The Hausdorff Metric Geometry

REU 2005 Report *

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Week 8

Abstract

The Hausdorff metric h gives us a method of measuring the distance between non-empty compact subsets of n-dimensional Euclidean space. Unlike the Euclidean concept of betweenness, there is not necessarily a unique set at each location between two other sets in the Hausdorff metric geometry. A configuration defines two sets (infinite or finite) for which it is possible to have a finite number of elements at each location between sets. We will first consider infinite and finite sets for which it is possible to have a finite number of sets at each location between two sets, noting that there is no configuration with exactly 19 elements at each location between two sets. We will conclude by connecting the number of sets at each location between two sets, making up specific finite configurations, and Fibonacci numbers.

1 Introduction

In this paper we will be discussing the geometry imposed on the collection of non-empty compact subsets of $\mathbb{R}^n$, which shall be denoted $\mathcal{H}(\mathbb{R}^n)$, by the Hausdorff metric, h. To gain a better understanding of the Hausdorff metric, and in fact, to understand what it means for subsets of $\mathbb{R}^n$ to be compact, we must become acquainted with some basic ideas in topology. Following the introduction to topology we will discuss the Hausdorff metric in more detail.

2 Topology

We define a topology in the following manner:

Definition 2.1. *Let $\mathbb{X}$ be a set of points. A collection of subsets $\mathcal{U} = \{U_\alpha : \alpha \in I\}$, where $U_\alpha \subseteq \mathbb{X}$ for all $\alpha \in I$, forms a topology on $\mathbb{X}$ if*

1. $\bigcup_{\beta \in I'} U_\beta \in \mathcal{U}$ *for any* $I' \subseteq I$.
2. *The intersection of any finite number of sets U_α in the collection $\mathcal{U}$ is another set in $\mathcal{U}$.*

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3. Both the empty set $\emptyset$ and the whole space $\mathbb{X}$ must be in $\mathcal{U}$.

Then $(X,\mathcal{U})$ is called a topological space and the sets U_α in the collection $\mathcal{U}$ are called open sets. A set C is closed if its compliment $X - C$ is open.

Example 2.1. *Recall that the Euclidean distance between two points $a, b \in \mathbb{R}^n$, where $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$, is defined by $|a - b| = \sqrt{(a_1 - b_1)^2 + ... + (a_n - b_n)^2}$. A set U in $\mathbb{R}^n$ will be open if given any $a \in \mathbb{R}$, there is a real number $\epsilon > 0$ such that $\{x \in \mathbb{R}^n : |x - a| < \epsilon\}$ is contained in U.*

This defines the usual topology on $\mathbb{R}^n$.

Example 2.2. *Suppose we define a set U in $\mathbb{R}$ to be open if given any $a \in \mathbb{R}$, there is a real number $\epsilon > 0$ such that $\{x : |x - a| < \epsilon\}$ is contained in U, or if $U = \{1\}$ or $U = \{2\}$*

This does not define a topology on $\mathbb{R}$ because $(1, 2) \cup \{1\} \cup \{2\} = [1, 2]$ is not an open set which violates the first condition of Definition 2.1.

Example 2.3. *The set $A = (0, 1)$ is open in the usual topology while the set $B = [0, 1]$ is closed because its compliment, $B' = (-\infty, 0) \cup (1, \infty)$, is open. Some sets, like $[0, 1)$, are neither open nor closed.*

Here we introduce some useful terminology, *open cover.* A collection $\Sigma = \{U_\alpha : \alpha \in I\}$, where I is an index set, of open sets in a topological space X is said to be an *open cover* of a subset $A \subseteq X$ if $A \in \bigcup_{\alpha \in I} U_\alpha$. A *subcover* of an open cover Σ is any subfamily $\{U_\beta : \beta \in I' \subset I\}$ of Σ where $A \in \bigcup_{\beta \in I'} U_\beta$ [1, 3]. This allows us to define what we meant by "compact" sets in the introduction.

Definition 2.2. *A subset A of a topological space X is compact if given any open cover Σ of A, there is a finite subcover, $\Sigma' \subset \Sigma$, of A.*

Topololgy extends the notion of a continuous function to contexts in which the limit version doesn't make sense [3]. This motivates the following definition of continuity.

Definition 2.3. *A function $f : X \longrightarrow Y$ is continuous, where X and Y are two topological spaces, if given any open set U in Y, then the inverse image $f^{-1}(U)$ in X must be open.*

We will now give one more definition, a statement of the Heine-Borel Theorem from [3], and conclude our discussion about topology with the idea of induced, or subspace, topology which will be useful to us in Section 4.

Definition 2.4. *A subset A is bounded in $\mathbb{R}^n$ if there is some fixed $r \in \mathbb{R}$ such that for all $x \in A$, $|x| < r$.*

Theorem 2.1. ***(Heine-Borel)*** *A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

For example, consider the set $A = [0, 1] \in \mathbb{R}$. This set is both closed and bounded and by Theorem 2.1 A is also compact, that is, for every open cover of A, there exists a finite subcover.

Definition 2.5. *Let A be a subset of a topological space $(\mathbb{X}, \mathcal{U})$. Then the induced topology on $A$is the collection, $\mathcal{U}_A = \{U \cap A : U \in \mathcal{U}\}$.*

Example 2.4. *Let $X = \{a, b, c, d, e\}$ and $\mathcal{U} = \{\emptyset, X, \{a, c, d\}, \{a, c\}, \{c, d\}, \{c\}\}$. If $A = \{a, c, e\}$, then the induced topology on A is $\mathcal{U}_A = \{\emptyset, A, \{a, c\}, \{c\}\}$. The subsets of $\mathcal{U}_A$ are relatively open sets with respect to the topology of X.*

3 Hausdorff Metric

A metric is a way of measuring distance between objects in space. We want the mathematical definition of this measure to agree with our natural sense of distance. That is, if we have objects A and B in some space X we would expect the distance from A to B to be the same as the distance B to A. It is also natural to think of distance as always positive, or zero if A and B are in the same location. Lastly, if we place an intermediate object, C, on our path between A and B, the distance through this new path will be greater than or equal to the distance between A and B, and the paths are equal if C lies on the original path from A to B. We now formalize this notion in the next definition.

Definition 3.1. *Let X be a set. A function $d : X \times X \to \mathbb{R}$ is a metric on X if for all $x, y, z \in X$,*

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$ *with equality if and only if* $x = y$
3. $d(x, z) \leq d(x, y) + d(y, z)$.

Notice that the Euclidean distance function in Example 2.1 is a metric on $\mathbb{R}^n$. The Euclidean metric is just one of many metrics and this paper will focus on what is called the Hausdorff metric. The Hausdorf metric was introduced in the early 20th centry by Felix Hausdorff as a way to measure the distance between sets [4]. Specifically, the Hausdorff metric measures the distance between non-empty compact subsets of $\mathbb{R}^n$. We will denote the collection of these sets by $\mathcal{H}(\mathbb{R}^n)$.

Definition 3.2. *Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$.*

- *If $x \in \mathbb{R}^n$, the "distance" from x to B is*

$$d(x, B) = \min_{b \in B}\{d_E(x, b)\}$$

  where d_E is the usual Euclidean metric.

- *The "distance" from A to B is*

$$d(A, B) = \max_{x \in A}\{d(x, B)\}.$$

  This is not yet a metric as described in definition 3.1 because $d(A, B)$ isn't necessarily equal to $d(B, A)$ as illustrated in Figure 1.

- *The **Hausdorff distance**, $h(A, B)$, between A and B is*

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

In order to guarantee that the Hausdorff distance function is always defined, we need to know that the smallest distance from any point in $\mathbb{R}^n$ to a set $B \in \mathcal{H}(\mathbb{R}^n)$ exists, more formally

Theorem 3.1. *Given any $x \in \mathbb{R}^n$ and any set $B \in \mathcal{H}(\mathbb{R}^n)$, the set $\{d_E(x, b) : b \in B\}$ contains a minimum value.*

Example 3.1. *Suppse $x = 1$ and $B = \{x : 4 \leq x \leq 18\}$ on R, the real number line. Then $\min\{d_E(x, b) : b \in B\} = 3$ since the smallest distance from x to some point in B is $d_E(1, 4) = 3$.*

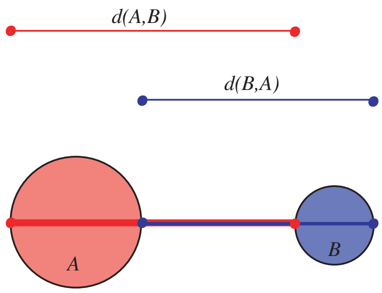


Figure 1: Here we have $d(B, A) < d(A, B)$.

The proof of Theorem 3.1 given in [5] relies on Definition 2.3 and Theorem 2.1 to show that this minimum value does exist. We are interested in the existence of this minimum value because if it always exists from any point $x \in \mathbb{R}^n$ to any set $B \in \mathcal{H}(\mathbb{R}^n)$ then the distance between any two sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ exists. This means that Hausdorff distance between any two sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ is always defined and we are able to show that the Hausdorff distance function is a metric on $\mathcal{H}(\mathbb{R}^n)$.

Theorem 3.2. *The Hausdorff distance function, h, as defined in Definition 3.2 is a metric on $\mathcal{H}(\mathbb{R}^n)$.*

Proof. Let $A, B, C \in \mathcal{H}(\mathbb{R}^n)$.

The Hausdorff distance function is symmetric because

$$\begin{aligned} h(A, B) &= \max\{d(A, B), d(B, A)\} \\ &= \max\{d(B, A), d(A, B)\} \\ &= h(B, A). \end{aligned}$$

Now we show that h satisfies the second condition of a metric given in Definition 3.1

From the Euclidean metric we know that if $x \in A$, then $d(x, B) = \min_{b \in B}\{d_E(x, b)\} \geq 0$, since $d_E(x, b) \geq 0$ for all $b \in B$. So, $d(A, B) = \max_{x \in A}\{d(x, B)\} \geq 0$. By a similar argument $d(B, A) \geq 0$, thus $h(A, B) = \max\{d(A, B), d(B, A)\} \geq 0$.

Suppose $h(A, B) = 0$, then

$$h(A, B) = \max\{d(A, B), d(B, A)\} = 0,$$

so it must be that

$$d(A, B) = 0 = d(B, A).$$

We have $d(A, B) = \max_{x \in A}\{d(x, B)\} = 0$, so for all $x \in A$, $d(x, B) = \min_{b \in B}\{d_E(x, b)\} = 0$. If $\min_{b \in B}\{d_E(x, b)\} = 0$ then for any $x \in A$ there exists a $b' \in B$ such that $d_E(x, b') = 0$. It follows that $x = b'$ therefore $x \in B$ and $A \subset B$. Likewise, since $d(B, A) = \max_{y \in B}\{d(y, A)\} = 0$ we have $d(y, A) = \min_{a \in A}\{d_E(y, a)\} = 0$. So, for any $y \in B$, there is some $a' \in A$ such that $d_E(y, a) = 0$. This means that $y = a'$ thus $y \in A$ and $B \subset A$. Therefore, $A = B$.

Now suppose $A = B$. If we let $x \in A$ then $x \in B$ and $d_E(x, x) = 0$. This gives us

$$\begin{aligned} d(x, A) &= \min_{a \in A}\{d_E(x, a)\} \\ &= 0 \\ &= \min_{b \in B}\{d_E(x, b)\} \\ &= d(x, B). \end{aligned}$$

Therefore,

$$\begin{aligned} d(A, B) &= \max\{0\} \\ &= 0 \\ &= d(B, A). \end{aligned}$$

so $h(A, B) = 0$.

We conclude by showing that h satisfies the third condition in Definition 3.1.

The triangle inequality,

$$d_E(a, c) \leq d_E(a, b) + d_E(b, c),$$

holds for any $a \in A, b \in B, c \in C$. Without loss of generality, suppose $d(A, C) \geq d(C, A)$ and choose $a' \in A, b' \in B, c' \in C$ such that

$$\begin{aligned} d(a', C) &= h(A, C) \\ d_E(a', b') &= d(a', B), \\ d_E(b', c') &= d(b', C). \end{aligned}$$

Then,

$$d_E(a', c') \leq d_E(a', b') + d_E(b', c') \tag{3.1}$$

with the appropriate substitutions into 3.1 gives

$$d_E(a', c') \leq d(a', B) + d(b', C). \tag{3.2}$$

By Definition 3.2,

$$\begin{aligned} h(A, C) = d(a', C) = \min_{c \in C}\{d_E(a', c)\} &\leq d_E(a', c'), \\ d(a', B) \leq \max_{x \in A}\{d(x, B)\} &= d(A, B) \leq h(A, B), \\ d(b', C) \leq \max_{y \in B}\{d(y, C)\} &= d(B, C) \leq h(B, C). \end{aligned}$$

Therefore, we can substitute these new values into 3.2 to obtain,

$$h(A, C) \leq h(A, B) + h(B, C).$$

$\square$

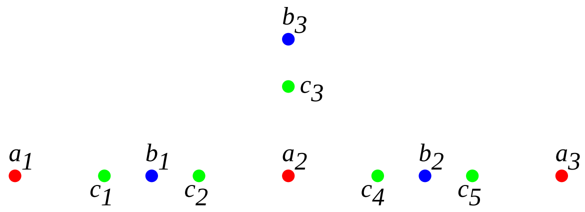


Figure 2: Four elements at each location between A and B.

4 Finding Elements Between Sets

In Euclidean geometry, we know that given points $a, b \in \mathbb{R}^n$, there exists a unique point $c \in \mathbb{R}^n$ such that $d_E(a, b) = d_E(a, c) + d_E(c, b)$ and we say c lies a distance $t = d_E(a, c)$ from a and $s = d_E(c, b)$ from b. In the Haudsorff geometry, however, given sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ it is possible to have more than one set $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, B) = h(A, C) + h(C, B)$. In fact, as we will see in the examples that follow, it is possible to have a finite number of elements at each location between two sets, and even an infinte number of elements.

The next part of our discussion will address the question of when we can find a finite number of elements $C \in \mathcal{H}(\mathbb{R}^n)$ at every location between two sets $A, B \in \mathcal{H}(\mathbb{R}^n)$. Just as in the Euclidean geometry, when we say an element $C \in \mathcal{H}(\mathbb{R}^n)$ lies between sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ we mean that the relationship, $h(A, B) = h(A, C) + h(C, B)$ holds. We will use the notation ACB to indicate that C lies between A and B. In the next definition we formalize what we mean for more than one element to be in the same location between two sets.

Definition 4.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. The elements $C \neq C' \in \mathcal{H}(\mathbb{R}^n)$ are said to be at the same location between A and B if C and C' satisfy ACB and $AC'B$ with $h(B, C) = h(B, C') = s$ for some $0 < s < h(A, B)$.*

The example in Figure 2, consisting of the sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, illustrates how it is possible to have more than one element at a given location between two sets. In this case, the sets $\{c_1, c_2, c_3, c_4, c_5\}$, $\{c_1, c_3, c_4, c_5\}$, $\{c_1, c_2, c_3, c_5\}$, and $\{c_1, c_3, c_5\}$ are all at the same location between A and B.

If for sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ it is true that

$$h(A, B) = d(b, A) = d(a, B) = r \text{ for all } a \in A \text{ and } b \in B,$$

we say that the sets A and B satisfy the PFAEL (Possibly Finite At Each Location) conditions. A set $X = A \cup B$ where A and B stasify the PFAEL conditions is called a *configuration*. If both $A, B \in X$ are finite sets, then X is a *finite configuration*. If either $A \in X$ or $B \in X$ is an infinite set, then X is an *infinite configuration*. As the following examples illustrate, the PFAEL conditions are necessary but not sufficient to guarantee that there are a finite number of elements at each location between sets A and B. We call a set $Y \subset X$ a subconfiguration of X if Y is a configuration.

The example in Figure 2 is a finite configuration because the sets A and B are finite sets. In Figure 3, the sets A, the union of the point at the center of the figure and the outer bold circle, and B, the inner bold circle, are infinite and therefore form an infinite configuration. Because

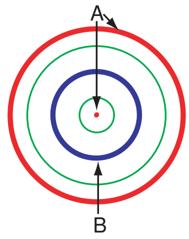


Figure 3: An infinite configuration with an infinite number of elements at each location between A and B.

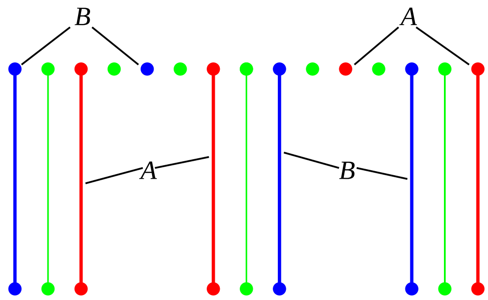


Figure 4: An infinite configuration with nine elements at each location btween A and B.

they are configurations, both the examples in Figures 2 and 3 satisfy the PFAEL conditions, however, as observed above, Figure 2 has a finite number of elements (four) at each location between A and B; there are an infinite number of elements at each location between sets A and B in Figure 3 since the largest element C, between A and B, consists of the union of the two thin circles *and* there are an infinite number of non-empty compact subsets C', consisting of any nonempty compact subset of inner thin circle united with the outer thin circle, which satisfy $AC'B$.

It is possible for an infinite configuration to have a finite number of elements at each location between sets. This is illustrated in Figure 4 where A consists of the union indicated line segments and single point and B is the union of the indicated line segments and single point. There are nine elements at each location between A and B that are subsets of the largest set C which is the union of the light colord line semgents and points.

We now give the following definition of the extension of a set because the idea is useful in identifying where a set $C \in \mathcal{H}(\mathbb{R}^n)$ must be located in order to satisfy ACB where $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions.

Definition 4.2. *The extension of the set $A \in \mathcal{H}(\mathbb{R}^n)$ by the real number $t > 0$ is the set*

$$A + t = \{x \in R^n : d_E(x, a) \leq t \text{ for some } a \in A\}.$$

We will now set the convention that, if $0 < s < r$ and $t = r - s$, then $C = (A+t) \cap (B+s)$. Then we know $h(A, C) = t$ and $h(B, C) = s$, in addion to the result that $C = (A+t) \cap (B+s)$ is the largest possible set that satsifies ACB and therefore if a set C' satisfies $AC'B$ then it must be true that $C' \subset C$ as shown in [2].

Since we know where to locate sets C that satisfy ACB at a given location, what can we say about the number of sets $C' \subset C$ such that $AC'B$ is satisfied at a given location? The next definitions give us the tools to answer this question more precisely.

Definition 4.3. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and let $x \in A$, $y \in B$. The point x is said to be adjacent to y, denoted $x \doteqdot y$, if $d_E(x, y) = h(A, B)$.*

Definition 4.4. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and let $a \in A$. The adjacency set, $[a]_C$, is defined as the set $\{c \in C : c \doteqdot a\}$. Let $[A]_C$ denote the set $\{[a]_C : a \in A\}$.*

Definition 4.5. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions.*

Let $q_A : C \longrightarrow [A]_C$ be defined by $q_A(c) = [a]_C$ where $c \in [a]_C$.

Let $q_B : C \longrightarrow [B]_C$ be defined by $q_B(c) = [b]_C$ where $c \in [b]_C$.

Example 4.1. *If we return to our configuration in Figure 2 we see that*

$$[a_1]_C = \{c_1\},$$
$$[a_2]_C = \{c_2, c_3, c_4\},$$
$$[b_2]_C = \{c_4, c_5\}.$$

In addition, we have

$$q_A(c_1) = [a_1]_C,$$
$$q_A(c_2) = q_A(c_3) = q_A(c_4) = [a_2]_C,$$
$$q_B(c_4) = [b_2]_C.$$

For a proof of Lemmas 4.1 and 4.2 (below) and that both q_A and q_B are well-defined and surjective functions see [7]. Also notice that from Example 4.1 it is not necessarily true that q_A and q_B are bijective.

Lemma 4.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and C' satisfy $AC'B$ with $C' \subseteq (A + t) \cap (B + s)$. If $a_0 \in A$, then $[a_0]_{C'} \neq \emptyset$.*

Lemma 4.2. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions. If $C' \subset C$, then $h(A, C) \leq h(A, C')$.*

Lemma 4.3. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and let $U \in \mathcal{U}_C$. If for all $[a]_C \in q_A(U)$ there exists $c \in [a]_C$ such that $c \notin U$ and for all $[b]_C \in q_B(U)$ there exists $c \in [b]_C$ such that $c \notin U$, then the set $C' = C - U$ satisfies $AC'B$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and $U \in \mathcal{U}_C$.

Suppose $U = \emptyset$. Then $C' = C - U = C$ which satisfies ACB.

Suppose now, that $U \neq \emptyset$ and for all $[a]_C \in q_A(U)$ there exists $c \in [a]_C$ such that $c \notin U$ and for all $[b]_C \in q_B(U)$ there exists $c \in [b]_C$ such that $c \notin U$.

First, notice that the set $C' = C - U = C \cap U^c$ is closed since it is the intersection of two closed sets. Further, C' is bounded because it is a subset of a bounded set, thus $C' \in \mathcal{H}(\mathbb{R}^n)$.

For each $a \in A$, if $[a]_C \notin q_A(U)$, then there exists $c \in [a]_C$ such that $c \notin U$ by Lemma 4.1. If $[a]_C \in q_A(U)$, then there exists $c \in [a]_C$ with $c \notin U$by our initial assumption. So, for all $a \in A$ there exists $c \in [a]_C$ such that $c \notin U$; in other words, $c \in C' = C - U$. Therefore, for all $a \in A$,

$a \curlyeqsucc c$ for some $c \in C'$ so $d(A, C') \leq t$. We already know $d(C', A) = t$ because $C' \subset C$, and for all $c \in C$, $d_E(c, a) \leq t$ for some $a \in A$. Further, by Lemma 4.2, $t = h(A, C) \leq h(A, C')$ and since $h(A, C') = \max\{d(A, C'), d(C', A)\}$ it must be true that $h(A, C') = t$.

A similar argument shows that $h(B, C') = s$ therefore C' satisfies $AC'B$. $\square$

If $A, B, C \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions, we will henceforth use Υ to denote the collection of sets $U \subset C$ such that $U \in \mathcal{U}_C$ with the property that

1. for all $[a]_C \in q_A(U)$ there exists $c \in [a]_C$ such that $c \notin U$ and
2. for all $[b]_C \in q_B(U)$ there exists $c \in [b]_C$ such that $c \notin U$.

We will also let $\mathcal{K} = \{C' : C' \text{ satisfies } AC'B\}$, the set of all possible elements at each location between A and B. We say that $\#(X) = |\mathcal{K}|$ in order to indicate the number of elements at each location between two sets in a configuration X. The next theorem shows that $|\Upsilon|$ is also equal to $\#(X)$ from the relationship between the set Υ and the set $\mathcal{K}$.

Theorem 4.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions. The mapping $f : \Upsilon \longrightarrow \mathcal{K}$ defined by $f(U) = C - U$ is a bijective function.*

Proof. Suppose $A, B \in \mathcal{H}(\mathbb{R}^n)$. First note that if $U \in \Upsilon$ then $f(U) = C - U \in \mathcal{K}$ by Lemma 4.3. It is also easy to see that f is well defined.

Let $C' \in \mathcal{K}$. Then, $C' \subseteq C$ and satisfies $AC'B$.

If $C' = C$ then let $U = \emptyset$ and $f(U) = C - U = C = C'$. If $C' \neq C$ then let $U = C - C'$. Since C' is closed, C'^c is open, which means $U = C - C' = C \cap C'^c \in \mathcal{U}_C$. Also, because C' satisfies $AC'B$ and $C \subseteq C = (A)_t \cap (B)_s$, by Lemma 4.1, $[a]_{C'} \neq \emptyset$ for all $a \in A$. Therefore, for all $[a]_C \in q_A(U)$ there exists $c \in [a]_C$ such that $c \in C'$ and hence $c \notin U$. Note that a similar argument holds for for all $[b]_C \in q_B(U)$. Therefore, f is surjective. Suppose $f(U) = f(V)$ for some $U, V \in \Upsilon$. Then we have $C - U = C - V$ thus $U = V$ so f is injective. Therefore, f is a bijective function. $\square$

Thus we have shown, for a configuration $X = A \cup B$ where $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions, that $\#(X) = |\mathcal{K}| = |\Upsilon|$.

Corollary 1. *If $|\Upsilon| = 1$ then $\Upsilon = \{\emptyset\}$*

Proof. Suppose $|\Upsilon| = 1$ and that $U \in \Upsilon$ where $U \neq \emptyset$. Then $f(U) = C - U = C'$ where $C' \neq C$. However, both $C, C' \in \mathcal{K}$ and so $|\mathcal{K}| = 2$. However, f is a bijection so $|\Upsilon| = 2$ which violates our assumption therefore it must be true that $U = \emptyset$. $\square$

5 Infinite Configurations

When we have an infinite configuration X satisfying the PFAEL conditions, it is possible that there are $k \in \mathbb{N}$ elements $C \in \mathcal{H}(\mathbb{R}^n)$ at each location between sets $A, B \in X$. The question arises then, if there are a fininte number of elements at each location between A and B in an infinite configuration, then is it possible to find a finite configuration with the same number of elements at each location? If this is possible, then if $k \in \mathbb{N}$ elements cannot be found at each location between two finite sets A and B, like 19 [4], then it is impossible for an infinite configuration to have k elements at each location between two sets as well. Therefore, we are interested in finding a way to write any infinite configuration as a finite configuration.

Finite Configuration Construction Algorithm 1. *(Finite Configuration Construction Algorithm) Let $X = A \cup B$ be an infinite configuration where $2 \leq |\Upsilon| \leq k$ such that $k \in \mathbb{N}$. Also, let*

- $Q_A = \{[a]_C : [a]_C \in q_A(U)$ *for some* $U \in \Upsilon\}$ *with* $|Q_A| = l$
- $Q_B = \{[b]_C : [b]_C \in q_B(U)$ *for some* $U \in \Upsilon\}$ *with* $|Q_B| = m$.

A finite configuration $X_F = Y \cup Z$ corresponding to X can be constructed in the following manner:

1. *For each $[a_i]_C \in Q_A$ (where $1 \leq i \leq l$), place a point $y_i \in Y$. Similarly, for each $[b_j] \in Q_B$ (where $1 \leq j \leq m$), place a point $z_j \in Z$.*
2. *For each $[a_i]_C \in Q_A$ (where $1 \leq i \leq l$), if there exists at least one point $c \in [a_i]_C$ such that $c \notin U$ for any $U \in \Upsilon$, place an endpoint $z_{y_i} \in Z$ and make $y_i \in Y$ adjacent to z_{y_i}.*
3. *For each $[b_j]_C \in Q_B$ (where $1 \leq j \leq m$), if there exists at least one point $c \in [b_j]_C$ such that $c \notin U$ for any $U \in \Upsilon$, place an endpoint $y_{z_j} \in Y$ and make $z_j \in Z$ adjacent to y_{z_j}.*
4. *For each $c \in [a_i]_C$, if c is in U for some $U \in \Upsilon$, make y_i adjacent to the $z_j \in Z$ which corresponds to the $b_j \in B$ such that $c \in [b_j]_C$.*

We need to know that it is always possible to build the kind of configuration described in the *Configuration Construction Algorithm* and this motivates the next theorem, a proof of which can be found in [8].

Theorem 5.1. *Given $\alpha, \beta \in \mathbb{N}$, we can create a configuration X consisting of sets A and B in $\mathbb{R}^{\beta+2}$ with*

- $|A| = \alpha$
- $|B| = \beta$
- *Every element in A is adjacent to a specified non-empty subset of elements in B (with every element in B adjacent to at least one element in A)*
- *A and B satisfy the PFAEL conditions*

Now that we have a way to construct a finite configuration corresponding to any infinite configuration with a finite number of elements at each location between A and B, we are ready to say that the number of elements at each location in the infinite configuration is conserved by the algorithm. More formally, we want to say that

Theorem 5.2. *If $X = A \cup B$ is an infinite configuration where $2 \leq |\Upsilon| \leq k$ such that $k \in \mathbb{N}$, then if X_F is the corresponding finite configuration constructed from the Configuration Construction Algorithm then $|\Upsilon| = |\Upsilon_F|$.*

The proof of this can also be found in [8].

In [4], the authors obtain the surprising result that

Theorem 5.3. *There is no finite configuration X such that $\#(X) = 19$.*

This fact, combined with the fact that every infinite configuration, X, can be written as a finite configuration, X_F, and $\#(X) = \#(X_F)$, it follows directly that

Theorem 5.4. *There is no configuration X, infinite or finite, such that $\#(X) = 19$.*

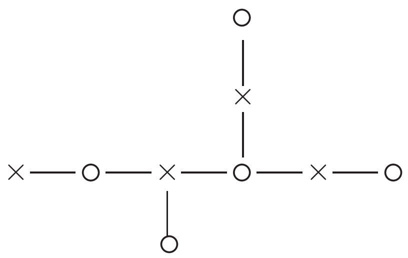
6 Counting Techniques for Some Specific Configurations

In this section we will look at some special configurations and an operation introduced in [4]. The specific configurations we will be considering are m-strings, denoted S_m where m indicates the length of the string. For example, we have S_6

$$\times — \circ — \times — \circ — \times — \circ$$

where the collection of $\circ$'s is the set A and the collection of $\times$'s is the set B. The lines indicate where there is a possible element $c \in C$ at each location between A and B. We give a point a location based upon it's position in the string. The first point on the left is the 1$^{\text{st}}$point, the second at the 2$^{\text{nd}}$, and so on.

The adjoinment operation we will be using is denoted by $\oplus$ and for our purposes here will be used only to adjoin strings to strings. If we consider $S_6 \oplus \{S_1[3]; S_2[4]\}$



we can see that the number in the square braces, like the 3 in $S_1[3]$, indicates the position on the "base string", in this case S_6, to which the new string should be attached.

We begin by making some observations about endpoints and then we will introduce a counting technique for specific kinds of adjoinments of strings.

6.1 Endpoints

In this section we will discuss some interesting properties of endpoints. These properties will be useful in the following section. We begin, naturally, with the definition of an endpoint.

Definition 6.1. *If a configuration X defines sets A and B, then we say that the point $a \in A$ is an end point if there is a unique $b' \in B$ such that $a \eqsim b'$.*

Theorem 6.1. *Let X be a configuration defining sets A and B where C satisfies ACB. If there exists $c \in [a]_C \cap [b]_C$ for some $a \in A$ and some $b \in B$, then c is unique.*

Proof. Suppose there exists $c \in [a]_C \cap [b]_C$ and that c is not unique. Then, there exists $c' \in [a]_C \cap [b]_C$ such that $c \neq c'$. Since $c' \in [a]_C$ we know that $c' \Bumpeq a$ thus $d_E(a, c') = t$ and similarly, $c' \in [b]_C$ so $c' \Bumpeq b$ and $d_E(b, c') = s$. Further, since $c \in [a]_C \cap [b]_C$, the same reasoning gives that $d_E(a, c) = t$ and $d_E(b, c) = s$ where $s + t = d_E(a, b)$. Therefore we conclude that c and c' are at the same location on a Euclidean line so $c = c'$ which is a contradiction, therefore c must be unique. $\square$

The following are two consequences of the previous theorem.

Corollary 2. *Let X be a configuration defining sets A and B where C satisfies ACB. If $a \in A$ is an endpoint, then $|[a]_C| = 1$.*

Proof. Suppose $a \in A$ is an endpoint. We know $|[a]_C| \neq 0$ because $a \Bumpeq b'$ for exactly one $b' \in B$ since a is an endpoint. Therefore, there exists $c \in [a]_C \cap [b']_C$ and c is unique by Theorem 6.1 so it must be true that $|[a]_C| = 1$. $\square$

Corollary 3. *Let X be a configuration defining sets A and B where C satisfies ACB and $c \in U$ for any $U \in \Upsilon$. If $c \in [a]_C$ for some $a \in A$, then a is not an endpoint.*

Proof. Suppose $c \in [a]_C$ and a is an endpoint, then there is a unique $b' \in B$ such that $a \Bumpeq b'$. This means that $c \in [a]_C \cap [b']_C$ and since c is unique by Theorem 6.1, there is no other $c' \in [a]_C$ such that $c' \notin U$, but this means $c \notin U$ which is a contradiction. $\square$

Corollary 4. *Let X be a configuration defining sets A and B where C satisfies ACB. If $a \in A$ is an endpoint and $c \in [a]_C$, then $c \notin U$ for any $U \in \Upsilon$.*

Proof. Suppose $a \in A$ is an endpoint and $c \in [a]_C$. If $c \in U$ then there must exist $c' \in [a]_C$ such that $c' \notin U$. However, by Corollary 2 we know that $|[a]_C| = 1$ so $c = c'$ and $c' \in U$ which is a contradiction. $\square$

We will see that Corollary 4 will be important for our counting technique in the following subsection. We conclude the current subsection with a proposition based on some observations made by the author.

Conjecture 1. *Let X be a finite configuration defining sets A and B where C satisfies ACB. If $[a]_C \in q_A(U)$ and $[b]_C \in q_B(V)$ for some $U, V \in \Upsilon$ and there exists $c \in [a]_C \cap [b]_C$, then $c \in W$ for some $W \in \Upsilon$.*

At this point, the author can only suggest that the proof of this conjecture must rely on the fact that the configuration is finite. For example, in Figure 5, the sets A and B are the unions of the indicated line segments and single points, and C, the remaining thin lines and singles points. Both $[a]_C \in q_A(U)$ and $[b]_C \in q_B(V)$ for some $U, V \in \Upsilon$; we can also see that $c_2 \in [a_1]_C \cap [b_2]_C$. However, $c_2 \notin W$ for any $W \in \Upsilon$ because $C - c_2$ is not a compact set.

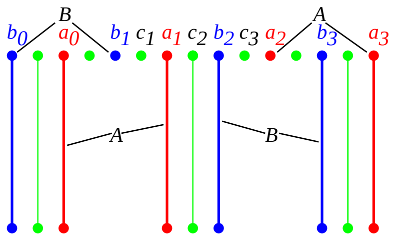


Figure 5: X must be a finite configuration.

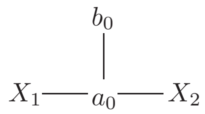


Figure 6: The configuration X.

6.2 Multiplicative Property of Configurations

In [4] it is observed that if two configurations X_1 and X_2 with $h(A_1, B_2) = h(A_2, B_2) = r$ where $\#(X_1) = k$ and $\#(X_2) = l$ for some $k, l \in \mathbb{N}$ are more than r units apart (that is, neither configuration shares any adjacencies), then for $X = X_1 \cup X_2$ we have $\#(X) = \#(X_1)\#(X_2)$. With this in mind, consider the next definition.

Definition 6.2. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions. If $X_1 = A_1 \cup B_1$ and $X_2 = A_2 \cup B_2$ such that for all $a_1 \in A_1$, $a_1 \not\Bumpeq b'_2$ for any $b'_2 \in B_2$ and for all $b_1 \in B_1$, $b_1 \not\Bumpeq a'_2$ for any $a'_2 \in A_2$ then we say that X_1 and X_2 are disjoint configurations.*

Now we will look more closely at the multiplicative property of disjoint configurations.

Theorem 6.2. *Let $X = X_1 \cup X_2 \cup \{a_0, b_0\}$ be a finite configuration, where the sets X_1 and X_2 are disjoint configurations, $a_0 \Bumpeq b_0$ where b_0 is an endpoint, $a_0 \Bumpeq b'_1$ for a unique $b'_1 \in X_1$ and $a_0 \Bumpeq b'_2$ for a unique $b'_2 \in X_1$, and $a_0 \not\Bumpeq y$ for any other $y \in (X - \{b'_1, b'_2\})$. Also let $X' = X'_1 \cup X'_2$ where $X'_1 = X_1 \cup \{a_1, b_1\}$ and $X'_2 = X_2 \cup \{a_2, b_2\}$ are disjoint configurations, $a_1 \Bumpeq b_1$ where b_1 is an endpoint, $a_2 \Bumpeq b_2$ where b_2 is an endpoint, $a_1 \Bumpeq b'_1$ for the unique $b'_1 \in X_1$ from above with $a_1 \not\Bumpeq y_1$ for any $y_1 \in (X_1 - \{b'_1\})$, and $a_2 \Bumpeq b'_2$ for the unique $b'_2 \in X_2$ from above with $a_2 \not\Bumpeq y_2$ for any $y_2 \in (X_2 - \{b'_2\})$. Then,*

$$\#(X) = \#(X') = \#(X'_1)\#(X'_2).$$

See Figures 6 and 7 for pictoral description of X and X', respectively.

Proof. Define configurations X and X' as above and let Υ and Υ' correspond to X and X' respectively. Also let Υ_1 and Υ_2 correspond to X_1 and X_2 repsectively.

Consider $W^\star = (U \cap \Upsilon_1) \cup (U \cap \Upsilon_2)$ for any $U \in \Upsilon$ and $W'^\star = (U' \cap \Upsilon_1) \cup (U' \cap \Upsilon_2)$ for any $U' \in \Upsilon'$. Because X and X' contain both of the disjoint configurations X_1 and X_2,

$$X_1 — a_1 — b_1 \qquad\qquad b_2 — a_2 — X_2$$

Figure 7: The configuration X'.

it must be true that $W^\star = W'^\star$. Therefore, we must look at sets $W \in \Upsilon$ and $W' \in \Upsilon'$ where

$$W = W^\star \cup = \begin{cases} \{c_1\} & \text{if } c_1 \in U, c_2 \notin U \\ \{c_2\} & \text{if } c_2 \in U, c_1 \notin U \\ \{c_1, c_2\} & \text{if } c_1 \in U, c_2 \in U \end{cases}$$

and

$$W' = W'^\star \cup = \begin{cases} \{d_1\} & \text{if } d_1 \in U', e_1 \notin U' \\ \{e_1\} & \text{if } e_1 \in U', d_1 \notin U' \\ \{d_1, e_1\} & \text{if } d_1, e_1 \in U' \\ \emptyset & \text{if } d_1, e_1 \notin U' \end{cases}$$

for

$$\begin{gathered} c_0 \Bumpeq a_0 \text{ and } c_0 \Bumpeq b_0 \\ c_1 \Bumpeq a_0 \text{ and } c_0' \Bumpeq b_1' \\ c_2 \Bumpeq a_0 \text{ and } c_0'' \Bumpeq b_2' \\ d_0 \Bumpeq a_1 \text{ and } d_0 \Bumpeq b_1 \\ d_1 \Bumpeq a_1 \text{ and } d_1 \Bumpeq b_1' \\ e_0 \Bumpeq a_2 \text{ and } e_0 \Bumpeq b_2 \\ e_1 \Bumpeq a_2 \text{ and } e_1 \Bumpeq b_2' \end{gathered}$$

where $c_0, c_1, c_2 \in C$ and $d_0, d_1, e_0, e_1 \in C'$. If we let $\mathcal{W} = \{W\}$ and $\mathcal{W}' = \{W'\}$ then $\mathcal{W} = \Upsilon$ and $\mathcal{W}' = \Upsilon'$.

Therefore, since we know that showing $\#(X) = \#(X')$ is the same as showing $|\Upsilon| = |\Upsilon'|$, we can also show that $|\mathcal{W}| = |\mathcal{W}'|$. However, since both of these sets contain $W^\star = W'^\star$, it is sufficient to show that

1. $c_0 \notin V$ for any $V \in \mathcal{W}$ iff $d_0, e_0 \notin V'$ for any $V' \in \mathcal{W}'$.
2. $c_1 \in V$ for any $V \in \mathcal{W}$ iff $d_1 \in V'$ for any $V' \in \mathcal{W}'$
3. $c_2 \in V$ for any $V \in \mathcal{W}$ iff $e_1 \in V'$ for any $V' \in \mathcal{W}'$

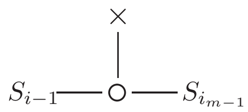


Figure 8: The configuration $S_m \oplus S_1[i_1]$

For the first condition we notice that c_0, d_0, e_0 are all endpoints and thus by Corollary 4, $c_0 \notin V$ for any $V \in \Upsilon$, and $d_0, e_0 \notin V'$ for any $V' \in \Upsilon'$. Therefore, the first condition is satisfied.

For the second condition, note first that $c_0 \in [a_0]_C$ but $c_0 \notin V$ for any $V \in \mathcal{W}$. This means that if $c_1 \in V$ then there must exist a $c_2 \in [b_1']_C$ such that $c_2 \notin V$. However, $c_2 \in C \cap C_2$. Similarly, we note that $d_0 \in [a_1]_C'$ but $d_0 \notin V'$ for any $V' \in \mathcal{W}'$, thus if $d_1 \in V'$ there must exist a $c_2' \in [b_1']_{C'}$ such that $c_2' \notin V'$. Here, $c_2' \in C' \cap C_2$. We know that $C_2 \subset C$ and $C_2 \subset C'$ thus $C \cap C_2 = C' \cap C_2$ so $c_2 \in C$ if and only if $c_2' \in C'$.

The proof for the third condition is similar.

Therefore, $|\mathcal{W}| = |\mathcal{W}'|$ and we can conclude that $\#(X) = \#(X')$. Further, since X_1' and X_2' are disjoint configurations, we know that $\#(X') = \#(X_1')\#(X_2')$, thus

$$\#(X) = \#(X') = \#(X_1')\#(X_2').$$

$\square$

Using the multiplicative property of disjoint configurations we can always calculate the number of elements at each location between two sets making up specific types of configurations consisting of 1-strings adjoined to m-strings where $m \geq 2$ is arbitrary. This calculation is explained in Theorem 6.3.

Theorem 6.3. *Let S_m be a string, $m \geq 2$ be arbitrary, and let $1 \leq i_1 \leq i_2 \leq ... \leq i_{k-1} \leq i_k \leq m$. Further, let $X = S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{k-1}]; S_1[i_k]\}$ where each $x \in S_m \oplus S_1[i_\alpha] \cap S_m$ is not adjacent to any $y \in S_m \oplus S_1[i_\beta] \cap S_m$ for any $2 \leq \alpha \leq m$ and $2 \leq \beta \leq m$ such that $\alpha \neq \beta$. Then,*

$$\#(X) = \#(S_{i_1+1})\#(S_{i_2-i_1+3}) \cdots \#(S_{i_k-i_{k-1}+3})\#(S_{m-i_k+2}).$$

Proof. Suppose $k = 1$, then we have $S_m \oplus S_1[i_1]$ as illustrated in Figure 8.

By Theorem 6.2, we also know that $\#(S_m \oplus S_1[i_1]) = \#(S_{i_1-1+2})\#(S_{m-i_1+2}) = \#(S_{i_1+1})\#(S_{m-i_1+2})$ as illustrated in Figure 9.

Suppose true for $k = l$, that is, for $S = S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{k-1}]; S_1[i_k]\}$, it is true that

$$\#(S) = \#(S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]\}) = \#(S_{i+1})\#(S_{i_2-i_1+3})...\#(S_{i_l-i_{l-1}+3})\#(S_{m-i_l+2}).$$

If we suppose $k = l + 1$, then we must consider the configuration

$$S' = S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]; S_1[i_{l+1}]\}.$$

$$S_{i_1-1} — \circ — \times \qquad\qquad \times — \circ — S_{m-i+2}$$

Figure 9: The configuration $S_m \oplus S_1[i_1]$ after using the multiplicative property

$$S_1' — \circ_1 — \times_1 \qquad\qquad \times_2 — \circ_2 — S_2'$$

Figure 10: The configuration S'' with S_1' and S_2'

If $i_l = i_{l+1}$, then since we already have S_1 adjoined at the i_l^{th} location, adjoining another S_1 to the $i_l^{\text{th}} = i_{l+1}^{\text{th}}$ location does not increase the number of elements at each location from the $\#(S)$, that is, $\#(S) = \#(S')$. Therefore, we have

$$\begin{aligned}
\#(S) &= \#(S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]\}) \\
&= \#(S_{i_1+1})\#(S_{i_2-i_1+3})...\#(S_{i_l-i_{l-1}+3})\#(S_{m-i_l+2}) \\
&= \#(S_{i_1+1})\#(S_{i_2-i_1+3})...\#(S_{i_l-i_{l-1}+3})\#(S_3)\#(S_{m-i_l+2}) \\
&= \#(S_{i_1+1})\#(S_{i_2-i_1+3})...\#(S_{i_l-i_{l-1}+3})\#(S_{i_{l+1}-i_l+3})\#(S_{m-i_{l+1}+2}) \\
&= \#(S') = \#(S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]; S_1[i_{l+1}]\}).
\end{aligned}$$

If $i_l < i_{l+1}$, then consider the configurations

$$S_1' = S_{i_{l+1}-1} \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]\} \text{ and } S_2' = S_{m-i_{l+1}}.$$

We see that this is the same as the configuration S'', the union of the disjoint configurations $S_1' \cup \{\circ_1, \times_1\}$ and $S_2' \cup \{\circ_2, \times_2\}$ as illustrated in Figure 10.

By Theorem 6.2 we obtain our original configuration S' in Figure 11 and so we know that

$$\begin{aligned}
\#(S') &= \#(S_m \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]; s_1[i_{l+1}]\}) \\
&= \#(S_1' \cup \{\circ_1, \times_1\})\#(S_2' \cup \{\circ_2, \times_2\}) \\
&= \#(S_{i_{l+1}+1} \oplus \{S_1[i_1]; S_1[i_2]; ...; S_1[i_{l-1}]; S_1[i_l]\})\#(S_{m-i_{l+1}+2}) \\
&= \#(S_{i_1+1})\#(S_{i_2-i_1+3})...\#(S_{(i_{l-1}+1)-i_l+2})\#(S_{m-i_{l+1}+2}) \\
&= \#(S_{i_1+1})\#(S_{i_2-i_1+3})...\#(S_{i_{l-1}-i_l+3})\#(S_{m-i_{l+1}+2})
\end{aligned}$$

$$\begin{tikzcd}
 & \times \arrow[d, dash] & \\
S_1' \arrow[r, dash] & \circ \arrow[r, dash] & S_2'
\end{tikzcd}$$

Figure 11: The configuration S'

□

Using this technique we can actually find $\#(S)$ for any

$$S = S_m \oplus \{S_{\iota_1}[i_1]; S_{\iota_2}[i_2]; ...; S_{\iota_{\kappa-1}}[i_{k-1}]; S_{\iota_\kappa}[i_k]\}.$$

Specifically, if we take advantage of the counting technique from [4] we see that finding $\#(S)$ for such an S involves adding together the $\#$'s of other configurations as in the followng example.

Example 6.1. *According to the counting technique in [4], if we have a configuration* $S = S_6 \oplus \{S_2[3]; S_2[4]\}$*, then*

$$\begin{aligned} S_6 \oplus \{S_2[3]; S_2[4]\} &= S_6 \oplus \{S_1[3]; S_2[4]\} + S_6 \oplus \{S_2[4]\} \\ &= S_6 \oplus \{S_1[3]; S_1[4]\} + S_6 \oplus \{S_1[3]\} + S_6 \oplus \{S_1[4]\} + S_6. \end{aligned}$$

Therefore it is always possible to calculate $\#(S)$ *because it simplifies into strings, or adjoinments of strings that can be calculated as described in Theorem 6.3.*

Finally, from [6] we know that $\#(S_m) = F_{m-1}$ were F_{m-1} is the $m-1^{\text{st}}$ Fibonacci number. Therefore we conclude this paper with a connection between any general string adjoinment,

$$S = S_m \oplus \{S_{\iota_1}[i_1]; S_{\iota_2}[i_2]; ...; S_{\iota_{\kappa-1}}[i_{k-1}]; S_{\iota_\kappa}[i_k]\}$$

and the Fibonacci numbers.

Theorem 6.4. *If* $S = S_m \oplus \{S_{\iota_1}[i_1]; S_{\iota_2}[i_2]; ...; S_{\iota_{\kappa-1}}[i_{k-1}]; S_{\iota_\kappa}[i_k]\}$*, then* $\#(S)$ *is going to be a sum of products of Fibonacci numbers.*

Proof. Using the counting technique from [4], as illustrated in Example 6.1, we see that any configuration

$$S = S_m \oplus \{S_{\iota_1}[i_1]; S_{\iota_2}[i_2]; ...; S_{\iota_{\kappa-1}}[i_{k-1}]; S_{\iota_\kappa}[i_k]\}$$

can be written as a sum of $\#(S')$'s where S' is of the form

$$S' = S_\mu \oplus \{S_1[l_1]; S_1[l_2]; ...; S_1[l_{\lambda-1}]; S_1[l_\lambda]\}.$$

By Theorem 6.3 we know that for all such strings S', $\#(S')$ is equal to a product of $\#(S_j)$'s where $j \in \{l_1 + 1, l_2 + 3, \dots, l_\lambda - l_{\lambda-1} + 3, \mu - l_\lambda + 2\}$. Since we know from [6] that $\#(S_j) = F_{j-1}$ where F_{j-1} is the $j-1^{\text{st}}$ Fibonacci number, we can conclude that for every configuration S, $\#(S)$ is the sum of products of Fibonacci numbers. □

7 Conclusion

We have noted in this paper that there is no configuration X such that $\#(X) = 19$. We have also seen that for any general string adjoinment S, $\#(S)$ is a sum of products of Fibonacci numbers. This leads to two very important questions. First, is it possible to find a relationship for the $\#(L)$ for any general adjoinment L of strings to loops (see [4] for more about loops) that is similar to the $\#(S)$? Secondly, if there is a general way to find the number of any finite configuration constructed from loops and strings (as introduced in Section 6 for strings, and not yet determined for loops), is it possible to use this information to prove that there is no finite configuration X such that $\#(X) = 19$? This final question might be answered with regard to a general string adjoinment by showing that not all sums of products of Fibonacci numbers can result from a string adjoinment, that is, by showing that the converse of Theorem 6.4 is false. If answering these two questions is tractable, then proving that converse of Theorem 6.4 is false for specific intigers might be an appropraite strategy to address whether or not there exist any other intergers k for which no configuration X exists with $\#(X) = k$.

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