Who's Line Is It Anyway?

Lines Defined by Two Point Sets in $\mathcal{H}(\mathbb{R}^n)$
and How They Differ From Euclidean Lines

GVSU 2003 REU Final Report*

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Abstract: The space composed of all non-empty, compact subsets in $\mathbb{R}^n$ is denoted as $\mathcal{H}(\mathbb{R}^n)$. When $\mathcal{H}(\mathbb{R}^n)$ is coupled with the Hausdorff metric, h, we have a complete metric space, and it is the geometry of this space which will be our focus. More specifically, we will look at the set of lines in $\mathcal{H}(\mathbb{R}^n)$ that are defined by two point sets. The properties of these lines are very surprising, for they do not act line the familiar lines in $\mathbb{R}^n$.

1 Background

Before we begin our main study, we need to define a few background terms that we will frequently use. Namely, we need to know the terms open neighborhood, boundary, and closed neighborhood in $\mathbb{R}^n$. Also, we need to understand what a metric is.

Definition 1. *Let $a \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Then,*

$$N_r(a) = \{x \in \mathbb{R}^n : d(x, a) < r\},$$

where $N_r(a)$ denotes the open neighborhood of radius r around a.

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Next, we can use the concept of a neighborhood to define the boundary of a set.

Definition 2. *Let X be a set of points in $\mathbb{R}^n$. The boundary of X, denoted ∂X, consists of all $x \in \mathbb{R}^n$ such that every open neighborhood around x contains at least one point in X and at least one point not in X.*

Finally, a closed neighborhood is the union of an open neighborhood with its boundary.

Definition 3. *Let $a \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Then,*

$$\overline{N_r(a)} = N_r(a) \cup \partial N_r(a) = \{x \in \mathbb{R}^n : d(x,a) \leq r\},$$

where $\overline{N_r(a)}$ denotes the closed neighborhood of radius r around a.

Now, let us move on to the notion of a metric.

Definition 4. *Let X be a set. A function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y, z \in X$,*

1. $d(x,y) = d(y,x)$
2. $d(x,y) \geq 0$ *with equality if and only if* $x = y$
3. $d(x,z) \leq d(x,y) + d(y,z)$.

A metric is used to provide a measure of distance between points in a set. The familiar Euclidean metric, d, is a great example of a metric, where the distance measured is between points in $\mathbb{R}^n$. In other words, if $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ are elements of $\mathbb{R}^n$, then the distance from x to y is

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}.$$

2 $\mathcal{H}(\mathbb{R}^n)$ and the Hausdorff Metric

Next, we will define the space $\mathcal{H}(\mathbb{R}^n)$ and build the Hausdorff metric, h. Then, we can show h is a metric on $\mathcal{H}(\mathbb{R}^n)$.

The space $\mathcal{H}(\mathbb{R}^n)$ is the collection of all the non-empty, compact subsets of $\mathbb{R}^n$. The definition of a compact subset of a topological space is quite complicated. However, thankfully the definition is much simpler in $\mathbb{R}^n$ as the Heine-Borel theorem shows [2].

Theorem 1. *A subset S of $\mathbb{R}^n$ is compact if and only if S is both closed and bounded.*

Therefore, $\mathcal{H}(\mathbb{R}^n)$ consists of all the subsets of $\mathbb{R}^n$ that are closed (the subset contains its boundary) and bounded (the subset can be contained in an n-sphere).

Now, recall that the Euclidean metric measured the distance between points in $\mathbb{R}^n$. In $\mathcal{H}(\mathbb{R}^n)$ a "point" is really a compact set. So, the next step that we need to take is to define a metric to measure the distance between sets.

Definition 5. *Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$.*

- *If $x \in \mathbb{R}^n$, the distance from x to B is*

$$d(x, B) = \min_{b \in B}\{d(x, b)\}.$$

An example of this can be seen in Figure 1.

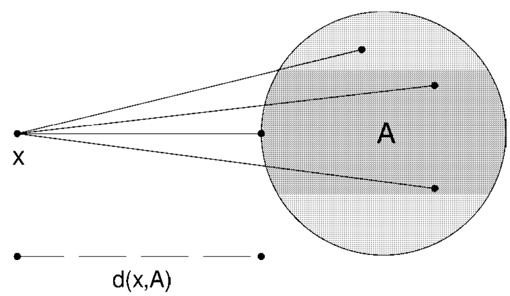

Figure 1: The distance from a point to a set

- *The "distance" from A to B is*

$$d(A, B) = \max_{x \in A}\{d(x, B)\}.$$

An example of this can be seen in Figure 2. Note that this is does not satisfy all of the necessary conditions to be classified as a metric, since it is possible to have $d(A, B) \neq d(B, A)$ (refer to Figure 3).

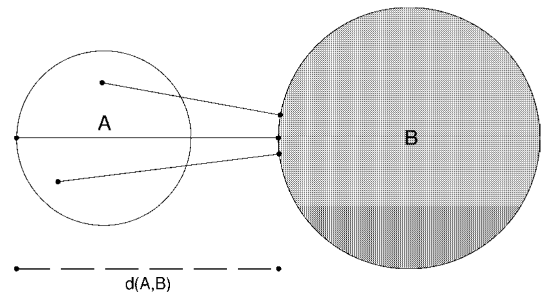

Figure 2: The distance from a set to a set

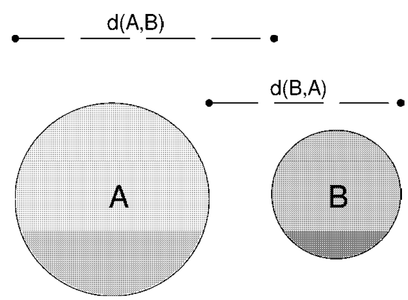

Figure 3: $d(A, B) > d(B, A)$

- *The Hausdorff distance, $h(A, B)$, between A and B is*

$$h(A, B) = max\{d(A, B), d(B, A)\}.$$

In order to use the function h to measure the distance between points in $\mathcal{H}(\mathbb{R}^n)$, h must be a metric on $\mathcal{H}(\mathbb{R}^n)$. The next theorem addresses this issue.

Theorem 2. *The function h as defined above is a metric on $\mathcal{H}(\mathbb{R}^n)$.*

Proof. Let $A, B, C \in \mathcal{H}(\mathbb{R}^n)$.

1. $h(A, B) = \max\{d(A, B), d(B, A)\} = h(B, A)$.

2. We now show that $h(A, B) \geq 0$ and that $h(A, B) = 0$ if and only if $A = B$. For any $a \in A$ and $b \in B, d(a, b) \geq 0$. Thus, $d(a, B) = \min\{d(a, b) \mid b \in B\} \geq 0$ for any $a \in A$ which implies that $d(A, B) =$

$\max\{d(a,B) \mid a \in A\} \geq 0$ also. Similarly, $d(B,A) \geq 0$. Therefore $h(A,B) = \max\{d(A,B), d(B,A)\} \geq 0$.

Suppose $h(A,B) = 0$. Then $d(A,B) = d(B,A) = 0$ which implies that $d(a,B) = 0$ for every $a \in A$ and $d(b,A) = 0$ for every $b \in B$. Now, for every $a \in A$ there exists a $b \in B$ such that $d(a,b) = 0$. Thus, $a = b$ and $A \subseteq B$. Similarly, $B \subseteq A$. Now suppose $A = B$. Then $d(a,B) = 0$ for every $a \in A$ which implies $d(A,B) = 0$. Similarly, $d(B,A) = 0$ so that $h(A,B) = 0$.

3. (Triangle Inequality) We first show that $d(A,C) \leq d(A,B) + d(B,C)$ for arbitrary sets A, B, and C in $\mathcal{H}(\mathbb{R}^n)$. Choose $a \in A, b \in B$, and $c \in C$. Then the triangle inequality for d guarantees that

$$d(a,c) \leq d(a,b) + d(b,c).$$

Choose $c' \in C$ such that $d(b,c') = d(b,C)$. Then,

$$d(a,c') \leq d(a,b) + d(b,C).$$

Since $d(a,C) \leq d(a,c')$,

$$d(a,C) \leq d(a,b) + d(b,C).$$

Now, choose $b' \in B$ such that $d(a,b') = d(a,B)$. Then,

$$d(a,C) \leq d(a,B) + d(b',C).$$

Since $d(b',C) \leq d(B,C)$,

$$d(a,C) \leq d(a,B) + d(B,C).$$

Choose $a' \in A$ such that $d(a',C) = d(A,C)$. Then,

$$d(A,C) \leq d(a',B) + d(B,C).$$

Since $d(a',B) \leq d(A,B)$, we have the final result that

$$d(A,C) \leq d(A,B) + d(B,C). \tag{1}$$

The Hausdorff distance from A to C is $h(A,C) = \max\{d(A,C), d(C,A)\}$. Applying (1) we see that

$h(A,C) \leq \max\{d(A,B) + d(B,C), d(C,B) + d(B,A)\}$
$h(A,C) \leq \max\{d(A,B), d(B,A)\} + \max\{d(B,C), d(C,B)\}$
$h(A,C) \leq h(A,B) + h(B,C)$. □

Therefore, when $\mathcal{H}(\mathbb{R}^n)$ is coupled with the Hausdorff metric, h, we find that h is a metric for $\mathcal{H}(\mathbb{R}^n)$.

3 Circles in $\mathcal{H}(\mathbb{R}^n)$

We will start our study of the geometry of $\mathcal{H}(\mathbb{R}^n)$ with the definition of a circle. In the summer of 2000, Dominic Braun and Steve Schlicker classified circles in $\mathcal{H}(\mathbb{R}^n)$ [1].

Definition 6. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $r \in \mathbb{R}^+$. Then,*

$$C_r(A) = \{B \in \mathcal{H}(\mathbb{R}^n) : h(A, B) = r\},$$

where $C_r(A)$ denotes the circle of radius r around A.

The next theorem shows the conditions a set B must fulfill in order to lie on $C_r(A)$.

Theorem 3. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $r \in \mathbb{R}^+$. Then $B \in C_r(A)$ if and only if $B \in \mathcal{H}(\mathbb{R}^n)$ and*

1. $B \subseteq \bigcup_{a \in A} \overline{N_r(a)}$.
2. $B \cap \overline{N_r(a)} \neq \emptyset$ *for each* $a \in A$.
3. *Either* $B \cap \partial \left(\bigcup_{a \in A} N_r(a) \right) \neq \emptyset$ *or there exists* $a \in A$ *such that* $B \cap \partial N_r(a) \neq \emptyset$ *but* $B \cap N_r(a) = \emptyset$.

Let us look at some examples to illustrate the previous theorem. In Figures 4 & 5, we let $A = \{a_1, a_2\}$. Figure 4 shows two examples of B where B does not lie on $C_r(A)$. On the left, $d(A, B) > r$ and so $h(A, B) > r$. On the right, $d(A, B) < r$ and $d(B, A) < r$, therefore $h(A, B) < r$. Figure 5 shows two examples of B where B lies on $C_r(A)$.

If we collect together every point that is within r units of any point in A, we obtain the largest set that is r units from B, which we will call $A + r$. By largest, we mean that if $B \in C_r(A)$, then $B \subseteq A + r$ [1].

Definition 7. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $r \in \mathbb{R}^+$. Then*

$$A + r = \{x \in \mathbb{R}^n : d(x, a) \leq r \text{ for some } a \in A\}.$$

Theorem 4. *The set $A + r$ is the largest element in $\mathcal{H}(\mathbb{R}^n)$ that is a distance r from A.*

Figure 6 shows an example of $A + r$.

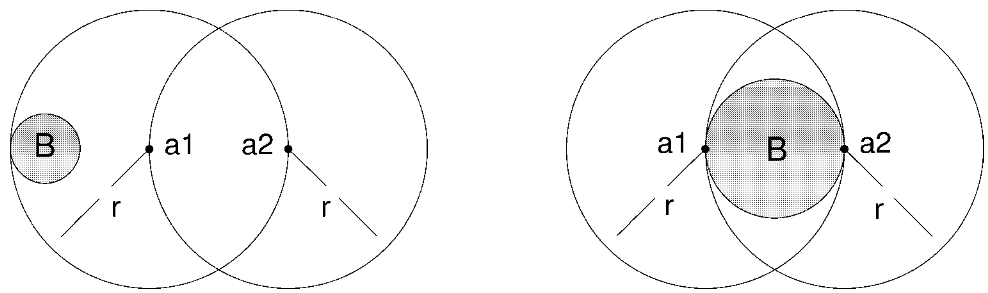

Figure 4: Condition 2 is not fulfilled (Left) Condition 3 is not fulfilled (Right)

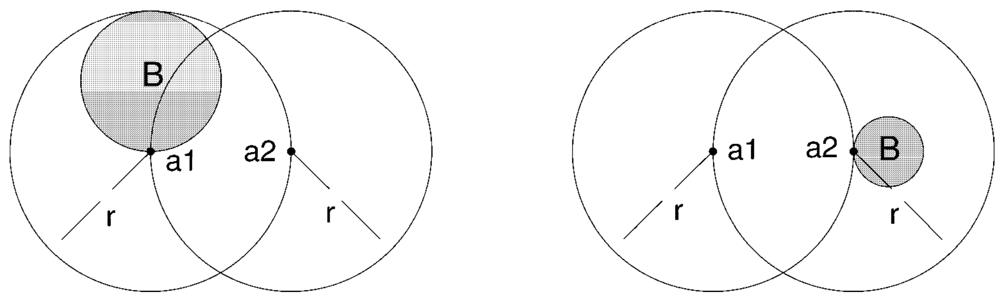

Figure 5: Examples of B that are on $C_r(A)$

4 Lines in $\mathcal{H}(\mathbb{R}^n)$

Since geometry is the study of points and lines, the next logical step would be to define lines in $\mathcal{H}(\mathbb{R}^n)$.

First, we need to look at lines in $\mathbb{R}^n$. We can determine the points on the Euclidean line connection points a and b as the set of all points c such that one of the following holds:

- $d(a,b) = d(a,c) + d(c,b)$ (c is between a and b)
- $d(c,b) = d(c,a) + d(a,b)$ (c is to the left of a)
- $d(a,c) = d(a,b) + d(b,c)$ (c is to the right of b)

We can extend this definition to $\mathcal{H}(\mathbb{R}^n)$ by using the Hausdorff metric, h, instead of the Euclidean metric, d.

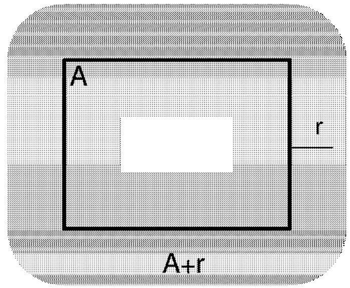

Figure 6: A is the black rectangle, and $A+r$ is the gray region

Definition 8. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$, $A \neq B$. The Hausdorff line defined by A and B, denoted $\overleftrightarrow{AB}$, is the set of all $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy one of the following:*

- $h(A,B) = h(A,C) + h(C,B)$ *(C is between A and B)*
- $h(C,B) = h(C,A) + h(A,B)$ *(C is to the left of A)*
- $h(A,C) = h(A,B) + h(B,C)$ *(C is to the right of B)*

Lines in $\mathcal{H}(\mathbb{R}^n)$ are very different from lines in $\mathbb{R}^n$. One example of this fact is already easy to illustrate. Recall that in $\mathbb{R}^n$, if c lies on $\overleftrightarrow{ab}$, then $\overleftrightarrow{ab} = \overleftrightarrow{ac} = \overleftrightarrow{bc}$. The same does not hold in $\mathcal{H}(\mathbb{R}^n)$. In other words, if C lies on $\overleftrightarrow{AB}$, we cannot automatically say that $\overleftrightarrow{AB} = \overleftrightarrow{AC} = \overleftrightarrow{BC}$. An example of this can be seen in Figure 7. In the figure, $A = \{(0,0)\}$, $B = \{(4,0)\}$, $C = \{(3,0)\}$, and $D = C_{0.5}((1.5,0))$. Since $h(A,B) = 4 = 3+1 = h(A,C) + h(C,B)$, we know that $C \in \overleftrightarrow{AB}$. Now we want to examine the set D and its relationship to $\overleftrightarrow{CB}$ and $\overleftrightarrow{AB}$. Since $h(D,B) = 3 = 2+1 = h(D,C) + h(C,B)$, we can see that D lies on $\overleftrightarrow{CB}$. However, $h(A,B) = 4$, $h(A,D) = 2$, and $h(D,B) = 3$. These values do not fulfill any of the equations needed to lie on a single line. Therefore, $D \notin \overleftrightarrow{AB}$, which means that $\overleftrightarrow{AB} \neq \overleftrightarrow{CB}$, even though C lies on $\overleftrightarrow{AB}$.

5 Lines Defined by One Point Sets

The simplest lines in $\mathcal{H}(\mathbb{R}^n)$ are those defined by one point sets, so it makes sense to start our study of lines in $\mathcal{H}(\mathbb{R}^n)$ with such lines.

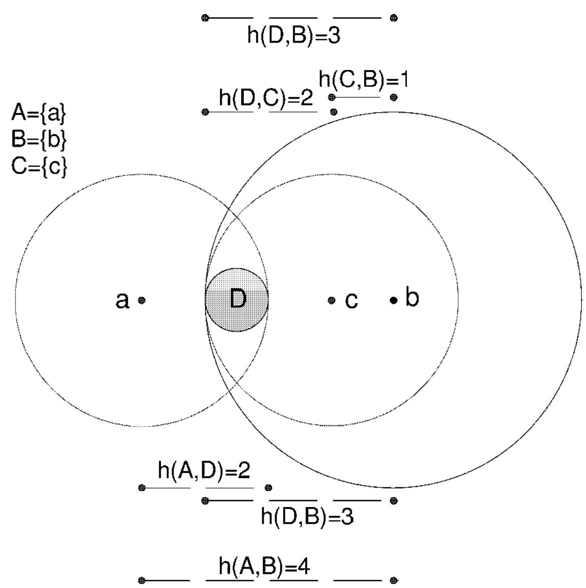

Figure 7: D lies on the line defined by C and B, but not the line defined by A and B

John Mayberry, Audrey Powers, and Steve Schlicker completely characterized the lines defined by one point sets in the summer of 2002. The characterization of the points in $\mathcal{H}(\mathbb{R}^n)$ that lie on such lines is summarized in the following theorem [1].

Theorem 5. *Let A and B be one point sets in $\mathcal{H}(\mathbb{R}^n)$, with $A = \{a\}$ and $B = \{b\}$. Then, $C \in \mathcal{H}(\mathbb{R}^n)$ lies on $\overleftrightarrow{AB}$ if and only if there exists $r, s \in \mathbb{R}^+$ such that:*

1. *$C \subseteq (A + r) \cap (B + s)$ and*

2. *there exists $c_0 \in C$ such that $c_0 \in \partial(A + r) \cap \partial(B + s) \cap \overleftrightarrow{ab}$.*

Figure 8 shows two examples that illustrate the previous theorem. In both pictures, $A = \{a\}$ and $B = \{b\}$. On the left is an example of a set that lies between A and B. We can see that $A + r$ and $B + s$ only intersect at one point, c_0, so C can only contain c_0. If we let $C = \{c_0\}$ then $h(A, B) = h(A, C) + h(C, B)$, so C does indeed lie between A and B. On the right we see an example of a set, C, that lies to the left of A (this time C is

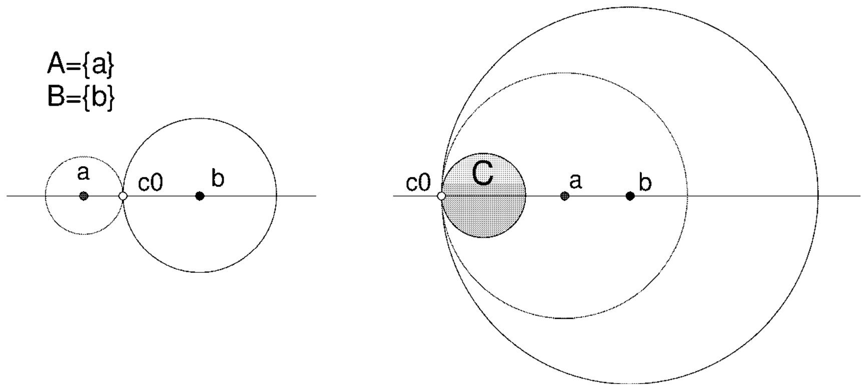

Figure 8: Example of a line defined by one point sets

the gray area). Note that $c_0 \in C$ and $C \subseteq (A+r) \cap (B+s)$, as the previous theorem requires.

At this point we can illustrate another big difference between line in $\mathbb{R}^n$ and lines in $\mathcal{H}(\mathbb{R}^n)$. In $\mathbb{R}^n$, the point that lies r units to the left of a on $\overleftrightarrow{ab}$ is a unique point. On the other hand, we can have more than one set that lies r units to the left of A on $\overleftrightarrow{AB}$ in $\mathcal{H}(\mathbb{R}^n)$. Look again at the right half of Figure 8. The set C is r units to the left of A on $\overleftrightarrow{AB}$. Now, if we let $D = \{c_0\}$, we can see that D is also r units to the left of A on $\overleftrightarrow{AB}$. Therefore, the set that lies r units to the right of A on $\overleftrightarrow{AB}$ is not unique. In fact, there are infinitely many sets that lies r units to the left of A on $\overleftrightarrow{AB}$.

Another interesting fact is that Euclidean geometry is embedded within the geometry of $\mathcal{H}(\mathbb{R}^n)$. In order to see this, we need to require that all the sets we work with are one point sets. Then, our lines look like those in Figure 8, where C always contains only c_0. This means that each of the points on the Euclidean line $\overleftrightarrow{ab}$ will correspond to the one point sets containing those points on the Hausdorff line $\overleftrightarrow{AB}$, and so the lines that we see in $\mathcal{H}(\mathbb{R}^n)$ are those of the familiar Euclidean space.

The 2002 REU was also able to determine how lines defined by one point sets in $\mathcal{H}(\mathbb{R}^2)$ intersect. They showed that two such distinct lines intersected in either no points (parallel lines), one point, or an infinite number of points [1]. This means that the geometry of $\mathcal{H}(\mathbb{R}^n)$ is not an incidence geometry (where distinct lines intersect in either no points, or one point). This is yet another difference between the geometry of $\mathcal{H}(\mathbb{R}^n)$ and Euclidean geometry, for Euclidean geometry is an incidence geometry.

For more information on lines defined by one point sets, go to

`http://faculty.gvsu.edu/schlicks/Hausdorff_paper.pdf`.

6 Lines Defined by Two Point Sets

The next step in studying lines in $\mathcal{H}(\mathbb{R}^n)$ is to look at lines that are more intricate than lines defined by one point sets. Therefore, we chose to move on to lines defined by two point sets.

6.1 Lemmas

Before we begin to examine lines defined by two point sets, we need to state a few lemmas that we will be using in the following theorems.

Lemma 1. *Let $a, b \in \mathbb{R}^n$, and let $r, s \in \mathbb{R}^+$ such that one of the following holds:*

- $d(a, b) = r + s$,
- $r = d(a, b) + s$, *or*
- $s = r + d(a, b)$.

Then, there exists $p \in \overleftrightarrow{ab}$ such that $\partial N_r(a) \cap \partial N_s(b) = \{p\}$.

Proof. Let $a, b \in \mathbb{R}^n$ and let $r, s \in \mathbb{R}^+$.

First, assume that $d(a, b) = r + s$, and let

$$p = a + \frac{r}{|b - a|}(b - a).$$

(Example in Figure 9) Then,

$$\begin{aligned} d(a, p) &= \left| a - (a + \frac{r}{|b - a|}(b - a)) \right| \\ &= \left| \frac{-r}{|b - a|}(b - a) \right| \\ &= \frac{r}{|b - a|}\,|b - a| \\ &= r \end{aligned}$$

and

$$\begin{aligned} d(b,p) &= \left| b - (a + \frac{r}{|b-a|}(b-a)) \right| \\ &= \left| b - a - \frac{r}{|b-a|}(b-a) \right| \\ &= \left| (b-a)(1 - \frac{r}{|b-a|}) \right| \\ &= |b-a| \left| 1 - \frac{r}{|b-a|} \right| \\ &= ||b-a| - r| \\ &= |d(a,b) - r| \\ &= |s| \\ &= s. \end{aligned}$$

Therefore, p is special because

$$p \in \partial N_r(a) \cap \partial N_s(b) \cap \overleftrightarrow{ab}.$$

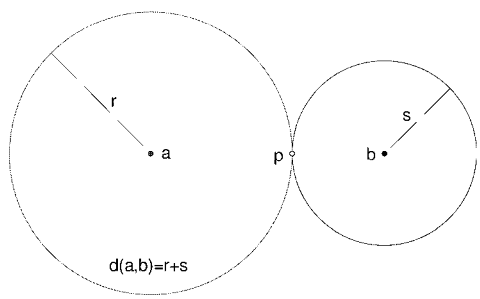

Figure 9: Example where $d(a,b) = r + s$

Next, assume that $r = d(a,b) + s$, and again let

$$p = a + \frac{r}{|b-a|}(b-a).$$

(Example in Figure 10) Then, $d(a,p) = r$ (like in the last case) and

$$\begin{aligned} d(b,p) &= |d(a,b) - r| \\ &= |-s| \\ &= s. \end{aligned}$$

Once again,

$$p \in \partial N_r(a) \cap \partial N_s(b) \cap \overleftrightarrow{ab}.$$

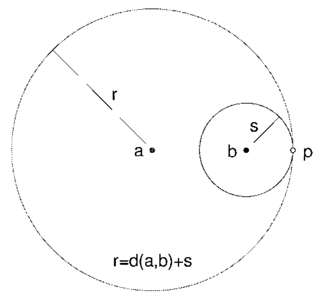

Figure 10: Example where $r = d(a, b) + s$

Finally, assume that $s = r + d(a, b)$, and this time let

$$p = a - \frac{r}{|b-a|}(b - a).$$

(Example in Figure 11) Then,

$$\begin{aligned} d(a, p) &= \left| a - (a - \frac{r}{|b-a|}(b - a)) \right| \\ &= \left| \frac{r}{|b-a|}(b - a) \right| \\ &= \frac{r}{|b-a|} \, |b - a| \\ &= r \end{aligned}$$

and

$$\begin{aligned} d(b,p) &= \left| b - (a - \frac{r}{|b-a|}(b-a)) \right| \\ &= \left| b - a + \frac{r}{|b-a|}(b-a) \right| \\ &= \left| (b-a)(1 + \frac{r}{|b-a|}) \right| \\ &= |b-a| \left| 1 - \frac{r}{|b-a|} \right| \\ &= ||b-a| + r| \\ &= |s| \\ &= s. \end{aligned}$$

Thus, yet again,

$$p \in \partial N_r(a) \cap \partial N_s(b) \cap \overleftrightarrow{ab}.$$

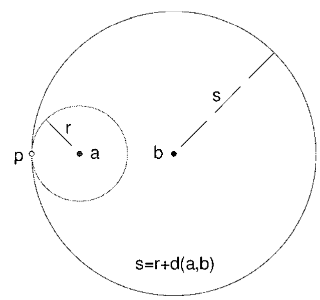

Figure 11: Example where $s = r + d(a, b)$

Therefore, there always exists $p \in \mathbb{R}^n$ such that

$$p \in \partial N_r(a) \cap \partial N_s(b) \cap \overleftrightarrow{ab}.$$

Now, we want to show that p is the only element in $\partial N_r(a) \cap \partial N_s(b)$. Suppose $p' \in \partial N_r(a) \cap \partial N_s(b)$. Then, $d(a, p') = r$ and $d(b, p') = s$. Since

- $d(a, b) = r + s$,
- $r = d(a, b) + s$, or

- $s = r + d(a,b)$

we know that $p' \in \overleftrightarrow{ab}$, making $p' = p$. Thus, $\partial N_r(a) \cap \partial N_s(b) = \{p\}$. □

Lemma 2. *Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$. If $d(a,b) = r+s$, then $\overline{N_r(a)} \cap \overline{N_s(b)}$ only contains one point.*

Proof. Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$ such that $d(a,b) = r + s$.

First, we want to show that $\overline{N_r(a)} \cap \partial N_s(b)$ only contains one point. Let $m \in N_r(a)$. Then, $d(b,m) \geq d(a,b) - d(a,m) > d(a,b) - r = s$. Therefore, $N_r(a) \cap \overline{N_s(b)} = \emptyset$, which means that

$$N_r(a) \cap \partial N_s(b) = \emptyset. \tag{2}$$

We know from Lemma 1 that

$$\partial N_r(a) \cap \partial N_s(b) = \{p\} \tag{3}$$

Thus, putting 2 and 3 together we see that

$$\overline{N_r(a)} \cap \partial N_s(b) = \{p\} \tag{4}$$

Next, we want to show that $\overline{N_r(a)} \cap N_s(b) = \emptyset$. Let $t \in N_s(b)$. Then, $d(t,a) \geq d(a,b) - d(b,t) > d(a,b) - s = r$. Therefore,

$$\overline{N_r(a)} \cap N_s(b) = \emptyset \tag{5}$$

Now, putting together 4 and 5, we can see that $\overline{N_r(a)} \cap \overline{N_s(b)} = \{p\}$. (An example can be seen in Figure 9) □

Lemma 3. *Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$. If $r = d(a,b) + s$, then*

- $\overline{N_s(b)} \subseteq \overline{N_r(a)}$ *and*
- $\overline{N_s(b)} \cap \partial N_r(a) = \{p\}$.

Proof. Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$ such that $r = d(a,b) + s$.

We will first prove that $\overline{N_s(b)} \subseteq \overline{N_r(a)}$. Let $m \in \overline{N_s(b)}$. Then, $d(a,m) \leq d(a,b) + d(b,m) \leq d(a,b) + s = r$. Therefore, $\overline{N_s(b)} \subseteq \overline{N_r(a)}$.

Now we will show that $\overline{N_s(b)} \cap \partial N_r(a) = \{p\}$. Let $t \in N_s(b)$. Then $d(a,t) \leq d(a,b) + d(b,m) < d(a,b) + s = r$, which means that

$$N_s(b) \cap \partial N_r(a) = \emptyset \tag{6}$$

By Lemma 1 we know that

$$\partial N_r(a) \cap \partial N_s(b) = \{p\} \tag{7}$$

Putting 6 and 7 together we see that $\overline{N_s(b)} \cap \partial N_r(a) = \{p\}$. (An example can be seen in Figure 10) $\square$

Lemma 4. *Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$. If $r > d(a,b) + s$, then $\overline{N_s(b)} \subseteq N_r(a)$.*

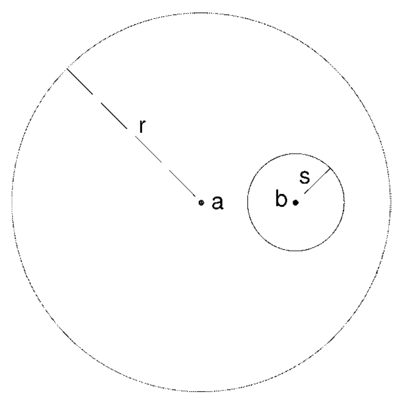

Figure 12: $\overline{N_s(b)}$ is enclosed in $N_r(a)$

Proof. Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$ such that $r > d(a,b) + s$.

Let $m \in \overline{N_s(b)}$. Then, $d(a,m) \leq d(a,b) + d(b,m) \leq d(a,b) + s < r$. Therefore, $\overline{N_s(b)} \subseteq N_r(a)$. (An example can be seen in Figure 12) $\square$

Lemma 5. *Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$. If $d(a,b) > r + s$, then $\overline{N_r(a)} \cap \overline{N_s(b)} = \emptyset$.*

Proof. Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$ such that $d(a,b) > r + s$.

We will proceed by contradiction. Suppose $m \in \overline{N_r(a)} \cap \overline{N_s(b)}$. Then $d(a,b) \leq d(a,m) + d(m,b) \leq r + s$. This is a contradiction, since $d(a,b) > r + s$, therefore $\overline{N_r(a)} \cap \overline{N_s(b)} = \emptyset$. (An example can be seen in Figure 13) $\square$

Lemma 6. *Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$. If $d(a,b) < r + s$, then there exists $c \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ such that $\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$.*

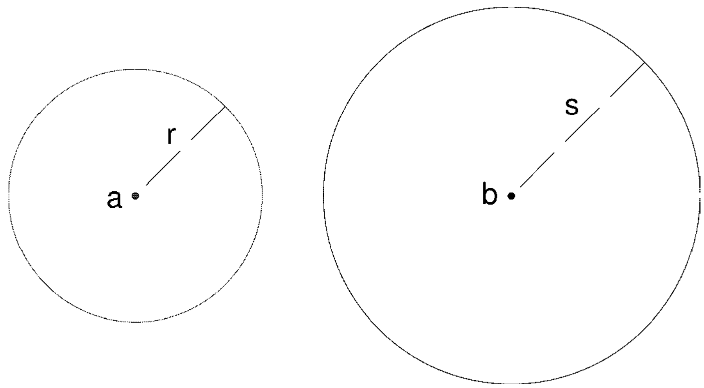

Figure 13: The neighborhoods do not intersect

Proof. Let $a, b \in \mathbb{R}^n$ and $r, s \in \mathbb{R}^+$ such that $d(a,b) < r + s$. There are two cases we must consider

1. $r \geq d(a,b) + s$ (which means that $d(a,b) \leq r - s < r + s$)

2. $r < d(a,b) + s$ (which means that $r - s < d(a,b) < r + s$)

Case 1 $r \geq d(a,b) + s$

Suppose $r > d(a,b) + s$. Then, by Lemma 4, $\overline{N_s(b)} \subseteq N_r(a)$ and therefore $\overline{N_s(b)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$. So, let $c = b$ and $t = s$, and then

$$\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}.$$

Suppose $r = d(a,b) + s$. Then, by Lemma 3, $\overline{N_s(b)} \subseteq \overline{N_r(a)}$ and therefore $\overline{N_s(b)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$. So, again let $c = b$ and $t = s$, and then

$$\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}.$$

Therefore, there always exists $c \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ such that $\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$.

Case 2 $r < d(a,b) + s$

In this case there are two subcases to consider:

1. $s \geq r + d(a,b)$ (which means that $d(a,b) \leq s - r < r + s$ and $r \leq s - d(a,b) < d(a,b) + s$)

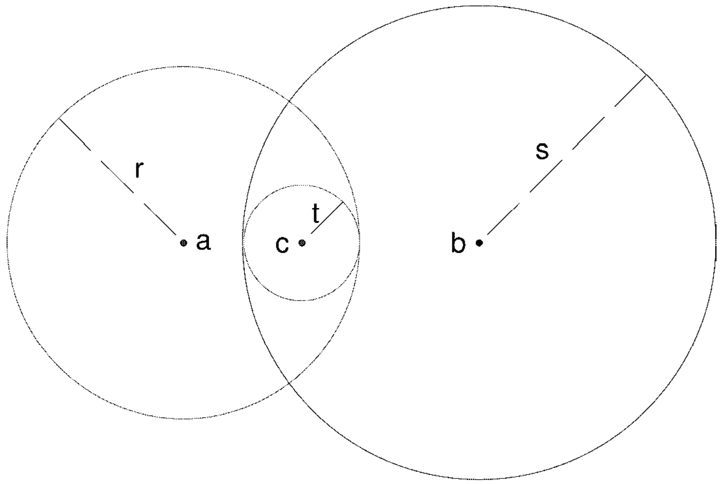

Figure 14: $\overline{N_r(a)} \cap \overline{N_s(b)}$ contains a whole neighborhood

2. $s < r + d(a,b)$ (which means that $s - r < d(a,b) < r + s$ and $s - d(a,b) < r < d(a,b) + s$)

Case 2.1 $r < d(a,b) + s$ and $s \geq r + d(a,b)$

Suppose $s > d(a,b) + r$. Then, by Lemma 4, $\overline{N_r(a)} \subseteq N_s(b)$ and therefore $\overline{N_r(a)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$. So, let $c = a$ and $t = r$, and then

$$\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}.$$

Suppose $s = r + d(a,b)$. Then, by Lemma 3, $\overline{N_r(a)} \subseteq \overline{N_s(b)}$ and therefore $\overline{N_r(a)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$. So, again let $c = a$ and $t = r$, and then

$$\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}.$$

Therefore, there always exists $c \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ such that $\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$.

Case 2.2 $r < d(a,b) + s$ and $s < r + d(a,b)$ (Note that in this case $d(a,b) - r > -s$ and $d(a,b) - s > -r$.)

First, we want to define c and t. Let $t = \frac{r+s-d(a,b)}{2}$. Note that $t \in \mathbb{R}^+$ and

$d(a, b) = r + s - 2t$. Then, let $c \in \overline{ab}$ such that $d(a, c) = r - t$. Since

$$\begin{aligned} r - t &= r - \frac{r + s - d(a, b)}{2} \\ &= \frac{2r - r - s + d(a, b)}{2} \\ &= \frac{r + d(a, b) - s}{2} \\ &> \frac{r - r}{2} \\ &= 0 \end{aligned}$$

we know that $d(a, c)$ is indeed a real distance. Then,

$$\begin{aligned} d(b, c) &= d(a, b) - d(a, c) \\ &= (r + s - 2t) - (r - t) \\ &= s - t. \end{aligned}$$

Since

$$\begin{aligned} s - t &= s - \frac{r + s - d(a, b)}{2} \\ &= \frac{2s - r - s + d(a, b)}{2} \\ &= \frac{s + d(a, b) - r}{2} \\ &> \frac{s - s}{2} \\ &= 0 \end{aligned}$$

we know that $d(b, c)$ is indeed a real distance.

Now, we want to create $\overline{N_t(c)}$ and see how it relates to $\overline{N_r(a)}$ and $\overline{N_s(b)}$. So, let $m \in \overline{N_t(c)}$. Then, $d(a, m) \leq d(a, c) + d(c, m) \leq (r - t) + (t) = r$ and $d(b, m) \leq d(b, c) + d(c, m) \leq (s - t) + (t) = s$. Thus, $\overline{N_t(c)} \subseteq \overline{N_r(a)} \cap \overline{N_s(b)}$. (An example can be seen in Figure 14) □

6.2 Segment Between A and B

Since lines in $\mathcal{H}(\mathbb{R}^n)$ act so differently from those in Euclidean space, a logical question would be to ask what these lines look like- since they may not resemble the lines we are used to. First, let's examine the line segment between the two point sets A and B. We know that there exists a set for every distance between A and B on $\overleftrightarrow{AB}$ as is demonstrated in the following theorem.

Theorem 6. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$. Then, for every $0 < r < h(A,B)$, there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A,C) = r$ and $h(A,B) = h(A,C) + h(C,B)$.*

Proof. Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ with $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Also, let $r, s \in \mathbb{R}^+$ such that $r + s = h(A,B)$. Without loss of generality, we will assume that

$$h(A,B) = d(B,A) = d(b_1, A) = d(b_1, a_1).$$

This means that $d(b_1, a_2) \geq h(A,B)$, leaving us with two cases to consider:

1. $d(b_1, a_2) > h(A,B)$ or
2. $d(b_1, a_2) = h(A,B)$.

Case 1 $d(b_1, a_2) > h(A,B)$

Since $d(a_2, B) \leq h(A,B)$, we know that $d(a_2, b_2) \leq h(A,B) = r + s$. If $d(a_2, b_2) = r + s$, then Lemma 2 states that $\overline{N_r(a_2)} \cap \overline{N_s(b_2)} \neq \emptyset$. If $d(a_2, b_2) < r + s$, the Lemma 6 states that $\overline{N_r(a_2)} \cap \overline{N_s(b_2)} \neq \emptyset$. So, let

$$m \in \overline{N_r(a_2)} \cap \overline{N_s(b_2)}.$$

Also, since $d(a_1, b_1) = h(A,B) = r + s$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}.$$

Next, we need to show that $p \in \partial(A+r)$ and $m \notin \overline{N_s(b_1)}$. Since $d(a_2, p) \geq d(a_2, b_1) - d(b_1, p) > h(A,B) - s = r$, we know that $p \notin \overline{N_r(a_2)}$, and so

$$p \in \partial(A + r).$$

Next, since $d(a_2, b_1) > h(A,B) = r+s$, Lemma 5 states that $\overline{N_r(a_2)} \cap \overline{N_s(b_1)} = \emptyset$. Therefore, since $m \in \overline{N_r(a_2)}$, we know that

$$m \notin \overline{N_s(b_1)}.$$

Note that this also means that $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,

2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and

3. $p \in \partial(A + r)$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,

2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and

3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$. Therefore, $h(A, C) = r$ and $h(A, B) = h(A, C) + h(C, B)$. (An example of this case can be seen in Figure 15)

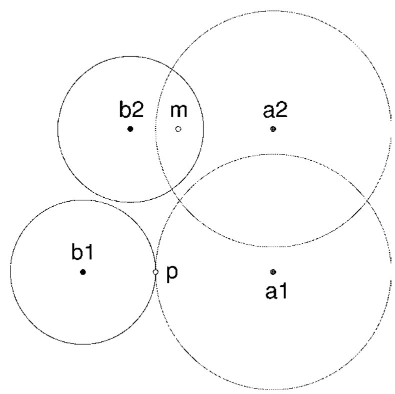

Figure 15: Example of Case 1

Case 2 $d(a_2, b_1) = h(A, B)$

Since $d(a_1, b_1) = h(A, B) = r + s$ and $d(a_2, b_1) = h(A, B) = r + s$, Lemma 1 states that $\partial N_r(a_1) \cap \partial N_s(b_1) = \{p_1\}$ and $\partial N_r(a_2) \cap \partial N_s(b_1) = \{p_2\}$.

Now, we want to show that $p_1 \neq p_2$. We will proceed by contradiction. Suppose $p_1 = p_2$. Then

$$a_1 = p_1 + \frac{r}{| b_1 - p_1 |}(b_1 - p_1) = p_2 + \frac{r}{| b_1 - p_2 |}(b_1 - p_2) = a_2.$$

However, since $a_1 \neq a_2$, this is a contradiction, and so $p_1 \neq p_2$.

We need to show that $p_1 \notin \overline{N_r(a_2)}$. We will proceed by contradiction. Suppose $p_1 \in \overline{N_r(a_2)}$. Then, $p_1 \in \overline{N_r(a_2)} \cap \partial N_s(b_1)$. However, according to Lemma 3, $\overline{N_r(a_2)} \cap \partial N_s(b_1) = \{p_2\}$ and so $p_1 = p_2$. However, this is a contradiction, and so $p_1 \notin \overline{N_r(a_2)}$. In a similar manner it can be shown that $p_2 \notin \overline{N_r(a_1)}$. This means that

$$p_1, p_2 \in \partial(A + r).$$

Next, since $d(b_2, A) \leq h(A, B)$ either $d(b_2, a_1) \leq h(A, B)$ or $d(b_2, a_2) \leq h(A, B)$. This gives us two subcases to consider:

1. $d(b_2, a_1) \leq h(A, B)$ or
2. $d(b_2, a_2) \leq h(A, B)$.

Case 2.1 $d(b_2, a_1) \leq h(A, B)$

If $d(b_2, a_1) = h(A, B) = r + s$, then Lemma 2 states that $\overline{N_r(a_1)} \cap \overline{N_s(b_2)} \neq \emptyset$. If $d(b_2, a_1) < h(A, B) = r + s$, then Lemma 6 states that $\overline{N_r(a_1)} \cap \overline{N_s(b_2)} \neq \emptyset$. Also, we need to notice that $p_2 \notin \overline{N_r(a_1)} \cap \overline{N_s(b_2)}$, since $p_2 \notin \overline{N_r(a_1)}$. So, let $m \in \overline{N_r(a_1)} \cap \overline{N_s(b_2)}$. Note that this means $m \neq p_2$ and $m \notin N_s(b_1)$, since $d(b_1, m) \geq d(b_1, a_1) - d(a_1, m) \geq h(A, B) - r = s$.

Let $C = \{m, p_2\}$. Then

1. $C \subseteq A + r$,
2. $m \in \overline{N_r(a_1)}$, $p_2 \in \overline{N_r(a_2)}$, and
3. $p_2 \in \partial(A + r)$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p_2 \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p_2 \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$. Therefore, $h(A, C) = r$ and $h(A, B) = h(A, C) + h(C, B)$. (An example of this case can be seen in Figure 16)

Case 2.2 $d(b_2, a_2) \leq h(A, B)$

If $d(b_2, a_2) = h(A, B) = r + s$, then Lemma 2 states that $\overline{N_r(a_2)} \cap \overline{N_s(b_2)} \neq \emptyset$. If $d(b_2, a_2) < h(A, B) = r + s$, then Lemma 6 states that $\overline{N_r(a_2)} \cap \overline{N_s(b_2)} \neq$

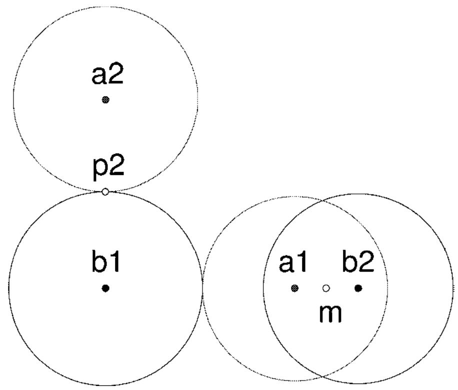

Figure 16: Example of Case 2.1

$\emptyset$. Also, we need to notice that $p_1 \notin \overline{N_r(a_2)} \cap \overline{N_s(b_2)}$, since $p_1 \notin \overline{N_r(a_2)}$. So, let $m \in \overline{N_r(a_2)} \cap \overline{N_s(b_2)}$. Note that this means $m \neq p_1$ and $m \notin N_s(b_1)$, since $d(b_1, m) \geq d(b_1, a_2) - d(a_2, m) \geq h(A, B) - r = s$.

Let $C = \{m, p_1\}$. Then

1. $C \subseteq A + r$,
2. $p_1 \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p_1 \in \partial(A + r)$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p_1 \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p_1 \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$. Therefore, $h(A, C) = r$ and $h(A, B) = h(A, C) + h(C, B)$. (An example of this case can be seen in Figure 17)

Therefore, we can always find $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(A, B) = h(A, C) + h(C, B)$. $\square$

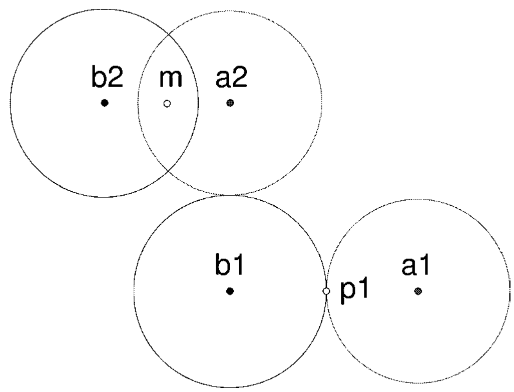

Figure 17: Example of Case 2.2

6.3 Lines with a Missing Ray

Now that we know that there is a complete line segment between A and B, we want to examine the rays to the right of B, or the left of A. It is here that we run into our most surprising result. We found that another huge difference between Euclidean and Hausdorff lines is that lines in $\mathcal{H}(\mathbb{R}^n)$ can act like rays. By this we mean that if we have a complete line in $\mathcal{H}(\mathbb{R}^n)$, it is possible that there are no points on the line that are more than a distance z to the right of B. We can separate lines in $\mathcal{H}(\mathbb{R}^n)$ into two categories: complete (there are sets on the line for any distance to the left of B or the right of A); or ray-like (the line only extends z units to the right of B and no further). We are also able to classify lines in $\mathcal{H}(\mathbb{R}^n)$ as being complete or ray-like by looking at the properties of their defining sets.

In the theorems in this section we are going to use the two point sets A and B whose elements are situated in a particular way. So, let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$, with $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Without loss of generality, we will assume $h(A, B) = d(B, A) = d(b_1, A) = d(b_1, a_1)$. In summary,

- $d(B, A) = h(A, B)$ which also means that $d(A, B) \leq h(A, B)$,
- $d(b_1, A) = h(A, B)$ which also means that $d(b_2, A) \leq h(A, B)$, and
- $d(b_1, a_1) = h(A, B)$ which also means that $d(b_1, a_2) \geq h(A, B)$.

Also, in the following theorems, the set $C \in \mathcal{H}(\mathbb{R}^n)$ is always a two point set.

First, we want to look at a specific set of requirements for which we know we have a ray-like line. Specifically, there are three conditions that we want, as stated in the next theorem.

Theorem 7. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If*

- $d(a_1, b_2) < h(A, B)$,
- $d(a_2, b_2) < h(A, B)$, *and*
- $d(a_2, b_1) < \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$,

then there exists $z \in \mathbb{R}^+$ such that

1. *if $C \in \mathcal{H}(\mathbb{R}^n)$ and $h(A, C) = h(A, B) + h(B, C)$, then $h(B, C) \leq z$ and*
2. *for every $0 < s \leq z$, there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $s > 0$ and $r = h(A, B) + s$. Before we begin the proofs of the two cases, we need to look at some background information.

Since $d(a_1, b_2) < h(A, B) = r - s$ and $d(a_2, b_2) < h(A, B) = r - s$, we know that $d(a_1, b_2) + s < r$ and $d(a_2, b_2) + s < r$. Then, according to Lemma 4

$$\overline{N_s(b_2)} \subseteq N_r(a_1) \text{ and } \overline{N_s(b_2)} \subseteq N_r(a_2).$$

The next step is to show that $\partial(A + r) \cap (B + s)$ can contain at most one point.

Since $\overline{N_s(b_2)} \subseteq N_r(a_1) \cap N_r(a_2)$, we know that there are no points in $\overline{N_s(b_2)}$ that are a distance r from a_1 or a_2, and therefore no points in $\overline{N_s(b_2)}$ on $\partial(A + r)$. Thus, if we are to find any points in $B + s$ that are a distance r from A, they will be found in $\overline{N_s(b_1)}$.

First, let's look at $\partial N_r(a_1) \cap \overline{N_s(b_1)}$. Let $q \in N_s(b_1)$. Then $d(a_1, q) \leq d(a_1, b_1) + d(b_1, q) < h(A, B) + s = r$. Therefore $d(a_1, q) < r$ for every $q \in N_s(b_1)$ and

$$\partial N_r(a_1) \cap N_s(b_1) = \emptyset.$$

However, by Lemma 1, we know that there exists a $p \in \mathbb{R}^n$ such that $\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}$. Thus,

$$\partial N_r(a_1) \cap \overline{N_s(b_1)} = \{p\}.$$

Also, we need to note that $\overline{N_s(b_1)} \subseteq \overline{N_r(a_1)}$ and $\partial N_r(a_1) \cap \overline{N_s(b_1)} = \{p\}$. Since $d(a_1, b_1) + s = r$, both these facts are drawn directly from Lemma 3.

Next, let's assume that $\partial N_r(a_2) \cap \overline{N_s(b_1)} \neq \emptyset$. Let $t \in \partial N_r(a_2) \cap \overline{N_s(b_1)}$, $t \neq p$. Then, $d(a_2, t) = r$, but $d(a_1, t) < r$, which means that $t \notin \partial(A + r)$. If we assume that $\partial N_r(a_2) \cap \overline{N_s(b_1)} = \emptyset$, then we know that there are no points in the intersection that line on $\partial(A + r)$. Overall, this means that there are no points in $\overline{N_s(b_1)}$ that are a distance r from a_2 that lie on $\partial(A + r)$.

Therefore, p is the only point in $B + s$ which can possibly be an element of $\partial(A + r)$.

The next thing that we need to do is simplify our problem by moving from n-space to 2-space. We can do this by taking the points a_1, a_2, b_1 and forming a plane with them. Then, we can set up our coordinate system by making $\overleftrightarrow{a_1 a_2}$ into the y-axis, and the perpendicular bisector of $\overline{a_1 a_2}$ into the x-axis. Also, we want to choose the orientation of the axes so that a_1 will have a positive y-coordinate, and allow for a reflection across the y-axis so that b_1 will have a positive x-coordinate. Then, we have a figure similar to that of Figure 18. Denote the location of a_1 as $(0, a)$, a_2 as $(0, -a)$, and b_1 as (x, y) where $a, x \in \mathbb{R}^+$ and $y \in \mathbb{R}$.

Note that this is a unique plane, because b_1 cannnot lie on $\overleftrightarrow{a_1 a_2}$. This can be shown by contradiction. Suppose b_1 was on $\overline{a_1 a_2}$. First, assume b_1 is to the right of a_2. Then $d(a_2, b_1) = d(a_1, b_1) - d(a_1, a_2) < h(A, B)$. However, since this is a contradiction, b_1 cannot lie to the right of a_2. Next, assume that b_1 is to the left of a_1. Then, $d(a_2, b_1) = d(a_2, a_1) + d(a_1, b_1) > \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$. However, since this is a contradiction, b_1 cannot lie to the left of a_1. Finally, assume that b_1 is between a_1 and a_2. Then, $d(a_1, a_2) = d(a_1, b_1) + d(b_1, a_2) \geq h(A, B) + h(A, B) = 2h(A, B)$. However, this would mean that if $r < h(A, B)$ then $r + r < h(A, B)$, and by Lemma 5 $\overline{N_r(a_1)} \cap \overline{N_r(a_2)} = \emptyset$. Yet, we know that this intersection contains a whole neighborhood for any $r > 0$, so b_1 cannot be between a_1 and a_2. Therefore, b_1 is not on $\overleftrightarrow{a_1 a_2}$, and so the plane defined by a_1, a_2, and b_1 is unique.

Now, recall that $h(A, B) \leq d(a_2, b_1) < \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$. We can use this fact to show that $0 \leq y < a$. We will continue by contradiction.

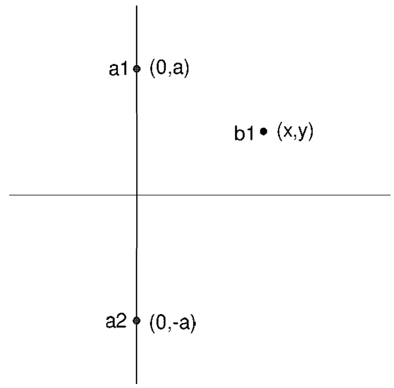

Figure 18: Plane formed by a_1, a_2, and b_1

Suppose $y < 0$. That would mean that $a - y > a + y$. Then

$$\begin{aligned} h(A,B) &= d(a_1, b_1) \\ &= \sqrt{x^2 + (a-y)^2} \\ &> \sqrt{x^2 + (a+y)^2} \\ &= d(a_2, b_1) \end{aligned}$$

However, this is a contradiction, because $h(A,B) \leq d(a_2, b_1)$. Thus, $y \geq 0$.

Next, suppose $y \geq a$. Then,

$$\begin{aligned} d(a_2, b_1) &= \sqrt{x^2 + (a+y)^2} \\ &= \sqrt{x^2 + (2a + y - a)^2} \\ &= \sqrt{x^2 + 4a^2 + 4a(y-a) + (y-a)^2} \\ &= \sqrt{4a^2 + 4a(y-a) + x^2 + (y-a)^2} \\ &\geq \sqrt{4a^2 + x^2 + (y-a)^2} \\ &= \sqrt{(d(a_1, a_2))^2 + (h(A,B))^2} \end{aligned}$$

However, this is also a contradiction, because $d(a_2, b_1) < \sqrt{(d(a_1, a_2))^2 + (h(A,B))^2}$. Thus, $y < a$.

Now, recall p is the only point in $B + s$ which can possibly be an element of $\partial(A + r)$. Note that since p is on $\overleftrightarrow{a_1 b_1}$, p is in the plane determined by a_1, a_2, and b_1.

Next, we want to show that if $s \leq z$, $p \in \partial(A+r)$ and if $s > z$, $p \notin \partial(A+r)$.

Let the coordinates of p be (f,g). Refer to Figures 19 in which $s < z$ on the left and $s > z$ on the right, and in both z is the distance from b_1 to the intersection of $\overleftrightarrow{a_1 b_1}$ and the x-axis. By similar triangles

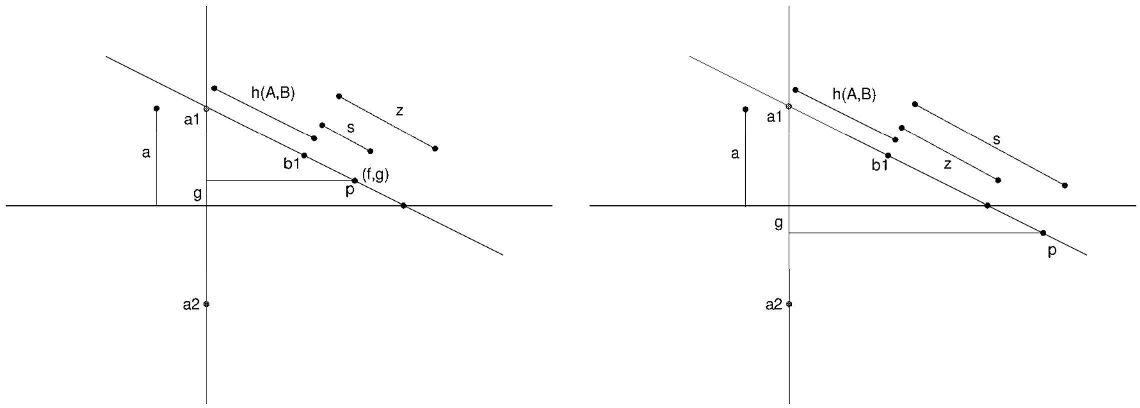

Figure 19: Location of p on the plane

$$\frac{h(A,B)+s}{h(A,B)+z} = \frac{a-g}{a}.$$

If $s \leq z$, then

$$\frac{h(A,B)+s}{h(A,B)+z} \leq 1 \text{ and } \frac{a-g}{a} \leq 1.$$

However, that means that $g \geq 0$ and $a - g \leq a + g$. However,

$$r = d(a_1, p) = \sqrt{f^2 + (a-g)^2} \leq \sqrt{f^2 + (a+g)^2} = d(a_2, p),$$

which means that $d(a_2, p) \geq r$ and $p \notin N_r(a_2)$. Therefore

$$p \in \partial(A+r).$$

If $s > z$, then

$$\frac{h(A,B)+s}{h(A,B)+z} > 1 \text{ and } \frac{a-g}{a} > 1.$$

However, that means that $g < 0$ and $a - g > a + g$. Then,

$$r = d(a_1, p) = \sqrt{f^2 + (a-g)^2} > \sqrt{f^2 + (a+g)^2} = d(a_2, p),$$

which means that $d(a_2, p) < r$ and $p \in N_r(a_2)$. Therefore

$$p \notin \partial(A + r).$$

Now we are ready to prove the two parts of the theorem.

1. Assume $C \in \mathcal{H}(\mathbb{R}^n)$ and $h(A, C) = h(A, B) + h(B, C)$. Also, let $r = h(A, C)$ and $s = h(B, C)$.

Since $C \in C_s(B)$, by part 2 of Theorem 3, $C \cap \overline{N_s(b_2)} \neq \emptyset$. Therefore, $C \cap N_r(a_1) \neq \emptyset$ and $C \cap N_r(a_2) \neq \emptyset$. However, since $C \in C_r(A)$, part 3 of Theorem 3 says that

i. $C \cap \partial(A + r) \neq \emptyset$ or

ii. there exists $a \in A$ such that $C \cap \partial N_r(a) \neq \emptyset$, but $C \cap N_r(a) = \emptyset$.

Yet, since $C \cap N_r(a) \neq \emptyset$ for any $a \in A$, we know that (ii) does not hold. Therefore, $C \cap \partial(A + r) \neq \emptyset$. This means that C must contain p (since we showed that it is the only point that can possibly lie on $\partial(A + r)$), and $p \in \partial(A + r)$. However, this only happens when $s \leq z$. Thus

$$h(B, C) \leq z.$$

2. Assume that $s \leq z$. Then, we know that $p \in \partial(A + r)$. Also, let $m \in \overline{N_s(b_2)}$. Then, note that $p \notin \overline{N_s(b_2)}$. This tells us that

$$p \neq m \text{ and } p \in \partial(B + s).$$

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,

2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and

3. $p \in \partial(A + r)$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,

2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and

3. $p \in \partial(B + s)$.

Thus, $C \in C_s(B)$.

Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. □

Now, since we know that the ray to the right of B is not complete, we want to know what happens to the left of A when the same three conditions hold.

Theorem 8. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If*

- $d(a_1, b_2) < h(A, B)$,
- $d(a_2, b_2) < h(A, B)$, *and*
- $d(a_2, b_1) < \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$,

then for every $r > 0$, there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $r > 0$ and $s = r + h(A, B)$.

First, we want to show that $\overline{N_r(a_1)} \cup \overline{N_r(a_2)} \subseteq N_s(b_2)$. Since $d(a_1, b_2) < h(A, B) = s - r$ and $d(a_2, b_2) < h(A, B) = s - r$, Lemma 3 states that $\overline{N_r(a_1)} \subseteq N_s(b_2)$ and $\overline{N_r(a_2)} \subseteq N_s(b_2)$ which means that

$$\overline{N_r(a_1)} \cup \overline{N_r(a_2)} \subseteq N_s(b_2).$$

Next, we want to define points p and m, which will eventually make up C. Since $s = r + d(a_1, b_1)$ Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}.$$

Note that since $\overline{N_r(a_1)} \subseteq N_s(b_2)$, we know that

$$p \in \overline{N_s(b_2)}.$$

Let $m \in \overrightarrow{b_1 a_2}$ such that $d(m, b_1) = d(m, a_2) + d(a_2, b_1)$ (i.e. m is to the left of a_2) and $d(m, a_2) = r$. Then, $d(m, b_1) = d(m, a_2) + d(a_2, b_1) \geq r + h(A, B) = s$, so

$$m \notin N_s(b_1).$$

Also, $d(m, a_1) \geq d(m, b_1) - d(b_1, a_1) \geq s - h(A, B) = r$, so

$$m \notin N_r(a_1).$$

In addition, since $\overline{N_r(a_2)} \subseteq N_s(b_2)$, we know that

$$m \in \overline{N_s(b_2)}.$$

Now, we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then

$$a_1 = p + \frac{r}{\mid b_1 - p \mid}(b_1 - p) = m + \frac{r}{\mid b_1 - m \mid}(b_1 - m) = a_2.$$

However, since $a_1 \neq a_2$, this is a contradiction, and so $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq (A + r)$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in \overline{N_r(a_1)}$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq (B + s)$,
2. $p \in \overline{N_s(b_1)} \cap \overline{N_s(b_2)}$, and
3. $p \in C \cap \overline{N_s(b_1)}$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$.

Therefore, $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.

□

Corollary 1. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If*

- $d(a_1, b_2) < h(A, B)$,

- $d(a_2, b_2) < h(A, B)$, *and*
- $d(a_2, b_1) < \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$,

then there exists a $z \in \mathbb{R}^+$ *such that*

1. *if* $C \in \mathcal{H}(\mathbb{R}^n)$ *and* $h(A, C) = h(A, B) + h(B, C)$, *then* $h(B, C) \leq z$,
2. *for every* $0 < s \leq z$, *there exists* $C \in \mathcal{H}(\mathbb{R}^n)$ *such that* $h(B, C) = s$ *and* $h(A, C) = h(A, B) + h(B, C)$, *and*
3. *for every* $r > 0$, *there exists* $C \in \mathcal{H}(\mathbb{R}^n)$ *such that* $h(A, C) = r$ *and* $h(C, B) = h(C, A) + h(A, B)$.

At this point we know that the three conditions give us a ray-like line. The next step would be to show that these are the only conditions that give us a ray like line. In order to show this, we will take one condition at a time, and show that if it fails, then our line is complete. First, we start with $d(a_1, b_2) \geq h(A, B)$.

Theorem 9. *Let A and B be two point sets in* $\mathcal{H}(\mathbb{R}^n)$ *as described in the beginning of the section. If* $d(a_1, b_2) \geq h(A, B)$, *then for every* $s > 0$ *there exists* $C \in \mathcal{H}(\mathbb{R}^n)$ *such that* $h(B, C) = s$ *and* $h(A, C) = h(A, B) + h(B, C)$.

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $s > 0$ and $r = h(A, B) + s$. There are two cases which we must consider

1. $d(a_1, b_2) > h(A, B)$ and
2. $d(a_1, b_2) = h(A, B)$.

Case 1 $d(a_1, b_2) > h(A, B)$

First we want to show that $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$. Since $d(a_1, b_2) > h(A, B)$, we know that $d(a_2, b_2) \leq h(A, B) = r - s$, and so $d(a_2, b_2) + s \leq r$. If $d(a_2, b_2) + r = s$, then by Lemma 3, $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$. If $d(a_2, b_2) + s < r$, then by Lemma 4, $\overline{N_s(b_2)} \subseteq N_r(a_2)$. Therefore, we know that

$$\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}.$$

Next, we want to define the points p and m, which will eventually make up C. Since $r = d(a_1, b_1) + s$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}.$$

Let $m \in \overrightarrow{a_1 b_2}$ such that $d(a_1, m) = d(a_1, b_2) + d(b_2, m)$ (i.e. m is to the right of b_2) and $d(b_2, m) = s$. Then, $d(a_1, m) = d(a_1, b_2) + d(b_2, m) \geq h(A, B) + s = r$, so

$$m \notin N_r(a_1).$$

Also, $d(m, b_1) \geq d(m, a_1) - d(a_1, b_1) \geq r - h(A, B) = s$, so

$$m \notin N_s(b_1).$$

Finally, since $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$, we know $m \in \overline{N_r(a_2)}$.

Now we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then,

$$b_1 = p + \frac{s}{|a_1 - p|}(a_1 - p) = m + \frac{s}{|a_1 - m|}(a_1 - m) = b_2.$$

However, since $b_1 \neq b_2$, we have a contradiction, and so $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. (An example of this case can be seen in Figure 20)

Case 2 $d(a_1, b_2) = h(A, B)$

First we want to define p_1 and p_2, which will eventually make up C. Since $d(a_1, b_1) + s = r$ and $d(a_1, b_2) + s = r$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p_1\}$$

and

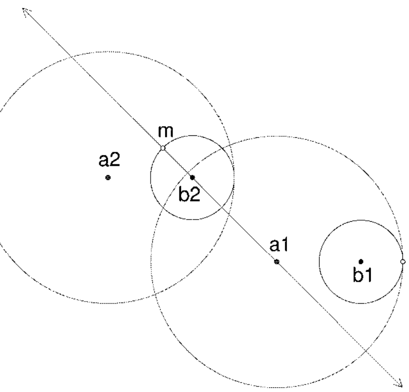

Figure 20: Example of Case 1

$$\partial N_r(a_1) \cap \partial N_s(b_2) = \{p_2\}.$$

We also need to show that $p_1 \neq p_2$. We will proceed by contradiction. Suppose $p_1 = p_2$. Then

$$b_1 = p_1 + \frac{s}{|a_1 - p_1|}(a_1 - p_1) = p_2 \frac{s}{|a_1 - p_2|}(a_1 - p_2) = b_2.$$

However, since $b_1 \neq b_2$, we have a contradiction, and so $p_1 \neq p_2$.

Also, we need to show that $p_2 \notin \overline{N_s(b_1)}$. We will proceed by contradiction. Suppose that $p_2 \in \overline{N_s(b_1)}$. Then, $p_2 \in \partial N_r(a_1) \cap \overline{N_s(b_1)}$. However, Lemma 3 states that $\partial N_r(a_1) \cap \overline{N_s(b_1)} = \{p_1\}$. This would mean that $p_1 = p_2$, which is a contradiction, and so

$$p_2 \notin \overline{N_s(b_1)}.$$

Now, we have two subcases to consider. Since $d(a_2, B) \leq h(A, B)$, either

1. $d(a_2, b_1) \leq h(A, B)$ or
2. $d(a_2, b_2) \leq h(A, B)$.

Case 2.1 $d(a_2, b_1) \leq h(A, B)$

First we want to show that $p_1 \in \overline{N_r(a_2)}$. Due to the way we named the elements of A and B, we said that $d(a_2, b_1) \geq h(A, B)$. Thus, in this case, $d(a_2, b_1) = h(A, B)$. This means that $d(a_2, b_1) = h(A, B) = r - s$, or $d(a_2, b_1) + s = r$. Then, Lemma 3 states that $\overline{N_s(b_1)} \subseteq \overline{N_r(a_1)}$. Therefore, $p_1 \in \overline{N_r(a_2)}$.

Let $C = \{p_1, p_2\}$. Then

1. $C \subseteq A + r$,

2. $p_1 \in \overline{N_r(a_1)} \cap \overline{N_r(a_2)}$, and

3. $p_1 \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,

2. $p_1 \in \overline{N_s(b_1)}$, $p_2 \in \overline{N_s(b_2)}$, and

3. $p_1 \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. (An example of this case can be seen in Figure 21)

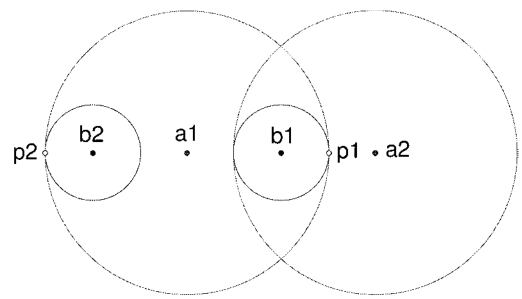

Figure 21: Example of Case 2.1

Case 2.2 $d(a_2, b_2) \leq h(A, B)$

First we want to show that $p_2 \in \overline{N_r(a_2)}$. Since $d(a_2, b_2) \leq h(A, B) = r - s$, we know that $d(a_2, b_2) + s \leq r$. If $d(a_2, b_2) + s = r$, then Lemma 3 states that $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$. If $d(a_2, b_2) + s < r$ then Lemma 4 states that $\overline{N_s(b_2)} \subseteq N_r(a_2)$. Therefore, $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$, and $p_2 \in \overline{N_r(a_2)}$.

Let $C = \{p_1, p_2\}$. Then

1. $C \subseteq A + r$,

2. $p_1 \in \overline{N_r(a_1)}$, $p_2 \in \overline{N_r(a_2)}$, and

3. $p_1 \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p_1 \in \overline{N_s(b_1)}$, $p_2 \in \overline{N_s(b_2)}$, and
3. $p_1 \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$.

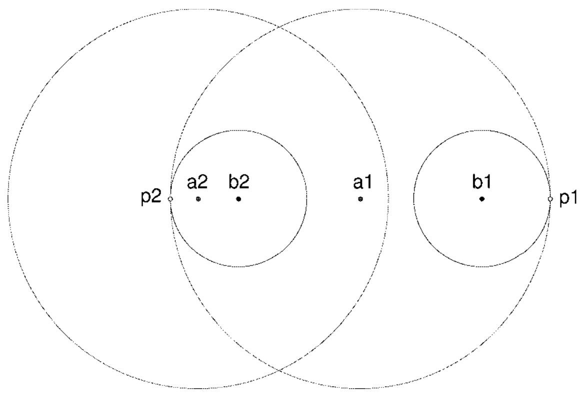

Figure 22: Example of Case 2.2

Therefore, we can always find $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. $\square$

Theorem 10. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_1, b_2) \geq h(A, B)$, then for every $r > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $r > 0$ and $s = h(A, B) + r$.

First, we want to define the points p and m, which will eventually make up C. Since $s = r + d(a_1, b_1)$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}.$$

Let $m \in \overrightarrow{b_1 a_2}$ such that $d(b_1, m) = d(b_1, a_2) + d(a_2, m)$ (i.e. m is to the left of a_2) and $d(a_2, m) = r$. Then, $d(b_1, m) = d(b_1, a_2) + d(a_2, m) \geq h(A, B) + r = s$, so

$$m \notin N_s(b_1).$$

Also, $d(m, a_1) \geq d(m, b_1) - d(b_1, a_1) \geq s - h(A, B) = r$, so

$$m \notin N_r(a_1).$$

Next, we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then

$$a_1 = p + \frac{r}{|b_1 - p|}(b_1 - p) = m + \frac{r}{|b_1 - m|}(b_1 - m) = a_2.$$

However, since $a_1 \neq a_2$, we have a contradiction, and so $m \neq p$.

Now, there are two cases which we must consider

1. $d(a_1, b_2) > h(A, B)$ and
2. $d(a_1, b_2) = h(A, B$.

Case 1 $d(a_1, b_2) > h(A, B)$

First, we want to show that $m \in \overline{N_s(b_2)}$. Since $d(b_2, a_1) > h(A, B)$, we know that $d(b_2, a_2) \leq h(A, B) = s - r$, and so $d(a_2, b_2) + r \leq s$. If $d(a_2, b_2) + r = s$, then Lemma 3 states that $\overline{N_r(a_2)} \subseteq \overline{N_s(b_2)}$. If $d(a_2, b_2) + r < s$, then Lemma 4 states that $\overline{N_r(a_2)} \subseteq N_s(b_2)$. Therefore, we know that $\overline{N_r(a_2)} \subseteq \overline{N_s(b_2)}$ and thus $m \in \overline{N_s(b_2)}$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. (An example of this case can be seen in Figure 23)

Case 2 $d(a_1, b_2) = h(A, B)$

First, we want to show that $p \in \overline{N_s(b_2)}$. Since $d(a_1, b_2) + r = s$, Lemma 3 states that $\overline{N_r(a_1)} \subseteq \overline{N_s(b_2)}$, and thus $p \in \overline{N_s(b_2)}$.

Let $C = \{m, p\}$. Then

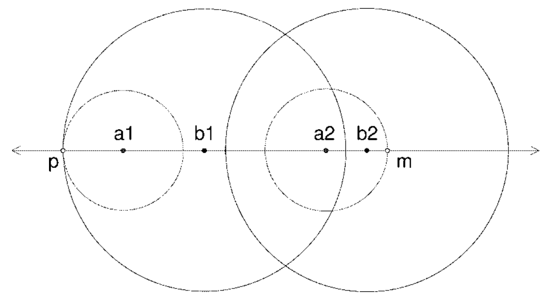

Figure 23: Example of Case1

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)} \cap \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B,C) = s$ and $h(A,C) = h(A,B) + h(B,C)$. (An example of this can be seen in Figure 24)

Therefore, we can always find $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A,C) = r$ and $h(C,B) = h(C,A) + h(A,B)$. □

Corollary 2. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_1,b_2) \geq h(A,B)$, then*

- *for every $s > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B,C) = s$ and $h(A,C) = h(A,B) + h(B,C)$, and*
- *for every $r > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A,C) = r$ and $h(C,B) = h(C,A) + h(A,B)$.*

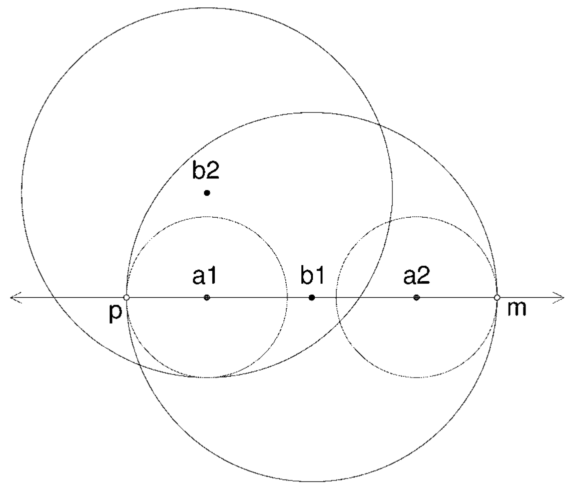

Figure 24: Example of Case 2

Therefore, if $d(a_1, b_2) \geq h(A, B)$, then the line defined by A and B is complete. Next, we will deal with $d(a_2, b_2) \geq h(A, B)$, and show that the line is once again complete.

Theorem 11. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_2, b_2) \geq h(A, B)$, then for every $s > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$.*

Proof. Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $s > 0$ and $r = h(A, B) + s$.

First, we need to note that since $d(a_2, B) \leq h(A, B)$ either $d(a_2, b_1) \leq h(A, B)$ (which means that $d(a_2, b_1) = h(A, B)$) or $d(a_2, b_2) \leq h(A, B)$ (which means that $d(a_2, b_2) = h(A, B)$). So, we have two cases to consider:

1. $d(a_2, b_1) = h(A, B)$ or
2. $d(a_2, b_1) > h(A, B)$ (and thus $d(a_2, b_2) = h(A, B)$).

Case 1 $d(a_2, b_1) = h(A, B)$

First, we want to define points p and m, which will eventually make up C. Since $d(a_2, b_1) + s = r$, Lemma 1 states that

$$\partial N_r(a_2) \cap \partial N_s(b_1) = \{p\}.$$

Also, since $d(a_1, b_1)+s = r$, Lemma 3 states that $\overline{N_s(b_1)} \subseteq \overline{N_r(a_1)}$. Therefore,

$$p \in \overline{N_r(a_1)}.$$

Let $m \in \overrightarrow{a_2b_2}$ such that $d(a_2, m) = d(a_2, b_2)+d(b_2, m)$ (i.e. m is to the right of b_2) and $d(b_2, m) = s$. Then $d(a_2, m) = d(a_2, b_2) + d(b_2, m) \geq h(A, B) + s = r$, so

$$m \notin N_r(a_2).$$

Also, $d(m, b_1) \geq d(m, a_2) - d(a_2, b_1) \geq r - h(A, B) = s$, so

$$m \notin N_s(b_1).$$

Next, we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then

$$b_1 = p + \frac{s}{|a_2 - p|}(a_2 - p) = m + \frac{s}{|a_2 - m|}(a_2 - m) = b_2.$$

However, since $b_1 \neq b_2$, we have a contradiction, and so $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)} \cap \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_2)$ but $C \cap N_r(a_2) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. (An example of this case can be seen in Figure 25)

Case 2 $d(a_2, b_1) > h(A, B)$

Note that in this case $d(a_2, b_2) = h(A, B)$.

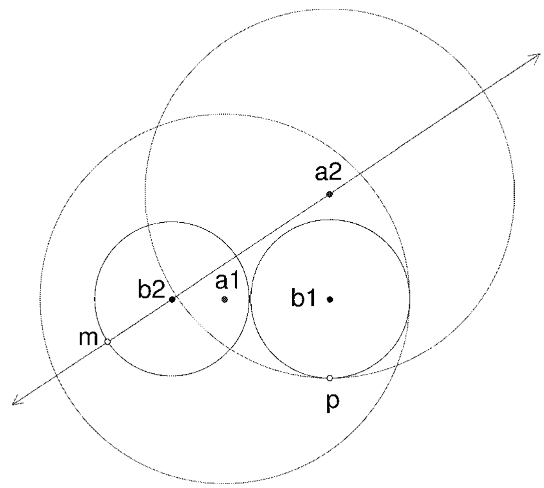

Figure 25: Example of Case 1

First, we want to define points p and m, which will eventually make up C. Since $d(a_2, b_2) + s = r$, Lemma 1 states that

$$\partial N_r(a_2) \cap \partial N_s(b_2) = \{p\}.$$

Let $m \in \overrightarrow{a_2 b_1}$ such that $d(a_2, m) = d(a_2, b_1) + d(b_1, m)$ (i.e. m is to the right of b_1) and $d(b_1, m) = s$. Then $d(a_2, m) = d(a_2, b_1) + d(b_1, m) > h(A, B) + s = r$, so

$$m \notin N_r(a_2).$$

Also, $d(m, b_2) \geq d(m, a_2) - d(a_2, b_2) > r - h(A, B) = s$, so

$$m \notin N_s(b_2).$$

In addition, since $d(a_1, b_1) + s = r$, Lemma 3 states that $\overline{N_s(b_1)} \subseteq \overline{N_r(a_1)}$, and so $m \in \overline{N_r(a_1)}$.

Next, we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then

$$b_1 = m + \frac{s}{|a_2 - m|}(a_2 - m) = p + \frac{s}{|a_2 - p|}(a_2 - p) = b_2.$$

However, since $b_1 \neq b_2$, we have a contradiction, and so $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,
2. $m \in \overline{N_r(a_1)}$, $p \in \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_2)$ but $C \cap N_r(a_2) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $m \in \overline{N_s(b_1)}$, $p \in \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_2)$ but $C \cap N_s(b_2) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. (An example of this case can be seen in Figure 26)

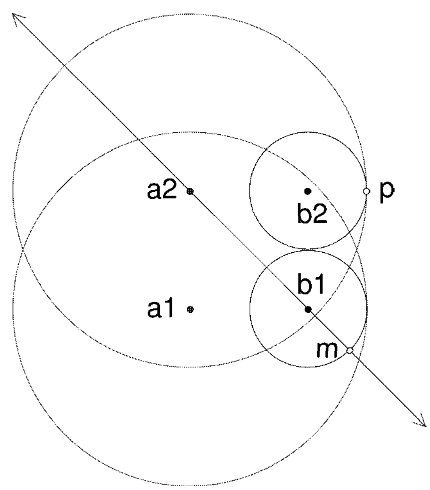

Figure 26: Example of Case 2

Therefore, we can always find $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$, and $h(A, C) = h(A, B) + h(B, C)$. $\square$

Theorem 12. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_2, b_2) \geq h(A, B)$, then for every $r > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.*

Proof. Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $r > 0$ and $s = r + d(A, B)$.

First, we need to note that since $d(a_2, B) \leq h(A, B)$ either $d(a_2, b_1) \leq h(A, B)$ (which means that $d(a_2, b_1) = h(A, B)$) or $d(a_2, b_2) \leq h(A, B)$ (which means that $d(a_2, b_2) = h(A, B)$). So, we have two cases to consider:

1. $d(a_2, b_1) = h(A, B)$ or
2. $d(a_2, b_1) > h(A, B)$ (and thus $d(a_2, b_2) = h(A, B)$).

Case 1 $d(a_2, b_1) = h(A, B)$

First we want to define points p_1 and p_2, which will eventually make up C. Since $r + d(a_1, b_1) = s$ and $r + d(a_2, b_1) = s$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p_1\}$$

and

$$\partial N_r(a_2) \cap \partial N_s(b_1) = \{p_2\}.$$

Next, we want to show that $p_1 \neq p_2$. We will proceed by contradiction. Suppose $p_1 = p_2$. Then,

$$a_1 = p_1 + \frac{r}{|b_1 - p_1|}(b_1 - p_1) = p_2 + \frac{r}{|b_1 - p_2|}(b_1 - p_2) = a_2.$$

However, since $a_1 \neq a_2$, we have a contradiction, and so $p_1 \neq p_2$.

Also, we need to show that $p_2 \notin \overline{N_r(a_1)}$. We will proceed by contradiction. Suppose that $p_2 \in \overline{N_r(a_1)}$. Then, $p_2 \in \partial N_s(b_1) \cap \overline{N_r(a_1)}$. However, Lemma 3 states that $\partial N_s(b_1) \cap \overline{N_r(a_1)} = \{p_1\}$. This would mean that $p_1 = p_2$, which is a contradiction, and so

$$p_2 \notin \overline{N_r(a_1)}.$$

Since $d(b_2, A) \leq h(A, B)$, we have two subcases to consider:

1. $d(a_1, b_2) \leq h(A, B)$ or
2. $d(a_2, b_2) \leq h(A, B)$ (which means that $d(a_2, b_2) = h(A, B)$)

Case 1.1 $d(a_1, b_2) \leq h(A, B)$

In this case we want to show that $p_1 \in \overline{N_s(b_2)}$. Since $d(a_1, b_2) \leq h(A, B) = s - r$, we know that $r + d(a_1, b_2) \leq s$. If $r + d(a_1, b_2) = s$, Lemma 3 states that

$\overline{N_r(a_1)} \subseteq \overline{N_s(b_2)}$. If $r + d(a_1, b_2) < s$, Lemma 4 states that $\overline{N_r(a_1)} \subseteq N_s(b_2)$. Therefore, $\overline{N_r(a_1)} \subseteq \overline{N_s(b_2)}$, and so

$$p_1 \in \overline{N_s(b_2)}.$$

Let $C = \{p_1, p_2\}$. Then

1. $C \subseteq A + r$,
2. $p_1 \in \overline{N_r(a_1)}$, $p_2 \in \overline{N_r(a_2)}$, and
3. $p_1 \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p_1 \in \overline{N_s(b_1)} \cap \overline{N_s(b_2)}$, and
3. $p_1 \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$. Therefore, $h(A, C) = r$ and $h(C, A) = h(C, A) + h(A, B)$. (An example of this case can be seen in Figure 27)

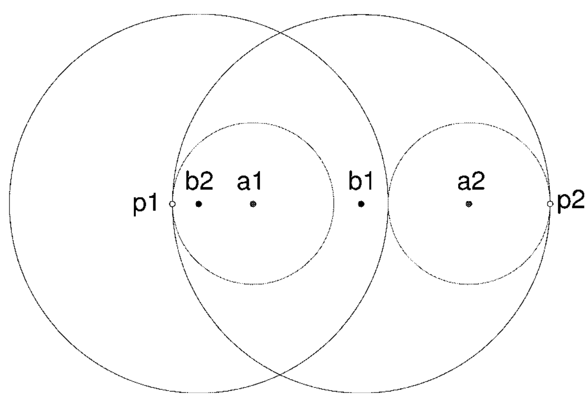

Figure 27: Example of Case 1.1

Case 1.2 $d(a_2, b_2) = h(A, B)$

In this case we want to show that $p_2 \in \overline{N_s(b_2)}$. Since $d(a_2, b_2) = h(A, B) = s - r$, we know that $r + d(a_2, b_2) = s$. Then, Lemma 3 states that $\overline{N_r(a_1)} \subseteq \overline{N_s(b_2)}$. Thus,

$$p_2 \in \overline{N_s(b_2)}.$$

Let $C = \{p_1, p_2\}$. Then

1. $C \subseteq A + r$,

2. $p_1 \in \overline{N_r(a_1)}$, $p_2 \in \overline{N_r(a_2)}$, and

3. $p_1 \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,

2. $p_2 \in \overline{N_s(b_1)} \cap \overline{N_s(b_2)}$, and

3. $p_1 \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$..

Thus, $C \in C_s(B)$. Therefore, $h(A, C) = r$ and $h(C, A) = h(C, A) + h(A, B)$. (An example of this case can be seen in Figure 28)

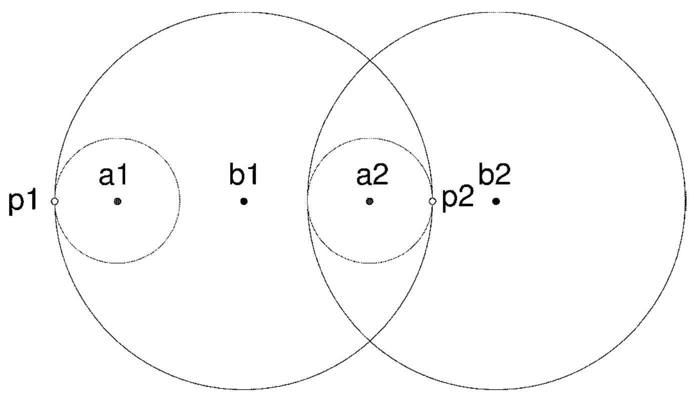

Figure 28: Example of Case 1.2

Case 2 $d(a_2, b_1) > h(A, B)$

Since $d(a_2, b_1) > h(A, B)$, we know that $d(a_2, b_2) = h(A, B)$.

First, we want to define points p and m, which will eventually make up C. Since $r + d(a_1, b_1) = s$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}.$$

Let $m \in \overrightarrow{b_1 a_2}$ such that $d(b_1, m) = d(b_1, a_2) + d(a_2, m)$ (i.e. m is to the left of a_2) and $d(a_2, m) = r$. Then, $d(b_1, m) = d(b_1, a_2) + d(a_2, m) > h(A, B) + r = s$, so $m \notin N_s(b_1)$. Also, $d(m, a_1) \geq d(m, b_1) - d(b_1, a_1) > s - h(A, B) = r$, so

$$m \notin N_r(a_1).$$

In addition, since $r + d(a_2, b_2) = s$, Lemma 3 states that $\overline{N_r(a_2)} \subseteq \overline{N_s(b_2)}$, and thus $m \in \overline{N_s(b_2)}$.

Next, we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then,

$$a_1 = p + \frac{r}{|b_1 - p|}(b_1 - p) = m + \frac{r}{|b_1 - m|}(b_1 - m) = a_2.$$

However, since $a_1 \neq a_2$, we have a contradiction, and so $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$. Therefore, $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$. (An example of this case can be seen in Figure 29)

Therefore, we can always find $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$. $\square$

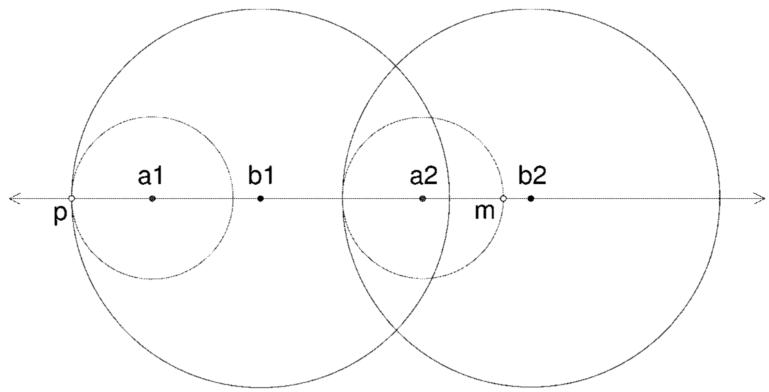

Figure 29: Example of Case 2

Corollary 3. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_2, b_2) \geq h(A, B)$, then*

- *for every $s > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$, and*
- *for every $r > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.*

Therefore, if $d(a_2, b_2) \geq h(A, B)$, our line is once again complete. Finally, we want $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$.

Theorem 13. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$, then for every $s > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$.*

Proof. Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $s > 0$ and $r = h(A, B) + s$.

Since $d(a_1, b_1) + s = r$, Lemma 1 states that $\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}$. This also means that $d(a_1, p) = d(a_1, b_1) + d(b_1, p)$ (i.e. p lies to the right of B). We want to show that $p \in \partial(A + r)$. However, first we need to simplify our problem by moving from n-space into 2-space. We can do this by taking our points a_1, a_2, and b_1 and forming a plane with them. Then, we can set up our coordinate system by making $\overleftrightarrow{a_1 a_2}$ into the y-axis, and the perpendicular bisector of $\overline{a_1 a_2}$ into the x-axis. Also, we want to chose the orientation of our axes so that a_1 has a positive y-coordinate and allow for a reflection over the y-axis so that b_1 has a positive x-coordinate. Then, the plane will look like Figure 30. Denote the location of a_1 as $(0, a)$, a_2 as $(0, -a)$, and b_1 as (x, y) where $a, x \in \mathbb{R}^+$ and $y \in \mathbb{R}$.

Note that in this case, it is possible for b_1 to lie on $\overleftrightarrow{a_1 a_2}$ to the left of a_1. If this happens, then just choose an arbitrary plane on which a_1, a_2, and b_1 all lie.

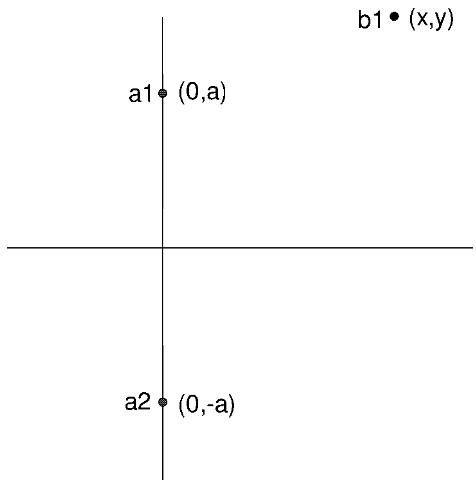

Figure 30: Plane formed by a_1, a_2, and b_1

Now, recall that $\sqrt{(d(a_1, a_2))^2 + (h(A, B))^2} \leq d(a_2, b_1)$. We can use this fact to show that $y \geq a$. We will proceed by contradiction. Suppose $y < 0$. Then

$$\begin{aligned} d(a_2, b_1) &= \sqrt{x^2 + (a + y)^2} \\ &\leq \sqrt{x^2 + (a - y)^2} \\ &= d(a_1, b_1) \\ &= h(A, B) \\ &< \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2} \end{aligned}$$

However, this is a contractiction, since $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$, so $y \geq 0$. Now, suppose $0 \leq y < a$. Then

$$\begin{aligned} d(a_2, b_1) &= \sqrt{x^2 + (a+y)^2} \\ &= \sqrt{x^2 + (2a + y - a)^2} \\ &= \sqrt{x^2 + 4a^2 + 4a(y-a) + (y-a)^2} \\ &= \sqrt{4a^2 + 4a(y-a) + x^2 + (y-a)^2} \\ &< \sqrt{4a^2 + x^2 + (y-a)^2} \\ &= \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2} \end{aligned}$$

However, this is a contradiction, since $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$, so $y \geq a$.

We also need to note that since p lies to the right of b_1 on $\overleftrightarrow{a_1 b_1}$, we know that p is on the plane formed by a_1, a_2, and b_1. The point p's location is illustrated in Figure 31.

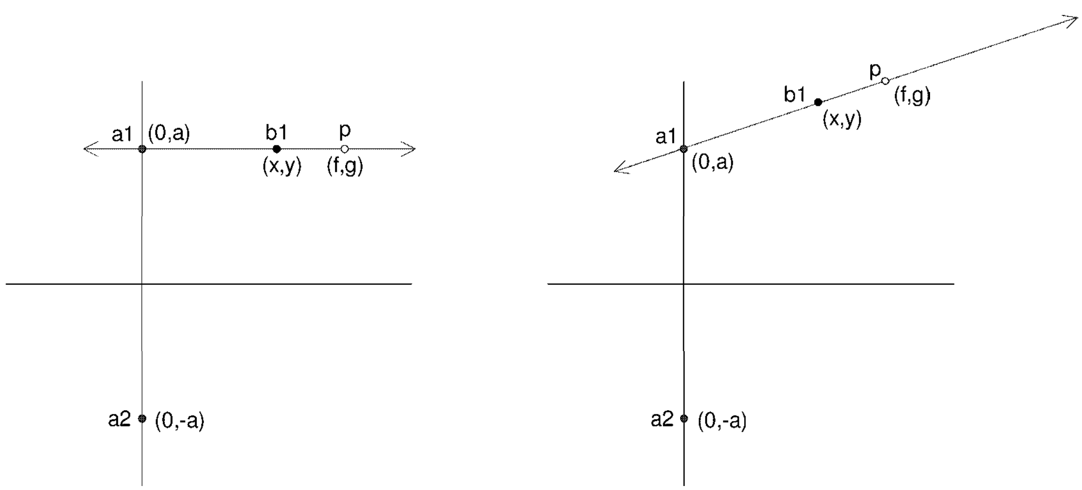

Figure 31: The location of p on the plane

Next, we want to show that $p \in \partial(A+r)$. Let $p = (f, g)$. Then $d(a_2, p) = \sqrt{f^2 + (a+g)^2} > \sqrt{f^2 + (g-a)^2} = d(a_1, p) = r$. Thus, $p \notin \overline{N_r(a_2)}$, which means that $p \in \partial(A + r)$.

Finally, we want to show that $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$. Since

$$d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2} > h(A, B),$$

we know that $d(a_2, b_2) \leq h(A, B) = r - s$, which means that $d(a_2, b_2) + s \leq r$. If $d(a_2, b_2) + s = r$, Lemma 3 states that $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$. If $d(a_2, b_2) + s < r$, Lemma 4 states that $\overline{N_s(b_2)} \subseteq N_r(a_2)$. Therefore $\overline{N_s(b_2)} \subseteq \overline{N_r(a_2)}$. Note that this means that $p \notin \overline{N_s(b_2)}$, and so

$$p \in \partial(B + s).$$

Let $C = \{m, p\}$, where $m \in \overline{N_s(b_2)}$ (note that $m \neq p$, since $p \notin \overline{N_r(a_2)}$). Then

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in \partial(A + r)$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p \in \partial(B + s)$.

Thus, $C \in C_s(B)$.

Therefore, $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$. □

Theorem 14. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$, then for every $r > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.*

Proof. Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Also, let $r > 0$ and $s = r + d(A, B)$.

First, we want to define points p and m, which will eventually make up C (pictured in Figure 32). Since $r = d(a_1, b_1) + s$, Lemma 1 states that

$$\partial N_r(a_1) \cap \partial N_s(b_1) = \{p\}.$$

Let $m \in \overrightarrow{b_1 a_2}$ such that $d(b_1, m) = d(b_1, a_2) + d(a_2, m)$ (i.e. m is to the left of a_2) and $d(a_2, m) = r$. Then $d(b_1, m) = d(b_1, a_2) + d(a_2, m) > h(A, B) + r = s$, so

$$m \notin N_s(b_1).$$

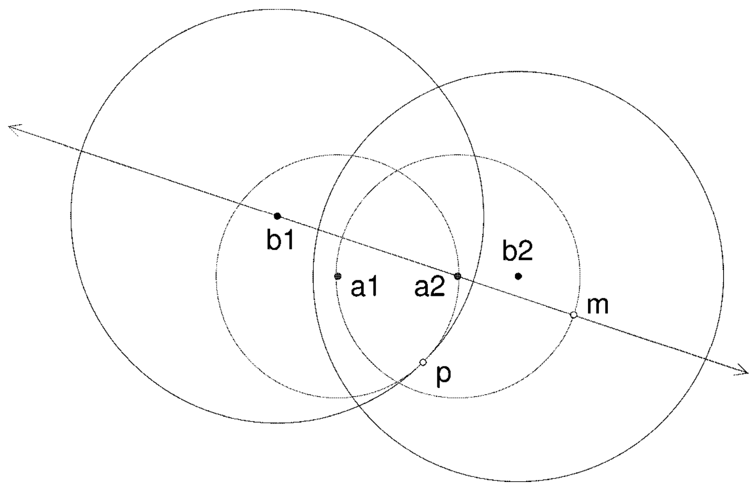

Figure 32: Example of p and m

Also, $d(m, a_1) \geq d(m, b_1) - d(b_1, a_1) \geq s - h(A, B) = r$, so

$$m \notin N_r(a_1).$$

Finally, since $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2} > h(A, B)$, we know that $d(a_2, b_2) \leq h(A, B) = s - r$, which means that $r + d(a_2, b_2) \leq s$. If $r + d(a_2, b_2) = s$, Lemma 3 states that $\overline{N_r(a_2)} \subseteq \overline{N_s(b_2)}$. If $r + d(a_2, b_2) < s$, Lemma 4 states that $\overline{N_r(a_2)} \subseteq N_s(b_2)$. Therefore, $\overline{N_r(a_2)} \subseteq \overline{N_s(b_2)}$, and so

$$m \in \overline{N_s(b_2)}.$$

Next, we want to show that $m \neq p$. We will proceed by contradiction. Suppose $m = p$. Then,

$$a_1 = p + \frac{r}{|b_1 - p|}(b_1 - p) = m + \frac{r}{|b_1 - m|}(b_1 - m) = a_2.$$

However, since $a_1 \neq a_2$, we have a contradiction, and so $m \neq p$.

Let $C = \{m, p\}$. Then

1. $C \subseteq A + r$,
2. $p \in \overline{N_r(a_1)}$, $m \in \overline{N_r(a_2)}$, and
3. $p \in C \cap \partial N_r(a_1)$ but $C \cap N_r(a_1) = \emptyset$.

Thus, $C \in C_r(A)$. Also,

1. $C \subseteq B + s$,
2. $p \in \overline{N_s(b_1)}$, $m \in \overline{N_s(b_2)}$, and
3. $p \in C \cap \partial N_s(b_1)$ but $C \cap N_s(b_1) = \emptyset$.

Thus, $C \in C_s(B)$.

Therefore, $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$. □

Corollary 4. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. If $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$, then*

- *for every $s > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$, and*
- *for every $r > 0$ there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.*

Thus, if $d(a_2, b_1) \geq \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$, we yet again have a complete line. Therefore, only lines that satisfy the three requirements are ray like. This is summed up in the final corollary.

Corollary 5. *Let A and B be two point sets in $\mathcal{H}(\mathbb{R}^n)$ as described in the beginning of the section. Then there exists a $z \in \mathbb{R}^+$ such that*

1. *if $C \in \mathcal{H}(\mathbb{R}^n)$ and $h(A, C) = h(A, B) + h(B, C)$, then $h(B, C) \leq z$,*
2. *for every $0 < s \leq z$, there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(B, C) = s$ and $h(A, C) = h(A, B) + h(B, C)$, and*
3. *for every $r > 0$, there exists $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = r$ and $h(C, B) = h(C, A) + h(A, B)$.*

if and only if

- *$d(a_1, b_2) < h(A, B)$,*
- *$d(a_2, b_2) < h(A, B)$, and*
- *$d(a_2, b_1) < \sqrt{(d(a_1, a_2))^2 + (h(A, B))^2}$.*

6.4 Specific z

It is also interesting to note that we create a line with any value of z that we want, as long as $z \geq 0$. To do this, we must first find a formula relating z to the y-coordinate of b_1. Let us start with A and B as in Figure 33. So,

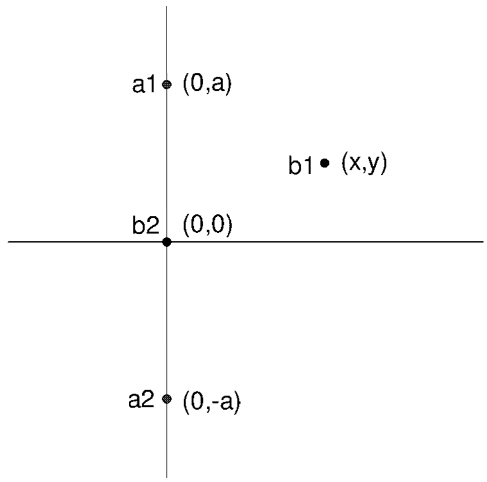

Figure 33: A and B for our specific z

- $A = \{a_1, a_2\}$ where $a_1 = (0, a)$ and $a_2 = (0, -a)$ where $a < h(A, B)$ and
- $B = \{b_1, b_2\}$ where $b_1 = (x, y)$ and $b_2 = (0, 0)$ where $0 \leq y < a$.

We also want $d(a_1, b_1) = h(A, B)$. Note that A and B fulfill the conditions of Theorem 7, so we know that the line will stop z units to the right of B. Recall that z is the distance from b_1 to the intersection of $\overleftrightarrow{a_1 b_1}$ and the x-axis. Then, due to similar triangles (refer to Figure 34)

$$\frac{h(A, B) + z}{h(A, B)} = \frac{a}{a - y}.$$

Solving for z shows us that

$$z = \frac{h(A, B)y}{a - y}.$$

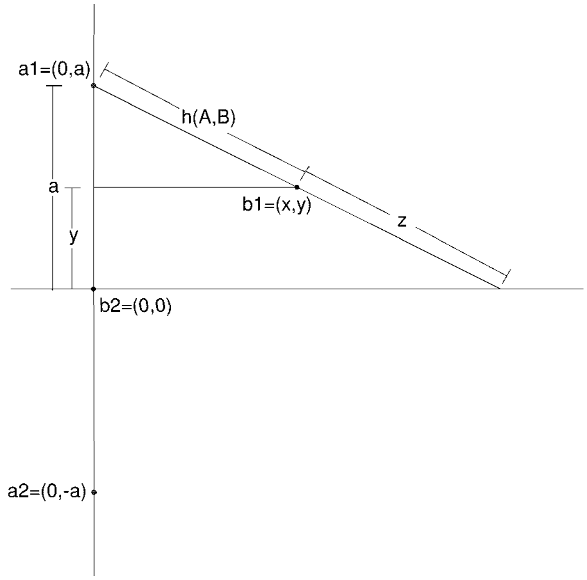

Figure 34: Similar Triangles

Note that z does not depend on x, and so we can change y in anyway we need, and just change x accordingly. Since $0 \leq y < a$ we can find the bounds on z. If $y \to 0$, then $z \to 0$; and if $y \to a$, then $z \to \infty$ (Note that when we change y, we also have to change x to preserve the fact that $d(a_1, b_1) = h(A, B)$). So, we can see that $z \geq 0$, which makes sense, since z is a distance.

Next, we want to show that we can form a line with any given value for z. Assume we want $z = z_0$, where $z_0 \geq 0$. Then, we can pick any value for $h(A, B) > 0$ (Note: $h(A, B) \neq 0$ because then $A = B$ and we cannot define a line with one point) and we can pick any $a < h(A, B)$. Then, if $y = \frac{az_0}{h(A,B)+z_0}$, we can show that y has the correct bounds, because

$$0 \leq \frac{az_0}{h(A, B) + z_0} = a\frac{z_0}{h(A, B) + z_0} < a.$$

Also, then

$$z = \frac{h(A, B)\frac{az_0}{h(A,B)+z_0}}{a - \frac{az_0}{h(A,B)+z_0}} = \frac{h(A, B)az_0}{h(A, B)a + az_0 - az_0} = z_0$$

which is exactly what we wanted. Finally, in order to assure that $d(a_1, b_1) = h(A, B)$ we need to pick $x = \sqrt{(h(A, B)^2 - (a - y)^2}$.

Now, since our points A and B fulfill the conditions of Theorem 7, we can find points on $\overline{AB}$ up to z units to the right of B, but no further.

7 And Beyond...

Through our study of the geometry of $\mathcal{H}(\mathbb{R}^n)$, specifically our focus on lines defined by two point sets, we have found many surprising and astonishing properties of lines in $\mathcal{H}(\mathbb{R}^n)$. Of these, the most intriguing find was that lines in $\mathcal{H}(\mathbb{R}^n)$ can act like rays under certain conditions. This was something that we never expected to find.

However, there are still many questions left unanswered, and conjectures left unproven. These questions and conjectures we leave open to further research.

1. In the lines defined by two point sets A and B, there is at least one $a \in A$ and one $b \in B$ such that $d(a, b) = h(A, B)$. Therefore, there exists at least one p as described in Lemma 1, and at most four p's (if $d(a_1, b_1) = d(a_1, b_2) = d(a_2, b_1) = d(a_2, b_2) = h(A, B)$). If we collect together all the p's formed by A and B, to create the set P, then we conjecture that if $C \in \mathcal{H}(\mathbb{R}^n)$ and $C \in \overleftrightarrow{AB}$, then $C \cap P \neq \emptyset$.

2. What do lines formed by sets with a finite number of points, but more than two points, look like? Can we still always find the segment $\overline{AB}$, and can the lines be classified as ray-like or complete?

3. How do the lines defined by two point sets intersect? Is there a way to extract an incidence geometry?

4. A study has also been done on lines in $\mathcal{H}(\mathbb{R}^n)$ formed by the general sets A and B (where there were no restrictions put on the number of points in A or B). Can the research done on lines defined by two point sets be connected to the research done on lines defined by general sets?

5. Can we begin to form shapes in $\mathcal{H}(\mathbb{R}^n)$? How would the shapes be defined (especially since lines in $\mathcal{H}(\mathbb{R}^n)$ can be ray-like)? Is it possible to create angles?

References

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[2] James R. Munkers, *Topology: Second Edition*, Prentice Hall, Upper Saddle River, 2000.