Bumpeq c_1$ but $b_0 \not\Bumpeq c_2$.*

We call the union of two sets $A$ and $B$ that satisfy the PFAEL conditions a **configuration**, which we usually denote $X$. Notice that if every point in $A$ is adjacent to some point in $B$ and every point in $B$ is adjacent to some point in $A$, the PFAEL conditions are satisfied and the union of these two sets will make a configuration: if the condition above is true, then for any point $a$ in $A$, there exists a point $b$ in $B$ with $d_E(a, b) = h(A, B)$ and no point $b_0 \in B$ closer to $a$ than $h(A, B)$, so $d(a, B) = h(A, B)$, and we can make a similar argument for each point $b \in B$. We will use the notation $\#(X)$ to indicate the number of elements at each location between the two sets in $X$.

Additionally, we will call a point in $A$ or $B$ which is only adjacent to one other point in the opposite set an **endpoint**. Endpoints are important because they are also only adjacent to one point in $C$. Now we begin to discuss topology again.

**Definition 4.5.** *Let $(X, \mathcal{U})$ be a topological space, and let $C \subset X$. The* ***subspace topology*** *of $C$, denoted $\mathcal{U}_C$, is the set $\{U \cap C : U \in \mathcal{U}\}$. If $U$ is in $\mathcal{U}_C$, we say that $U$ is a* ***relatively open*** *set.*

The subspace topology $\mathcal{U}_C$ on a set $C$ will actually satisfy all of the conditions necessary to be a topology on $C$ [4], so $(C, \mathcal{U}_C)$ is a topological space. We will eventually be able to connect the subspace topology of $C$ with the elements at each location between $A$ and $B$, but we need to make several more definitions in order to have the necessary tools to make this connection.

**Definition 4.6.** *Let $A, C \in \mathcal{H}(\mathbb{R}^n)$, and let $a \in A$. The* ***adjacency set*** *of $a$ relative to $C$, denoted $[a]_C$, is defined as the set $\{c \in C : c \Bumpeq a\}$. We will let $[A]_C$ denote the set $\{[a]_C : a \in A\}$.*

Referring to the configuration in Figure 4, we can see that $[a_1]_C = \{c_1, c_2\}$ and $[b_0]_C = C_1$.

**Definition 4.7.** *Let $A, B, C$ satisfy the PFAEL conditions.*

1. *Let $q_A : C \to [A]_C$ be defined by $q_A(c) = [a]_C$ where $c \in [a]_C$.*

2. *Let $q_B : C \to [B]_C$ be defined by $q_B(c) = [b]_C$ where $c \in [b]_C$.*

**Example 4.3.** *Consider the configuration $X$ in $\mathbb{R}^2$ defined by $A = \{(0, 0)\}$ and $B = \{(x, y) : x^2 + y^2 = 1\}$. The only element between $A$ and $B$ at $s = t = \frac{1}{2}$ is the set $C = \{(x, y) : x^2 + y^2 = \frac{1}{4}\}$. Then $[(0, 0)]_C$ will be the set of all points on $C$, since $(0, 0)$ is adjacent to every once of these points. Also, for any $b \in B$, we can see that $[b]_C$ will be the single point on $C$ that intersects a line drawn between $b$ and the origin. For any point $c \in C$, $q_A(c) = [(0, 0)]_C$. The adjacency set given by $q_B(c)$ will be different for every $c$.*

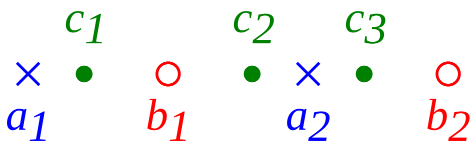
In [12], the author demonstrates that these two functions are both well-defined and onto. Now we can begin to define the function that will take an element of the subspace topology of $C$ to an element between $A$ and $B$.

Let $A$ and $B$ satisfy the PFAEL conditions. Define $\Upsilon$ as the set of all $U \in \mathcal{U}_C$ such that

1. for all $[a]_C \in q_A(U)$, there exists $c \in [a]_C$ such that $c \notin U$.
2. for all $[b]_C \in q_B(U)$, there exists $c \in [b]_C$ such that $b \notin U$.

Now let $\mathcal{K}$ be the set of all $C^\star$ such that $C^\star$ satisfies $AC^\star B$ with $h(A, C^*) = h(A, C)$. Note that every $C^\star$ must be a subset of $C$ [2]. Finally, define $f : \Upsilon \to \mathcal{K}$ by $f(U) = C - U$.

**Example 4.4.** *Let $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2, c_3\}$ as shown below. We can see that $A$ and $B$ satisfy the PFAEL conditions:*




*For this configuration, we have $\Upsilon = \{\emptyset, \{c_2\}\}$. It would be impossible to have $c_1$ in any $U \in \Upsilon$, because $q_A(c_1) = [a_1]_C = \{c_1\}$, which would mean that there would not exist another $c \in [a_1]_C$ such that $c \notin U$. A similar argument shows that $c_3$ cannot be in any $U \in \Upsilon$. In this example, $f(\emptyset) = C$ and $f(\{c_2\}) = \{c_1, c_3\}$, both of which lie at the location shown between $A$ and $B$.*

We can see that $f$ is clearly well-defined, but how do we know that $f$ actually takes elements from $\Upsilon$ to $\mathcal{K}$? Suppose that $A$ and $B$ satisfy the PFAEL conditions, and let $a \in A$ and $U \in \Upsilon$. If $[a]_C \in q_A(U)$, then there exists $c \in [a]_C$ such that $c \notin U$. On the other hand, if $[a]_C \notin q_A(U)$, we know $[a]_C$ is non-empty from [12], so there is some $c \in [a]_C$ such that $c \in C$ with $c \notin U$. Either way, $a \Bumpeq c$ for some $c \in C$ with $c \notin U$, so $c \in C - U$. A similar argument shows that for any $b \in B$, $b \Bumpeq c$ for some $c \in C$ with $c \notin U$, so $c \in C - U$. Thus, every point in $A$ and $B$ is adjacent to some point in $C - U$, and $C - U$ satisfies $A(C - U)B$. We also know that $(C - U) \subset C$, so $h(A, C - U) \geq h(A, C)$ [9], and for every $a \in A$, we have $d_E(a, c) = h(A, C)$ for some $c \in C - U$, so $h(A, C - U) \leq h(A, C)$. Then we have necessarily that $h(A, C - U) = h(A, C)$.

Furthermore, $C - U = C \cap U^C$, where $U^C$ is the complement of $U$. Since $C - U$ is the intersection of two closed sets and one of these sets is bounded, $C - U$ is also closed and bounded, and thus it must be in $\mathcal{H}(\mathbb{R}^n)$.

The function $f$ is very important for the following reason:

**Theorem 4.2.** *Let $A$ and $B$ satisfy the PFAEL conditions, and let $\Upsilon$ and $\mathcal{K}$ be defined as above. Then the function $f : \Upsilon \to \mathcal{K}$ by $f(U) = C - U$ is a bijective mapping.*

*Proof.* First we must show that $f$ is 1-1. Suppose $U_1, U_2 \in \Upsilon$. If $f(U_1) = f(U_2)$, then $C - U_1 = C - U_2$ and $U_1 = U_2$.

Now we must demonstrate that $f$ is onto. Let $C^* \in \mathcal{K}$, and let $V = C - C^*$. Suppose that for all $c_* \in [a]_C$, we have $c_* \in V$. Then there does not exist $c_* \in C^*$ such that $a \Bumpeq c_*$, but if this is the case, then $C*$ cannot satisfy $AC * B$, which is a contradiction. Hence, we see that for every $[a_C] \in V$, there exists a $c \in [a]_C$ such that $c \notin V$. By a similar argument, we can show that for every $[b]_C \in V$, there exists a $c \in [b]_C$ such that $b \notin V$.

We know that $V = C - C^* = C \cap C^{*C}$, so as the intersection of an open set with $C$, the set $V$ must be relatively open. Therefore, $V \in \Upsilon$, and $f$ is onto, from which it follows that $f$ is bijective.

□

This new function will be very useful; it allows us to count the number of elements at each loaction between two sets $A$ and $B$ not by looking at the sets $C'$ that satisfy $AC'B$ but rather by examining the removable points from the set $C$. We will see that to connect certain infinite and finite configurations we can compare these sets of removable points in an effective manner.

## 5 Building Configurations

We would like to be able to create certain configurations with an algorithm, but we must first know that this creation will be possible. Is it possible to configure two sets $A$ and $B$ in $\mathbb{R}^n$ with whatever adjacencies we desire? The following theorem answers that question and brings us one step closer to eventually identifying each infinite configuration (having a finite number of elements at each location between two sets) with a finite configuration.

**Configuration Construction Theorem.** *Given $\alpha, \beta \in \mathbb{N}$, we can create a configuration $X$ consisting of sets $A$ and $B$ in $\mathbb{R}^{\beta+2}$ with*

- $|A| = \alpha$
- $|B| = \beta$
- *Every element in $A$ is adjacent to a specified non-empty subset of elements in $B$ (with every element in $B$ adjacent to at least one element in $A$).*
- *$A$ and $B$ satisfy the PFAEL conditions.*

*Proof.* Let $\alpha, \beta \in \mathbb{N}$. If $\beta = 1$ the problem is trivial (we simply place points in $A$ in a circle around a single point in $B$), so suppose $\beta \geq 2$. Fix $c \in \mathbb{N}$ such that $c > \sqrt{\beta}$. Then define $b_1 = (\frac{1}{c}, 0, ..., 0)$ and $b_2 = (0, \frac{1}{c}, 0, ..., 0)$. Continuing in this manner, for $1 \leq i \leq \beta$, define $b_i = (u_1, ..., u_{\beta+2})$ such that $u_i = \frac{1}{c}$ and all other coordinates $u_1$ through $u_{\beta+2}$ equal 0. Let $B = \{b_i : 1 \leq i \leq \beta\}$.

Now, consider the $(\beta+1)$-sphere around each point $b_i$, named $\mathcal{B}_i$, equal to the set of points $(x_1, ..., x_{\beta+2})$ satisfying

$$x_i = \frac{1}{c}, \sum_{\substack{j=1 \\ j \neq i}}^{\beta+2} x_j^2 = 1 - \frac{1}{c^2}$$

Every point lying on each $\mathcal{B}_i$ will be equidistant from the correlating $b_i$. The intersection of the boundary of any two of these $(\beta+1)$-spheres, call them $\mathcal{B}_{i_1}$ and $\mathcal{B}_{i_2}$, will contain a $\beta$-sphere. The equation defining such a $\beta$-sphere is as follows:

$$x_{i_1}, x_{i_2} = \frac{1}{c}, \sum_{\substack{j=1 \\ j \neq i_1, i_2}}^{\beta+2} x_j^2 = 1 - \frac{2}{c^2}.$$

To verify that this $\beta$-sphere lies on $\mathcal{B}_{i_1}$, notice that if there is a point $y = (y_1, ..., y_{\beta+2})$ on this $\beta$-sphere, then $y_{i_1} = \frac{1}{c}$, and

$$\begin{aligned}
\sum_{\substack{j=1 \\ j \neq i_1}}^{\beta+2} y_j^2 &= \left( \sum_{\substack{j=1 \\ j \neq i_1, i_2}}^{\beta+2} y_j^2 \right) + y_{i_2}^2 \\
&= \left(1 - \frac{2}{c^2}\right) + \frac{1}{c^2} \\
&= 1 - \frac{1}{c^2}
\end{aligned}$$

Thus, we can see that the point $y$ satisfies the conditions that define $\mathcal{B}_{i_1}$. A similar series of equations can show that $y$ is on $\mathcal{B}_{i_2}$.

Now suppose that we have a set of $m$ of these $(\beta+1)$-spheres $\{\mathcal{B}_{i_1}, \mathcal{B}_{i_2}, ..., \mathcal{B}_{i_m}\}$, where $2 \leq m \leq \beta$. Consider the $(\beta+2-m)$-sphere, defined by

$$x_{i_1}, x_{i_2}, ..., x_{i_m} = \frac{1}{c}, \quad \sum_{\substack{j=1 \\ j \neq i_1, ..., i_m}}^{\beta+2} x_j^2 = 1 - \frac{m}{c^2}$$

Notice that since $c > \sqrt{\beta}$ and $m \leq \beta$, we have $c^2 > m$, or $1 > \frac{m}{c^2}$, which means that the radius of this $(\beta+2-m)$-sphere, $\sqrt{1-\frac{m}{c^2}}$, will always be defined. Furthermore, this sphere will be contained in the intersection of all of the $(\beta+1)$-spheres $\mathcal{B}_{i_1}$ through $\mathcal{B}_{i_m}$. Pick any point $z = (z_1, ..., z_{\beta+2})$ on the given $(\beta+2-m)$-sphere, noting that $z_{i_1}, ..., z_{i_m} = \frac{1}{c}$. To verify that $z$ is on the sphere $\mathcal{B}_{i_k}$ (where $1 \leq k \leq m$), we can see that

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq i_k}}^{\beta+2} z_j^2 &= \left( \sum_{\substack{j=1 \\ j \neq i_1, ..., i_m}}^{\beta+2} z_j^2 \right) + \left( \sum_{l=1}^{m} z_{i_l}^2 \right) - z_{i_k}^2 \\ &= \left(1 - \frac{m}{c^2}\right) + \left(m \cdot \frac{1}{c^2}\right) - \frac{1}{c^2} \\ &= 1 - \frac{1}{c^2} \end{aligned}$$

Hence, we have $z_{i_k} = \frac{1}{c}$, and the equation for $\mathcal{B}_{i_k}$ is satisfied, so our $(\beta+2-m)$-sphere lies on $\mathcal{B}_{i_k}$ for all $1 \leq k \leq m$. Note that if we take the intersection of all of our $(\beta+1)$-spheres, $\mathcal{B}_1$ through $\mathcal{B}_\beta$, we will get the circle defined by

$$x_1, ..., x_\beta = \frac{1}{c}, \; x_{\beta+1}^2 + x_{\beta+2}^2 = 1 - \frac{\beta}{c^2}$$

So any intersection of a collection of our $(\beta+1)$-spheres will contain an infinite number of points.

For each natural number $p$, where $1 \leq p \leq \alpha$, we want a point $a_p$ adjacent to a specified subset of points in $B$, $\{b_{s_1}, ..., b_{s_t}\}$, where $t \leq \beta$. Place each $a_p = (w_1, ..., w_{\beta+2})$ on the intersection of the $(\beta+1)$-spheres $\mathcal{B}_{s_1}$ through $\mathcal{B}_{s_t}$, under the following conditions:

$$\sqrt{1 - \frac{t+1}{c^2}} < w_{\beta+2} < \sqrt{1 - \frac{t}{c^2}}, \text{ and } -\sqrt{1 - \frac{t}{c^2}} < w_q < 0 \text{ for all } 1 \leq q \leq \beta+1 \text{ such that } q \neq s_1, ..., s_t.$$

We also want $a_{p_1} \neq a_{p_2}$ if $p_1 \neq p_2$, which is always possible because there will be infinitely many points satisfying the above conditions. Let $A = \{a_p : 1 \leq i \leq \mu\}$. Notice that the integer $c > \sqrt{\beta}$, so $c^2 > \beta$, or $c^2 \geq \beta+1 \geq t+1$. Thus, $1 \geq \frac{t+1}{c^2} > 0$ and $w_{\beta+2}$ is always defined.

How close will the point $a_p$ be to any $b_v \in B$ if the two points are not meant to be adjacent? Remember that the $u_v$-coordinate of $b_v$ is $\frac{1}{c}$, and all other coordinates are zero. Thus, the Euclidean distance between

$a_p$ and $b_v$ will be

$$\begin{aligned}\sum_{j=1}^{\beta+2}(u_j-w_j)^2 &= \left(\sum_{\substack{j=1\\ j\neq v}}^{\beta+2} w_j^2\right)+\left(\frac{1}{c}-w_v\right)^2\\ &= \left(\left(\sum_{\substack{j=1\\ j\neq v,s_1,\ldots,s_t}}^{\beta+1} w_j^2\right)+\left(\sum_{k=1}^{t} w_{s_k}^2\right)+w_{\beta+2}^2\right)+\left(\frac{1}{c}-w_v\right)^2\\ &= \left(\left(\sum_{\substack{j=1\\ j\neq v,s_1,\ldots,s_t}}^{\beta+1} w_j^2\right)+\left(t\cdot\frac{1}{c^2}\right)+w_{\beta+2}^2\right)+\left(\frac{1}{c}-w_v\right)^2.\end{aligned}$$

We can perform the last step because $a_p$ is on the intersection of $\mathcal{B}_{s_1}$ through $\mathcal{B}_{s_t}$, which means $w_{s_k}=\frac{1}{c}$ for all $1\leq k\leq t$. Now, substituting the inequalites $w_{\beta+2}>\sqrt{1-\frac{t+1}{c^2}}$ and $w_v<0$, we get

$$\begin{aligned}\left(\left(\sum_{\substack{j=1\\ j\neq v,s_1,\ldots,s_t}}^{\beta+1} w_j^2\right)+\left(t\cdot\frac{1}{c^2}\right)+w_{\beta+2}^2\right)+\left(\frac{1}{c}-w_v\right)^2 &\geq \left(\left(t\cdot\frac{1}{c^2}\right)+w_{\beta+2}^2\right)+\left(\frac{1}{c}-w_v\right)^2\\ &> t\cdot\frac{1}{c^2}+1-\frac{t+1}{c^2}+\left(\frac{1}{c}-0\right)^2=1\end{aligned}$$

Therefore, the distance between any two points in $A$ and $B$ that are not meant to be adjacent is greater than 1 unit. Additionally, the distance between any two adjacent points is $\sqrt{1-\frac{1}{c^2}}$, and no point from $A$ can be closer than this distance to a point in $B$. Finally, we know that every point in $B$ is adjacent to at least one point in $A$ and vice versa. Therefore, we have $d(a,B)=\sqrt{1-\frac{1}{c^2}}$ for all $a\in A$, $d(b,A)=\sqrt{1-\frac{1}{c^2}}$ for all $b\in B$, and finally $h(A,B)=\sqrt{1-\frac{1}{c^2}}$, so our configuration $X$ satifies the PFAEL conditions. □

# 6 Finite Conversion Algorithm

If we have an infinite configuration $X$ with a finite number of elements at each location between its two sets, we can convert $X$ into a finite configuration using the algorithm given below. This algorithm is convenient because, as we will see later, it will preserve the number of elements at each location between two sets. Thus, we find that for our purposes, we can limit ourselves to finite configurations; if we cannot find any configuration with exactly $k$ elements at each location between sets in the finite case, we cannot find any such configuration, finite or infinite.

**Finite Conversion Algorithm 1.** *Let $A$ and $B$ in a configuration $X$ satisfy the PFAEL conditions, and let there exist $k\in\mathbb{N}$ such that there are $k\geq 2$ elements at each location between $A$ and $B$. Define $\Upsilon$ as before, and let $Q_A=\{[a]_C:[a]_C\in q_A(U)$ for some $U\in\Upsilon\}$ and $Q_B=\{[b]_C:[b]_C\in q_B(U)$ for some $U\in\Upsilon\}$. We will convert $X$ into a new configuration $X_F$, consisting of sets $Y$ and $Z$. Let $l=|Q_A|$ and $m=|Q_B|$.*

*Step 1: For each $[a_i]_C\in Q_A$ (where $1\leq i\leq l$), place a point $y_i\in Y$. Similarly, for each $[b_j]\in Q_B$ (where $1\leq j\leq m$), place a point $z_j\in Z$.*

*Step 2: For each $[a_i]_C \in Q_A$ (where $1 \le i \le l$), if there exists at least one point $c \in [a_i]_C$ such that $c \notin U$ for any $U \in \Upsilon$, place an endpoint $z_{y_i} \in Z$ and make $y_i \in Y$ adjacent to $z_{y_i}$.*

*Step 3: For each $[b_j]_C \in Q_B$ (where $1 \le j \le m$), if there exists at least one point $c \in [b_j]_C$ such that $c \notin U$ for any $U \in \Upsilon$, place an endpoint $y_{z_j} \in Y$ and make $z_j \in Z$ adjacent to $y_{z_j}$.*

*Step 4: For each $c \in [a_i]_C$, if $c \in U$ for some $U \in \Upsilon$, make $y_i$ adjacent to the $z_j \in Z$ which corresponds to the $b_j \in B$ such that $c \in [b_j]_C$.*

From [12], we know that $|U|$ is finite for all $U \in \Upsilon$. We also know from our bijective mapping from $\Upsilon$ to $\mathcal{K}$ that $|\Upsilon| = k$, and thus we have that $|Q_A|$ and $|Q_B|$ are finite. Finally, for any $[a]_C$, let $[a]_{C(\Upsilon)} = \{c \in [a]_C : c \in U \text{ for some } U \in \Upsilon\}$. If $|[a]_{C(\Upsilon)}| = \infty$, then we would have either an infinite $U$ or an infinite number of $U$'s containing a finite number of elements, both of which contradict that $|\Upsilon| = k$. Thus, $|[a]_{C(\Upsilon)}|$ is finite for every $[a_i]_C$. So we have all we need to ensure that $|Y|$ and $|Z|$ are finite. Additionally, $Y$ and $Z$ must be bounded, and as the finite union of sets of single points, they must also be closed. Therefore, $Y$ and $Z$ are in $\mathcal{H}(\mathbb{R}^n)$.

We also know that each point in $Y$ is adjacent to a certain subset of points in $Z$. Furthermore, any point $z_j$ will either be adjacent to some point $y_i$ or $y_{z_j}$ in $Y$, so by the Configuration Construction Theorem, $X_F$ satisfies the PFAEL conditions.

**Example 6.1.** *If we apply the Finite Conversion Algorithm to the configuration $X$ shown in Figure 6, where $A = A_1 \cup \{a_0\}$ and $B = B_1 \cup \{b_0\}$, we have $Q_A = \{[a_0]_C\}$ and $Q_B = \{[b_0]_C\}$, which creates points $y_0$ in $Y$ and $z_0$ in $Z$ by step 1. We also have endpoints $z_{y_0}$ and $y_{z_0}$ created by steps 2 and 3, respectively. When we adjoin all these points as prescribed, we get the configuration $X_F$ depicted in Figure* **??**, *which is equivalent to $S_4$. Notice that $X$ and $X_F$ both have two elements at each location between their two sets.*

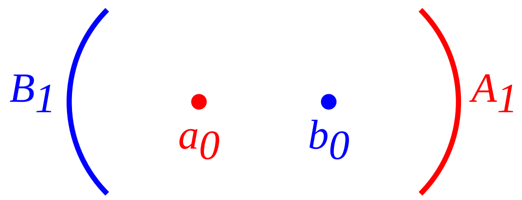



Figure 5: Infinite configuration $X$


Figure 6: Finite Configuration $X_F$

## 7 Matching Configurations

Now that we have the conversion process, all we must show is that the conversion preserves $\#(X)$ for a converted configuration $X$. This is the final part of our demonstration that all configurations have finite configurational representations.

Suppose that we convert a configuration $X$ of sets $A$ and $B$ to a finite configuration $X_F$ of sets $Y$ and $Z$ as described above. We will set the convention that $r' = h(Y, Z)$, and if $0 < s' < r'$, then $t' = r' - s'$. Let $W = (Y + t') \cap (Z + s')$. Then $h(Y, W) = t'$ and $h(W, Z) = s'$.

We know $X$ and $Y$ satisfy the PFAEL conditions. Define $\Upsilon_F$ as the set of all $U_F \in \mathcal{U}_W$ such that

1. for all $[y]_W \in q_Y(U_F)$, there exists $w \in [y]_W$ such that $w \notin U_F$.

2. for all $[z]_W \in q_Z(U_F)$, there exists $w \in [z]_W$ such that $w \notin U_F$.

Now we are ready for the next lemma.

**Theorem 7.1.** *Suppose that we convert a configuration $X$ with $\#(X) < \infty$ to a finite configuration $X_F$ as described in the algorithm, defining $\Upsilon$ and $\Upsilon_F$ as above. Then $|\Upsilon| = |\Upsilon_F|$.*

*Proof.* We know that the null set is in both $\Upsilon$ and $\Upsilon_F$, so we can focus our attention on the nonempty elements of both of these sets. Suppose $U \in \Upsilon$. We know that $|U| = \eta$ for some $\eta \in \mathbb{N}$, so $U = \{c_1, ..., c_\eta\}$, where $c_\gamma \in C$ for all $1 \leq \gamma \leq \eta$. So $c_\gamma \Bumpeq a_i$ and $c_\gamma \Bumpeq b_j$ for some $a_i \in A$ and $b_j \in B$. This means that $a_i \Bumpeq b_j$, implying that the corresponding points in $X_F$, $y_i$ and $z_j$ are adjacent by step 4 of our algorithm. Thus, each pair of these points in $X_F$ must have a $w_\gamma \in W$ which lies between them.

We will construct a set $U_F$ corresponding to $U$ and show that each $[y]_W \in q_Y(U_F)$ contains a point not in $U_F$. Let $U_F = \{w_\gamma : 1 \leq \gamma \leq \eta\}$. If $[y]_W \in q_Y(U_F)$, then $[y]_W = [y_i]_W$ for some $y_i$ adjacent to $w_\gamma$ where $1 \leq \gamma \leq \eta$. We know that the corresponding $[a_i]_C$ contains some set of $\tau$ points $\{c_k : 1 \leq k \leq \tau\}$ which are all not in $U$. If $c_k$ is in some other $U' \in \Upsilon$ for each $1 \leq k \leq \tau$, then $c_k$ is between $a_i$ and some $b_{j_k} \in B$, where $[b_{j_k}]_C \in q_B(U')$. Thus we have corresponding adjacent points $y_i$ and $z_{j_k}$ in $X_F$, and a point between them, $w_k$, in $W$. If $w_k$ were in $U_F$ for all $1 \leq k \leq \tau$, this would imply that $c_k \in U$ for all $k$, which is a contradiction. Hence there exists a $w_k$ adjacent to $y_i$ with $w_k \notin U_F$. On the other hand, if, for some $k$ with $1 \leq k \leq \tau$, $c_k$ is not in $U'$ for every $U' \in \Upsilon$, then we know that $y_i$ is adjacent to the endpoint $z_{y_i}$, and $[z_{y_i}]_W \notin q_Z(U_F)$. Once again, there exists a $w_k$ between $y_i$ and some point in $Z$ with $w_k \notin U_F$. In either case, every $[y]_W \in q_Y(U_F)$ contains a $w_k \notin W$. A similar series of steps will demonstrate that every $[z]_W \in q_Z(U_F)$ contains a $w_k \notin W$. Hence, $U_F \in \Upsilon_F$.

Now suppose that we have $U \neq U' \in \Upsilon$. Without loss of generality, we assume that there exists some $c' \in U'$ with $c' \notin U$. Suppose we make a set $U'_F$ out of $U'$ with the process given above. We know that $c' \Bumpeq a'$ and $c' \Bumpeq b'$ for some $a' \in A$ and $b' \in B$, so $U'_F$ will contain a point $w'$ between the corresponding points $y'$ and $z'$ in $X_F$. However, no such point will be created in $U_F$, since the point $c' \notin U$. Therefore, we have $U_F \neq U'_F$. This means that every element of $\Upsilon$ corresponds to some unique element of $\Upsilon_F$, so

$$|\Upsilon| \leq |\Upsilon_F| \tag{7.1}$$

On the other hand, suppose we have a set $U_F \in \Upsilon_F$. We will construct a set $U$ from $U_F$ and show that $U \in \Upsilon$. Since $Y$ and $Z$ are finite and satisfy the PFAEL conditions, we know that $U_F$ is finite, so $U_F = \{w_1, ..., w_\mu\}$, where $\mu \in \mathbb{N}$. Since $U_F \in \Upsilon_F$, we know that

1. for all $[y]_W \in q_Y(U_F)$, there exists $w \in [y]_W$ such that $w \notin U_F$.

2. for all $[z]_W \in q_Z(U_F)$, there exists $w \in [z]_W$ such that $w \notin U_F$.

Now for every $1 \leq \nu \leq \mu$, we know that $w_\nu \Bumpeq y_i$ and $w_\nu \Bumpeq z_j$ for some $y_i \in Y$ and $z_j \in Z$. This implies that for all such $\nu$, we have $y_i \Bumpeq z_j$. There are three scenarios in which this adjacency relationship could have been created by the conversion algorithm applied to the configuration $X$:

1. $y_i \Bumpeq z_{y_i}$ (where $z_j = z_{y_i}$)

2. $y_{z_j} \Bumpeq z_j$ (where $(y_i = y_{z_j})$

3. $y_i \Bumpeq z_j$

In the first case, the only point adjacent to $z_{y_i}$ would be $w_\nu$, so $[z_{y_i}]_W = \{w_\nu\}$, which contradicts our condition (2) on $U_F$. The second case will create a similar contradiction to condition (1) on $U_F$. Hence, this adjacency relationship must have been created by two adjacent points $a_i$ and $b_j$ in $X$ that have some point $c_\nu$ between them, where $c_\nu \in U$ for some $U \in \Upsilon$. Let $U^* = \{c_\nu : 1 \leq \nu \leq \mu\}$. Now suppose that for every $c \in [a_i]_C$, we have $c \in U^*$. This means that $a_i$ is only adjacent to a set of $b_{j_\rho}$'s, where $1 \leq \rho \leq \mu$. This means that the corresponding point $y_i$ is only adjacent to a set of $z_{j_\rho}$'s. We also know that there is a $c_\rho \in U^*$ between $a_i$ and every $b_{j_\rho}$, so there is a $w_\rho$ between $y_i$ and every $z_{j_\rho}$. However, then $[y_i]_W$ is the set of all the $w_\rho$'s, but this contradicts our condition (1) on $U_F$. If $[a]_C \in q_A(U)$, then $[a]_C = [a_i]_C$ for some $a_i$ adjacent to some $c_\nu$, and every $[a_i]_C$ contains some point not in $U$. Thus, we have that for all $[a]_C \in q_A(U^*)$, there exists $c \in [a]_C$ such that $c \notin U^*$. A similar series of statements will show that for all $[b]_C \in q_B(U^*)$, there exists $c \in [b]_C$ such that $b \notin U^*$.

Now, suppose that we have $U_F \neq U'_F \in \Upsilon_F$. Without loss of generality, we assume that there exists a $w' \in U'_F$ with $w' \notin U_F$. Suppose we create a set $U^* \in \Upsilon$ using the process given above. We know that there will exist a point $c' \in U^{*\prime}$ that corresponds to $w'$, but because $w' \notin U_F$, we have $c' \notin U^*$. Thus, $U^* \neq U^{*\prime}$, and we have that each $U_F \in \Upsilon_F$ corresponds to a unique $U^* \in \Upsilon$. Hence,

$$|\Upsilon_F| \leq |\Upsilon| \tag{7.2}$$

Finally, if we combine the inequalities (7.1) and (7.2), we find that $|\Upsilon| = |\Upsilon_F|$.

□

In [9], the authors prove that there exists no finite configuration with exactly 19 elements at each location between its sets. Hence, the next theorem follows directly:

**Theorem 7.2.** *There exists no configuration $X$ of sets $A$ and $B$ with 19 elements at each location between $A$ and $B$.*

*Proof.* By contradiction, suppose such an $X$ did exist. Then we could convert $X$ into $X_F$ using the conversion algorithm. Defining $\Upsilon$ and $\Upsilon_F$ as above and applying our lemma, we have $|\Upsilon| = |\Upsilon_F| = 19$. However, this is impossible, since no finite configuration with 19 elements at a given location exists. Therefore, there is no such configuration $X$. □

We will see that 7.1 has sweeping implications in terms of our search for other numbers with the same property as 19.

# 8 Finite Configurations and Counting Techniques

As a result of our infinite configuration conversion process, we can narrow our focus to only finite configurations with a finite number of elements at each location between two sets and still make generalizations about configurations in general. We now know that if there is some $k \in \mathbb{N}$ such that there is no finite configuration $X$ with $\#(X) = k$, then in general there is no such configuration, finite or infinite. In the following sections, we will focus on methods of counting the number of elements at each location for finite sets and talk about patterns we see and questions that arise.

First, we need to introduce some notation. In our diagrams, points labeled $\times$ will represent points from $A$ and points labeled $\circ$ will represent points from $B$. If two points are connected with a line segment, the two points are adjacent.

The information in [9] and [8] gives us several valuable tools for counting the total number of elements between two sets $A$ and $B$. There are two fundamental components to each configuration of points in $A$ and $B$, the *m-string* $S_m$ and the *2m-loop* $L_{2m}$. These are clearly defined in [9]; let it suffice to say that $S_m$ is a string of $m$ alternating elements in $A$ and $B$, and $L_{2m}$ is a $2m$-gon with alternating vertices from $A$ and $B$.

In [8], the authors demonstrate that for each $S_m$, we have $\#(S_m) = F_{m-1}$, the $m-1^{\text{st}}$ Fibonacci number. In a like fashion, $\#(L_{2m}) = l_{2m}$, the $2m^{\text{th}}$ Lucas number, for each loop $L_{2m}$. The Lucas numbers are defined by

$$l_1 = 1,\ l_2 = 3,\ l_n = l_{n-1} + l_{n-2}$$

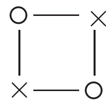
These are the first building blocks for our methods of counting arrangements in each configuration.

Recall that an endpoint in a configuration $X$ is adjacent to only one other point. We will use the notions of endpoints when we define counting for configurations which are neither strings nor loops. In general, many configurations are not of the form $S_m$ or $L_{2m}$. In [9], we learn an algorithm for counting arrangements after attaching points to strings and loops. If we add a new point to a point adjacent to an endpoint, the number of arrangements will not change. If we attach a point to a point $a \in A$, with $a$ not adjacent to any endpoints, in a configuration $X$ to make a new configuration $X'$, [9] tell us that

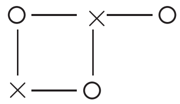
$$\#(X') = \#(X) + \#(X - \{a\}) \tag{8.1}$$

I will use the same nomenclature as in [9]. A configuration $Y$ defined by a string or loop $X_1$ and a string $X_2$ as $X_1 \oplus X_2[n]$ will consists of $X_1$ with $X_2$ attached at the $n^{\text{th}}$ point of $X_1$, the $n^{\text{th}}$ point of the string when viewed horiontally from left to right or $n$th point beginning in the upper right of a loop and rotating counterclockwise around the loop.

**Example 8.1.** *Let $X = L_4$ and let $X' = L_4 \oplus S_1[1]$. Then $X$ looks like*



*and $X'$ looks like*



*We know that $\#(X) = l_4 = 7$, so then $\#(X') = \#(X) + \#(S_3) = 7 + 1 = 8$.*

What happens if we keep adjoining points to the first point of $L_4$? Let $X'' = L_4 \oplus S_2[1]$ and let $X''' = L_4 \oplus S_3[1]$. Then, by (8.1), $\#(X'') = \#(L_4 \oplus S_1[1]) + \#(L_4) = \#(X') + \#(X) = 8 + 7 = 15$ and $\#(X''') = \#(L_4 \oplus S_2[1]) + \#(L_4 \oplus S_1[1]) = \#(X'') + \#(X') = 15 + 8 = 23$.

In general, let $Y_0$ be any loop or string, and let $Y_1 = Y_0 \oplus S_1[k]$ for some fixed $k^{\text{th}}$ point, call it $a$, on $Y_0$. If $a$ is adjoined to an endpoint, then $\#(Y_1) = \#(Y_0)$. If not, then by (8.1), $\#(Y_1) = \#(Y_0) + \#(Y_0 - \{a\})$.

Now let $Y_n = Y_0 \oplus S_n[k]$. We can see that

$$\begin{aligned}\#(Y_2) &= \#(Y_0 \oplus S_2[k]) \\ &= \#(Y_0 \oplus S_1[k]) + \#(Y_0) \\ &= \#(Y_1) + \#(Y_0).\end{aligned}$$

Then, by induction,

$$\begin{aligned}\#(Y_n) &= \#(Y_0 \oplus S_n[k]) \\ &= \#(Y_0 \oplus S_{n-1}[k]) + \#(Y_0 \oplus S_{n-2}) \\ &= \#(Y_{n-1}) + \#(Y_{n-2})\end{aligned}$$

and the sequence $\{\#(Y_n)\}$ is a Fibonacci-type sequence in that each term is the sum of the two terms previous to it.

## 9 Generating Sequences

In [9], the authors generate the configurations which yield every integer number of elements at each location between $A$ and $B$ for all integers between 1 and 35 except for 19. The multiplicative property of configurations [9] also provides configurations which will give all composite integer number of elements at each location between $A$ and $B$, granted that we can find configurations for the prime factors of each integer. Thus, after we discovered the Fibonacci-type property of adding consecutive elements of a string to configurations, we decided to use this feature to try to generate all of the other primes and unaccounted-for composite numbers between 36 and 100. The primes are $37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89,$ and $97$. Here is a list of the primes and the configurations which produced them:

- 37: ??
- 41: ??
- 43: $S_8 \oplus S_3[4]$
- 47: $L_8$
- 53: $S_7 \oplus \{S_2[4]; S_3[4]\}$
- 59: ??
- 61: $L_4 \oplus S_5[1]$
- 67: ??
- 71: $S_8 \oplus S_4[4]$
- 73: $S_{10} \oplus S_2[4]$
- 79: ??
- 83: $S_6 \oplus L_4[3]$
- 89: $S_{12}$
- 97: ??

After an exhaustive search we could not find a way to configure points in order to produce with 37, 41, 59, 67, 79, or 97 elements at each location between two sets, and we are fairly convinced that no such configuration exists for five of these numbers (we later found 79 using a configuration in $\mathbb{R}^3$). The next question we asked was based on the primes we could not find, are there any composite numbers less than 100 which do not describe the number of elements at a given location between any two sets $A$ and $B$? The set of these possible composite numbers is $\{38, 57, 74, 82, 95\}$. Here are our findings:

- 38 $(19 \cdot 2)$: $L_4 \oplus S_4[1]$
- 57 $(19 \cdot 3)$: ??
- 74 $(37 \cdot 2)$: $S_{10} \oplus S_2[5]$
- 82 $(41 \cdot 2)$: ??
- 95 $(19 \cdot 5)$: $S_6 \oplus \{S_2[3]; S_2[3]; S_2[3]; S_2[3]\}$

So I can add 57 and 82 to the list of numbers I have been unable to find using current methods. I also catalogued the various Fibonacci-type sequences I was able to find in order to look for patterns in which sequences exist and which seem not to exist. I list the triple $(\#(Y_1) - \#(Y_0), \#(Y_0), \#(Y_1))$ - I think the $\#(Y_1) - \#(Y_0)$ term will help identify patterns - which will generate each sequence, and the pair of configurations $(Y_0, Y_1)$ which served as the basis for finding each sequence (I leave out sequences which are part of sequences already listed; for example, $(1, 2, 3)$ contains $(2, 3, 5)$, so I do not include $(2, 3, 5)$):

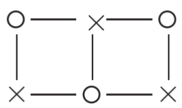
- $(1, 2, 3) : (S_4, S_5)$
- $(1, 3, 4) : (S_5, S_5 \oplus S_1[3])$
- $(1, 5, 6) : (S_6, S_6 \oplus S_1[3])$
- $(1, 7, 8) : (S_5 \oplus S_2[3], S_5 \oplus \{S_2[3]; S_1[3]\})$ or $(L_4, L_4 \oplus S_1[1])$

- $(1, 8, 9) : (S_7, S_7 \oplus S_1[4])$ or $(L_4 \oplus S_1[1], L_4 \oplus \{S_1[1]; S_1[3]\})$
- $(1, 11, 12) : (S_6 \oplus S_2[3], S_6 \oplus \{S_2[3]; S_1[3]\})$
- $(1, 15, 16) : (S_5 \oplus \{S_2[2]; S_2[2]\}, S_5 \oplus \{S_2[2]; S_2[2]; S_1[2]\})$ or $(L_4 \oplus S_2[1], L_4 \oplus \{S_2[1]; S_1[1]\})$
- $(1, 17, 18) : (S_7 \oplus S_2[4], S_7 \oplus \{S_2[4]; S_1[4]\})$ or $(L_4 \oplus \{S_2[1]; S_1[3]\}, L_4 \oplus \{S_2[1]; S_1[1]; S_1[3]\})$
- $(2, 6, 8) : (S_6 \oplus S_1[3], S_6 \oplus \{S_1[3]; S_1[4]\})$
- $(2, 8, 10) : (S_7, S_7 \oplus S_1[3])$ or $(L_4 \oplus S_1[1], L_4 \oplus \{S_1[1]; S_1[2]\})$
- $(2, 10, 12) : (S_7 \oplus S_1[3], S_7 \oplus \{S_1[3]; S_1[4]\})$ or $(S_7 \oplus S_1[3], S_7 \oplus \{S_1[3]; S_1[5]\})$ or $(L_4 \oplus \{S_1[1]; S_1[2]\}, L_4 \oplus \{S_1[1]; S_1[2]; S_1[3]\})$
- $(2, 13, 15) : (S_8, S_8 \oplus S_1[4])$
- $(2, 14, 16) : (S_6 \oplus \{S_2[3]; S_1[4]\}, S_6 \oplus \{S_2[3]; S_1[3]; S_1[4]\})$
- $(2, 15, 17) : (L_4 \oplus S_2[1], L_4 \oplus \{S_2[1]; S_1[3]\})$
- $(2, 16, 18) : (S_8 \oplus S_1[3], S_8 \oplus \{S_1[3]; S_1[5]\})$
- $(3, 9, 12) : (S_7 \oplus S_1[4], S_7 \oplus \{S_1[3]; S_1[4]\})$ or $(L_4 \oplus \{S_1[1]; S_1[3]\}, L_4 \oplus \{S_1[1]; S_1[2]; S_1[3]\})$
- $(3, 11, 14) : (S_6 \oplus S_2[3], S_6 \oplus \{S_2[3]; S_1[4]\})$
- $(3, 13, 16) : (S_8, S_8 \oplus S_2[3])$
- $(3, 15, 18) : (S_8 \oplus S_1[4], S_8 \oplus \{S_1[4]; S_1[5]\})$ or $(S_8 \oplus S_1[4], L_4 \oplus \{S_1[4]; S_1[6]\})$ or $(L_4 \oplus S_2[1], L_4 \oplus \{S_2[1]; S_1[2]\})$
- $(4, 12, 16) : (S_7 \oplus \{S_1[3]; S_1[4]\}, S_7 \oplus \{S_1[3]; S_1[4]; S_1[5]\})$ or $(L_4 \oplus \{S_1[1]; S_1[2]; S_1[3]\}, L_4 \oplus \{S_1[1]; S_1[2]; S_1[3]; S_1[4]\}$
- $(4, 16, 20) : (S_8 \oplus S_1[3], S_8 \oplus \{S_1[3]; S_1[4]\})$ or $(S_8 \oplus S_1[3], S_8 \oplus \{S_1[3]; S_1[6]\})$
- $(5, 15, 20) : (S_8 \oplus S_1[4], S_8 \oplus \{S_1[3]; S_1[4]\})$
- $(5, 17, 22) : (S_7 \oplus S_2[4], S_7 \oplus \{S_1[3]; S_2[4]\})$ or $(L_4 \oplus \{S_2[1]; S_1[3]\}, L_4 \oplus \{S_2[1]; S_1[2]; S_1[3]\})$

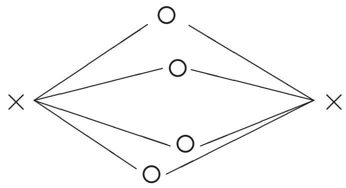
I was unable to recognize any real patterns in this data, although it may help to expand the list further. I would also like to examine the sequences which do not seem to exist, namely sequences such $(1, 4, 5)$, which would produce 37, and $(1, 9, 10)$, which would produce 19.

## 10 The Looping Algorithm

We know how to count the number of elements at each location between two sets created by adding single points to a configuration, but how do we examine what happens when we connect two points that already exist in a configuration? We already can count the number of elements at each location for polygonal configurations from [8], but we lack the tools to characterize configurations such as



or



In these cases we must use the Looping Algorithm, which gives us a precise method of determining $\#(X)$ for configurations with multiple loops and/or in 3-dimensional real space and above.

**Theorem 10.1.** *Let $a \in A$ and $b \in B$ be two non-adjacent points in a finite configuration $X$. If a new configuration $X'$ is formed by making these two points adjacent, then*

$$\#(X') = \#(X) + \#(X^*), \tag{10.1}$$

*where $X^*$ is the configuration formed by the two sets $A^* = A \cup \{a_*\}$ and $B^* = B \cup \{b_*\}$, for some endpoint $b_* \in \mathbb{R}^n$ adjacent to $a \in A$ and some endpoint $a_* \in \mathbb{R}^n$ adjacent to $b \in B$, with $a_* \not\approx b_*$.*

*Proof.* Let $h(A, B) = r$, let $0 < s < r$, and let $t = r - s$. In our new configuration $X'$, let the two sets of points (in which $a \approx b$) be called $A'$ and $B'$. Note that $h(A', B') = h(A, B)$. Then let $C = A' + t \cap B' + s$. This is the largest element of $\mathcal{H}(\mathbb{R}^n)$ which lies at the location $t$ units from $A'$ [9], so each element $\hat{C}$ that satisfies $A\hat{C}B$ will be a subset of $C$. Note that there is a point $c_0 \in C$ between $a$ and $b$, because we have moved these two points a distance $h(A, B)$ apart.

Let $C' = \{C^\dagger \in \mathcal{H}(\mathbb{R}^n) : c_0 \in C^\dagger$ and $C^\dagger$ satisfies $A'C^\dagger B'\}$, and let $C'' = \{C^{\dagger\dagger} \in \mathcal{H}(\mathbb{R}^n) : c_0 \notin C^{\dagger\dagger}$ and $C^{\dagger\dagger}$ satisfies $A'C^{\dagger\dagger}B'\}$. Then

$$\#(X') = |C''| + |C'|$$

Every element between $A$ and $B$ in $X$ will be at the same location between $A'$ and $B'$ in $X'$ and will not contain the point $c_0$, and every element of $C''$ will satisy $AC''B$, so $|C''| = \#(X)$, and we have

$$\#(X') = \#(X) + |C'|$$

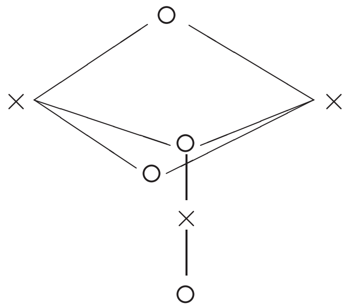
In the configuration $X^*$, let $a_*, A^*, b_*$, and $B^*$ be defined as above. Since $a_*$ and $b_*$ are endpoints, any $C^*$ at some location between $A^*$ and $B^*$ in $X^*$ must contain a point $c_1$ between $a$ and $b_*$ and another point $c_2$ between $b$ and $a_*$. Let $C_1^\dagger = C^* - \{c_1, c_2\}$. Then we are assured that $C_1^\dagger$ contains a point adjacent to every point in $A$ and $B$, except for $a$ and $b$. If we create a set $C_2^\dagger$ by placing every point in $C_1^\dagger$ that is between two points in $A$ and $B$ into $C_2^\dagger$ between the corresponding points in $A'$ and $B'$, and then we add the point $c_0$ to $C_2^\dagger$, we can see that every point in $A'$ and $B'$ will be adjacent to some point in $C_2^\dagger$, and thus $C_2^\dagger$ will satisfy $A'C_2^\dagger B'$. Hence, $C_2^\dagger$ is in $C'$.

This process is reversible, so for every $C^\dagger \in C'$, if we let the set $C_1^* = C^\dagger - \{c_0\}$, and then we create a new set $C_2^*$ with adjacencies corresponding to the sets $A$ and $B$, we can add $c_1$ and $c_2$ to $C_2^*$ and show that the resulting set will be between the two sets in $X^*$. Thus, $|C'| = \#(X^*)$, and finally, we have

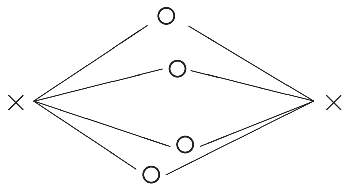
$$\#(X') = \#(X) + \#(X^*)$$

□

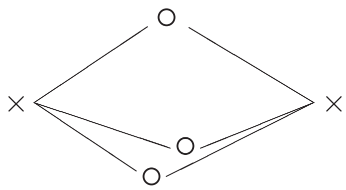
Now, we can use this algorithm combined with the point-adding algorithm in [9] to count the number of elements at each location between any two finite sets $A$ and $B$. This new counting method helped us find previously unknown configurations in $\mathbb{R}^3$ which yield 57:



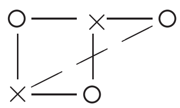
and 79:



elements at each location between $A$ and $B$. To determine $\#(X)$ for each of these configurations, we had to know the number of elements at each location between $A$ and $B$ on the three dimensional configuration consisting of two $\times$'s and three $\circ$'s, pictured here:



We will call this configuration $L_{2\times3}$, where $L_{2\times3}[1]$ and $L_{2\times3}[3]$ are $\times$'s; all other points are $\circ$'s. We use this notation because our configuration has two points in $A$ and three points in $B$, with each point adjacent every point in the opposing set. Now we can use the algorithm for counting loops to find $\#(L_{2\times3})$ by connecting the non-loop point of $L_4 \oplus S_1[1]$ to the $\times$ to which it is not already connected, as shown:



The looping algorithm tells us

$$\begin{aligned}\#(L_{2\times3}) &= \#(L_4 \oplus S_1[1]) + \#(L_4 \oplus \{S_2[1]; S_1[3]\}) \\ &= 8 + 17 = 25.\end{aligned}$$

Now we can use this information to compute the number of each elements at each location for both configurations in our drawings. For the first drawing, we only have to use the point-adding algorithm (note that when we refer to $L_{2\times3} \oplus S_2[2]$ the [2] is arbitrary; we could have just as well used [4] or [5]):

$$\begin{aligned}\#(L_{2\times3} \oplus S_2[2]) &= \#(L_{2\times3} \oplus S_1[2]) + \#(L_{2\times3}) \\ &= 2 \cdot \#(L_{2\times3}) + \#(L_4) \\ &= 2 \cdot 25 + 7 = 57.\end{aligned}$$

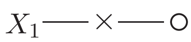
For the second configuration, we will have to use the looping algorithm once more, connecting the point in $L_{2\times3} \oplus S_1[1]$ to the $\times$ to which it is not already adjacent. We will call this new configuration $L_{2\times4}$, and we find that

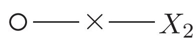
$$\begin{aligned}
\#(L_{2\times4}) &= \#(L_{2\times3} \oplus S_1[1]) + \#(L_{2\times3} \oplus \{S_2[1]; S_1[3]\}) \\
&= \#(L_{2\times3} \oplus S_1[1]) + \#(L_{2\times3} \oplus S_2[1]) + \#(S_4) \\
&= 2 \cdot \#(L_{2\times3} \oplus S_1[1]) + \#(L_{2\times3}) + \#(S_4) \\
&= 2 \cdot (25 + 1) + 25 + 2 = 89
\end{aligned}$$

So now we see that we have all of the tools to build and count every finite configuration. Any configuration can be completely constructed by adding points and looping points, and now we have algorithms to determine the new number of elements between two sets $A$ and $B$ created by each process. Furthermore, every configuration has a finite configurational representation, so we can limit our search for the possibilites of $\#(X)$ to the finite cases. The next step is to systematically find and prove that other numbers such as 19 that do not exist in the Hausdorff Metric Geometry.

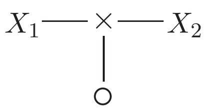
We can limit our search by eliminating certain configurations, as demonstrated by the following argument:

Suppose that we have two configurations $X_1'$ and $X_2'$ as shown below. Note that $X_1'$ is the union of the configuration $X_1$ and the points depicted adjacent to $X_1$, and $X_2'$ is the union of the configuration $X_2$ and the points depicted adjacent to $X_2$.




We claim that $\#(X_1') \cdot \#(X_2') = \#(X_{1,2}')$, where $X_{1,2}'$ is depicted below. Note that $X_{1,2}'$ is the union of $X_1$, $X_2$, and the points depicted adjacent to these two configurations.




The argument is as follows: Let $\Upsilon_1$, $\Upsilon_2$, and $\Upsilon_{1,2}$ be defined for $X_1'$, $X_2'$, and $X_{1,2}'$ (respectively) as before. Let $U_1 \in \Upsilon_1$ and $U_2 \in \Upsilon_2$. Let $c_1$ be the point between $X_1$ and the $\times$ in $X_1'$ and let $c_2$ be the point between $X_2$ and the $\times$ in $X_2'$. Clearly, $U_1$ will contain no point between the $\times$ and $\circ$ in $X_1'$, so we can translate it to be in the same position relative to $X_1$ in $X_{1,2}$ as it was to $X_1$ in $X_1'$. Call this translated set $U_1'$. Similarly, we can translate $U_2$ to be in the same position relative to $X_2$ in $X_{1,2}$ as it was to $X_2$ in $X_2'$. Call this translated set $U_2'$.

We claim that $U_1' \cup U_2' \in \Upsilon_{1,2}$. This fact is obvious if $c_1 \notin U_1$ and and $c_2 \notin U_2$: since both sets satisfy

1. for all $[a]_C \in q_A(U)$, there exists $c \in [a]_C$ such that $c \notin U$
2. for all $[b]_C \in q_B(U)$, there exists $c \in [b]_C$ such that $b \notin U$

for their respective configurations, their translations must satisfy these conditions in $X_{1,2}'$. On the other hand, suppose that $c_1 \in U_1$, then we can that for the translated point, $c_1'$, we know that $q_A(c_1)$ must contain the point between the $\times$ and $\circ$ shown in $X_{1,2}$ (this point will never be in any $U \in \Upsilon_{1,2}$ because it is adjacent to an endpoint). Clearly, $U_1$ will also satisfy condition (2). We can present a similar argument if $c_2 \in U_2$. Basically, we can say that $U_1'$ and $U_2'$ act independently of each other because the elements in $U_2'$ will never prevent the elements in $U_1'$ from satisfying the above conditions, and vice versa. Thus, every $U_1' \cup U_2'$ will be a distinct element of $\Upsilon_{1,2}$.

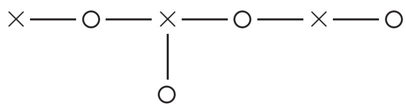
Furthermore, we can see that every $U_{1,2} \in \Upsilon_{1,2}$ is composed of two such disjoint sets $U_1'$ and $U_2'$, which are independent and disjoint for the reasons listed above. If we translate $U_1'$ to be in the same position relative to $X_1$ in $X_1'$ as it was to $X_1$ in $X_{1,2}$, we can see that this translated set will be satisfy the above conditions, so this translated set will be in $\Upsilon_1$. We can perform a similar translation with the other disjoint set $U_2'$ to get a set in $\Upsilon_2$. Thus, each element in $V_{1,2}$ can be dissected into a distinct pair of elements in $\Upsilon_1$ and $\Upsilon_2$. Hence, we have

$$|\Upsilon_1| \cdot |\Upsilon_2| = |\Upsilon_{1,2}|.$$

We can easily see that if this equality holds, then the statement $\#(X_1') \cdot \#(X_2') = \#(X_{1,2}')$ that we claimed earlier must be true.

**Example 10.1.** *We will let $X_1' = S_4$ and $X_2' = S_5$ and unite them in the manner described above. We see that*

*through the adjoining process becomes*



*We verify that $\#(X_1') \cdot \#(X_2') = 2 \cdot 3 = 6$, and we can see that $X_{1,2}'$ is the same as $S_6 \oplus S_1[3]$, so we have $\#(X_{1,2}') = \#(S_6) + 1 = 5 + 1 = 6$. Thus, the formula works for this example.*

This concept can be extended to more than two sets $X_1'$ and $X_2'$; if we unite three or more sets in this manner, the above arguments still hold. Additionally, the arguments make sense if the $\times$'s shown have more than one adjacency with $X_1$ and/or $X_2$. This information can be very useful in searching for configurations $X$ with $\#(X)$ prime because if the above structure is present in $X$, we know that $\#(X)$ is composite.

# 11 SPACK Numbers

Our investigation led to the definition of a special type of numbers, which we call SPACK numbers:

**Definition 11.1.** *A number $p$ is a **SPACK-**$n$ number if and only if there exists a configuration of two sets $A$ and $B$ in $\mathbb{R}^n$ with $p$ elements at each location between $A$ and $B$, and no such configuration exists in $\mathbb{R}^{n-1}$. If no such configuration exists in any dimension, then $p$ is called a SPACK-0 number. If a SPACK-$n$ number $p$ is prime, than we call $p$ a SPACK-$n$ prime.*

The only configurations which can be realized on the real number line ($\mathbb{R}^1$) are single or multiple $m$-strings. The multiplicative property of configurations demonstrates that if we have more than one string on the real line, the number of elements at each location will be a composite number. So our SPACK-1 numbers will only be those that can be written as Fibonacci numbers or as the product of Fibonacci numbers. We can easily see that the only SPACK-1 primes will be those numbers that are Fibonacci primes. It is widely suspected but yet to be proven that there are infinitely many SPACK-1 primes [7].

Furthermore, we can realize every Lucas number in $\mathbb{R}^2$ (see our discussion of Fibonacci-type sequences), so it is also thought that there are an infinite number of SPACK-2 primes. As of our research to this point, the SPACK-3 numbers listed below are only conjectured, and 19 is the only confirmed SPACK-0 number, although we believe that 37 falls into this category as well. Here is a list of some of the SPACK-$n$ numbers we have determined:

- SPACK-1 numbers: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 15, 16, 18, 20, 21, ...
  SPACK-1 primes: 2, 3, 5, 13, 89 ,233, 1597, 28657, ...
- SPACK-2 numbers: 7, 11, 14, 17, 22, 23, 28, 29, 31, 33, 35, 38, 43, 44, ...
  SPACK-2 primes: 7, 11, 17, 23, 29, 31, 43, 47, ...
- SPACK-3 numbers: 57, 79 ?
  SPACK-3 primes: 79 ?
- SPACK-4+ numbers: ?
- SPACK-0 numbers: 19, 37 ?

This distinction is an interesting new way to partition the natural numbers (every number will be a SPACK-$n$ number for some $n$), and these numbers may have number theoretic applications of which we are currently unaware.

## 12 Graph Theory

Our interpretation of finite configurations has strong ties to graph theory. In fact, we often gather data by treating these configurations as graphs rather than compact sets in $\mathbb{R}^n$. We can interpret our results concerning the number 19 in a graph theoretical manner. In this section, we demonstrate the strong correlation between Hausdorff Geometry and a specific type of graph, about which we will go into greater detail later. First, some definitions:

**Definition 12.1.** *A **bipartite graph** is a graph in which the set of vertices can be divided into two disjoint sets with two vertices of the same set never sharing an edge.*

All of the finite configurations we have drawn can, in fact, be interpreted as bipartite graphs - the $\times$ symbols form one vertex set, the $\circ$'s form the other, the edge set is created from the adjacencies in the configuration, and points in the same set are never adjacent to each other.

**Definition 12.2.** *We call a graph in which there are no isolated vertices a **linked** graph.*

In a linked graph, every vertex will have a **degree** of at least one. The degree of a vertex is the number of edges connected to it. We will only be dealing with linked graphs, because isolated vertices are irrelevant to our geometrical interpretation.

**Definition 12.3.** *A **simple** graph has no more than one edge between any two vertices and no edges from a vertex to itself.*

From this point forward, any time we use the term "graph" we will mean "simple graph."

**Definition 12.4.** *A **labeled** graph is a graph in which each vertex is assigned a label.*

We use labeled graphs because each of our compact sets contain specific points in $\mathbb{R}^n$ - each point is unique, so it makes sense to make each vertex unique as well.

We know now that any finite configuration can be represented by a linked labeled bipartite graph. However, are there linked labeled bipartite graphs which do not have equivalent configurational representations? The Configuration Construction Theorem can answer this question.

If $G$ is a linked labeled bipartite graph, it will have two finite sets of vertices, which will become $A$ and $B$ from the theorem, and if we wish to make the adjacencies in our configuration correspond to edges in $G$, we know that each vertex has an edge with at least one other vertex from the opposing set. Hence, this theorem tells us we can create a configuration out of $G$ that satisfies the PFAEL conditions. Now we have all the necessary tools to apply our Hausdorff findings to these graphs.

We will set the convention if we convert a graph $G$ to a configuration $X$ of sets $Y$ and $Z$ as described in the Configuration Construction Theorem, then $r' = h(Y, Z)$, and if $0 < s' < r'$, then $t' = r' - s'$. Let the set $W = (Y + t') \cap (Z + s')$. Then $h(Y, W) = t'$ and $h(W, Z) = s'$. Furthermore, we will let $\mathcal{G}$ be the set of all linked subgraphs of $G$ with the same vertex set as $G$ (we will consider $G$ a subgraph of itself).

**Theorem 12.1.** *Every linked labeled bipartite graph $G$ has an equivalent configuration $X$ of sets $Y$ and $Z$ with $\#(X)$ equal to the number of linked subgraphs of $G$ with the same vertex set as $G$.*

*Proof.* Because $G$ is bipartite, we can separate its vertices into two disjoint sets $A$ and $B$, letting $|A| = \alpha$ and $|B| = \beta$. Since $G$ is a labeled graph, we will have vertices $a_,..., a_\alpha \in A$ and $b_1, ..., b_\beta \in B$. Let $\mathcal{G}$ denote the set of all linked subgraphs of $G$ with the same vertex set as $G$. Now convert $G$ into a finite configuration $X$ with the Configuration Construction Theorem. We will show that $|\mathcal{G}| = \#(X)$.

Consider $G' \in \mathcal{G}$. We know that $G'$ has the same number of vertices as $X$ has points, because $G'$ has the same vertex set as $G$. We will construct a set $W'$ which lies between $Y$ and $Z$ from $G'$. For each edge between some $a_i$ and $b_j$ in $G'$, the corresponding $y_i \in Y$ and $z_j \in Z$ will be adjacent, so there will be some point $w_k$ between them. Let $W$ be the set of all such points $w_k$. Since $G'$ is a linked subgraph, we must have that every vertex $a_i$ and $b_j$ is adjacent to at least one point in the opposing set. Hence, each $y_i$ and $z_k$ must be adjacent to some point in the opposing set, which means each of these points must be adjacent to some $w_k \in W'$, so we know that $h(W', Y) \leq h(W, Y)$. Furthermore, $W' \subset W$, so we know that $h(W', Y) \geq h(W, Y)$. Therefore, we have $h(W', Y) = h(W, Y)$, and $W'$ lies between $Y$ and $Z$ at the same location as $W$.

Finally, suppose that $G^\dagger \neq G'$ for some $G^\dagger \in \mathcal{G}$. Without loss of generality, we can assume that $G^\dagger$ contains an edge $E$ which is not a part of $G$. Hence, $E$ adjoins two vertices $a_\dagger$ and $b_\dagger$ in $G^\dagger$ that have no edge between them in $G'$. If $G^\dagger$ is converted to a configuration to a set $W^\dagger$ by the process given above, then $W^\dagger$ will contain a point $w_\dagger$ between adjacent points $y_\dagger$ and $z_\dagger$, but $w_\dagger$ will not be in $W$, so we have $W^\dagger \neq W$. This means that each element of $\mathcal{G}$ corresponds to a unique element in $X$, and thus

$$|\mathcal{G}| \leq \#(X). \tag{12.1}$$

Now consider some element $W^*$ that lies between $Y$ and $Z$. Since $\#(X)$ is finite, we have $|W^*|$ is also finite, so let $W = \{w_1, ..., w_l\}$. We will construct a linked subgraph $G^* \in \mathcal{G}$ with the same vertex set as $G$ out of $W^*$. Thus, let $G^*$ be a graph with an empty set of edges and the same vertex set as $G$. For every point $w_k \in W$, where $1 \leq k \leq l$, we know that $w_k \Leftrightarrow y_i$ and $w_k \Leftrightarrow z_j$ for some $y_i \in Y$ and $z_j \in Z$. For every such pair of adjacent points $y_i$ and $z_j$ in $X$, make an edge between the vertices $a_i$ and $b_j$ in $G^*$. These vertices must exist in $G^*$, since they exist in $G$. We already know that $G^*$ is a subgraph of $G$ with the same vertex set as $G$, so all we must show is that $G^*$ is linked. For any vertex $a_i \in G^*$, the corresponding point $y_i$ must be adjacent to some $w_k \in W$. This means that $y_i$ must have $w_k$ between $y_i$ and some $z_j$, so means there must be an edge between $a_i$ and the corresponding $b_j$. A similar series of arguments will show that every vertex $b_j \in G$ will not be isolated. Thus, $G^*$ is a linked subgraph of $G$, and $G^* \in \mathcal{G}$.

Finally, suppose $\hat{W} \neq W^*$ for some $\hat{W}$ at the same location between $Y$ and $Z$ in $X$. Without loss of generality, we suppose there exists $\hat{w} \in \hat{W}$ such that $\hat{w} \notin W^*$. We know that $\hat{w}$ is adjacent to some $\hat{y} \in Y$ and $\hat{z} \in Z$, so if we convert $\hat{W}$ to a linked subgraph $\hat{G}$ of $G$ by the process given above, there will be an edge between the corresponding $\hat{a}$ and $\hat{b}$ in $\hat{G}$. However, this edge will not be contained in $G^*$, and we have $\hat{G} \neq G^*$. Hence, any two distinct elements $W^*$ and $\hat{W}$ between $Y$ and $Z$ at the same location will create distinct elements of $\mathcal{G}$, so each $W^*$ corresponds to a unique linked subgraph of $G$ with the same vertex set as $G$, which means

$$|\mathcal{G}| \geq \#(X). \tag{12.2}$$

Combining the two inequalities (12.1) and (12.2) yields the equation $|\mathcal{G}| = \#(X)$. Hence, the configuration $X$ has a number of elements at each location between $Y$ and $Z$ equal to the number of linked subgraphs of $G$ with the same vertex set as $G$.

□

**Corollary 1.** *There exists no linked labeled bipartite graph $G$ with exactly 19 linked subgraphs with the same vertex set as $G$.*

*Proof.* By contradiction, suppose that such a graph $G$ did exist. Then, by the preceding theorem, $G$ could be converted to a configuration $X$ with exactly 19 elements at each location between the two sets that make up $X$. However, we have proved that there exists no such configuration $X$. Therefore, we must also have that no graph $X$ can exist. □

Although we have limited ourselves to simple graphs, we suspect that Corollary 1 will hold true even if we allow multiple edges between two vertices and/or edges that connect the same vertex.

## 13 Future Research

We would like to pose some unanswered questions which we may address in the future or could be addressed by another research group:

- Are there other numbers such as 19 which never appear as the exact number of elements between $A$ and $B$ at a given location (37, 41, etc...)?
- Could we somehow fit an algebraic group-like structure to these configurations to explain why certain numbers are missing from the Hausdorff Metric Geometry?
- Does this geometry change if, instead of working in $\mathbb{R}^n$, we make configurations on the surface of a torus? On a torus, the graphs that correspond to our configurations will no longer have to be simple graphs. Could we experiment with other surfaces?
- We strongly believe that there is no configuration which will have exactly a number of elements at each location between two sets that has the cardinality of the integers. How can we prove this statement?
- Could there be a more elegant proof of the non-existence of the the number 19?
- Can we find some sort of combinatorial approach to computing the number of elements at each location for a given configuration?
- Can we prove that 57 and 79 are indeed SPACK-3 numbers, and are there any SPACK-$n$ numbers where $n > 3$?

In conclusion, this research project proved to be very worthwhile and enjoyable, and we are happy to have been given the opportunity to work with, learn about, and struggle through the Hausdorff Metric Geometry during our research experience at Grand Valley State University.

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