The Geometry of $\mathcal{H}(\mathbb{R}^n)$*

Christopher Bay and Steven Schlicker

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Abstract

If X is a complete metric space, then the collection of all non-empty compact subsets of X is denoted $\mathcal{H}(X)$. The Hausdorff metric h provides a way to measure distances between the elements of $\mathcal{H}(X)$ and generates the complete metric space $(\mathcal{H}(X), h)$. Circles and lines are defined in $\mathcal{H}(\mathbb{R}^n)$ and their geometric structure is explored. We find that under certain conditions, lines in $\mathcal{H}(\mathbb{R}^n)$ are actually rays. This surprising situation leads to further results concerning where points exist on Hausdorff lines.

1 Introduction

Euclidean geometry is a familiar topic. All mathematicians know what circles, lines, spheres, etc. look like in Euclidean space. It is well known how these geometric objects behave and interact. The goal of this paper is to explore simple geometric structures in a space known as $\mathcal{H}(\mathbb{R}^n)$. The Euclidean metric d can be used to define the Hausdorff metric on the space $\mathcal{H}(\mathbb{R}^n)$. The resulting geometry turns out to be interesting in many respects. Primarily, this is so because we find situations that occur in the geometry of $\mathcal{H}(\mathbb{R}^n)$ which are contrary to what happens Euclidean space.

Before constructing a geometry on $\mathcal{H}(\mathbb{R}^n)$, several basic topological concepts are introduced. In Section 3, we define circles and lines and begin our exploration of the geometry of $\mathcal{H}(\mathbb{R}^n)$ with an example. This seemingly simple example produces an unexpected situation that motivates the rest of our work. In Section 5, results are given that describe the existence of points on Hausdorff lines. We focus on situations in which Hausdorff lines are actually rays with reference to examples that illustrate the results of our theorems.

As stated above, we start by providing a solid background for the topic we are studying.

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2 Background

2.1 Metric Topology

The concepts and arguments presented here rely on a basic understanding of the concepts of topology and metric spaces. The most important definitions are those of topology, compactness, metric, and boundary; the rest are here for completeness. All come from [4].

Definition 1. *A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

1. *$\emptyset$ and X are in $\mathcal{T}$.*
2. *The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*
3. *The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.*

The simplest topology on a set X, known as the *indiscrete topology*, is $\mathcal{T} = \{\emptyset, X\}$. The reader can easily verify that $\mathcal{T}$ satisfies all three necessary conditions. Another example is the *discrete topology*, which consists of the collection of all subsets of a set X.

A ***topological space*** is the ordered pair $(X, \mathcal{T})$ consisting of the set X and a specific topology $\mathcal{T}$ on X. The elements of $\mathcal{T}$ are called ***open sets***. The next definition will give us an easier way to describe the open sets of a topology.

Definition 2. *If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements**) such that*

1. *For each $x \in X$, there is at least one basis element B containing x.*
2. *If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.*

A set $U \subset X$ is open in $\mathcal{T}$ if for each $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$. The collection of all such open sets makes up the ***topology $\mathcal{T}$ generated by the basis $\mathcal{B}$***. It is usually more convenient to speak of a basis that generates a topology than to consider the topology itself. For example, a basis for the discrete topology on X is the collection of all singleton sets. That is, $\mathcal{B} = \{\{x\} \mid x \in X\}$. From this basis we can construct any subset of X. Clearly, for every $x \in X$ there is at least one $B \in \mathcal{B}$ containing x. The second requirement for a basis is also satisfied, for if $x \in B_1 \cap B_2$, then $x \in \{x\} \subseteq B_1 \cap B_2$.

One particular topology that will be mentioned is the ***order topology***, which requires the definition of an ***ordered set***.

Definition 3. *A relation C on a set A is called an **order relation** (or a **simple order**, or a **linear order**) if it has the following properties:*

1. (Comparability) *For every x and y in A for which $x \neq y$, either xCy or yCx.*

2. (Non-reflexivity) *For no x in A does the relation xCx hold.*

3. (Transitivity) *If xCy and yCz, then xCz.*

If a set X has an order relation, it is called an ***ordered set.***

An example of an ordered set is the set of all positive integers $\mathbb{Z}_+$ with the standard $<$ relation. The reader may check that $(\mathbb{Z}_+, <)$ satisfies all three of the conditions listed above. Another ordered set is the set of real numbers $(\mathbb{R}, <)$.

Definition 4. *Let X be a set with a simple order relation $<$; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:*

1. *All open intervals $(a, b) = \{x \mid a < x < b\}$ in X.*
2. *If there exists a smallest element a_0 of X, all intervals of the form $[a_0, b) = \{x \mid a_0 \leq x < b\}$.*
3. *If there exists a largest element b_0 of X, all intervals of the form $(a, b_0] = \{x \mid a < x \leq b_0\}$.*

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the ***order topology.***

We will not verify that $\mathcal{B}$ is a basis. For a proof, see [4]. From this definition, a topology can easily be generated for ordered sets such as $\mathbb{R}$. The order topology on $\mathbb{R}$ turns out to be equivalent to the metric topology on $\mathbb{R}$. The definition of the metric topology and a metric space comes after that of a metric itself.

Definition 5. *A* ***metric*** *on a set X is a function*

$$d : X \times X \longrightarrow \mathbb{R}$$

having the following properties:

1. *$d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.*
2. *$d(x, y) = d(y, x)$ for all $x, y \in X$.*
3. (Triangle inequality) *$d(x, y) + d(y, z) \geq d(x, z)$, for all $x, y, z \in X$.*

Most readers are familiar with the ***Euclidean metric***, which we will use to define the Hausdorff metric.

Definition 6. *Given $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ in $\mathbb{R}^n$ the Euclidean metric d on $\mathbb{R}^n$ is given by*

$$d(\mathbf{x}, \mathbf{y}) = [(x_1 - y_1)^2 + \cdot\cdot\cdot\cdot + (x_n - y_n)^2]^{\frac{1}{2}}.$$

Given a set X and a metric d on X, we can construct sets of the form

$$N_r(x) = \{y \mid d(x, y) < r\}.$$

This is the set of all points y whose distance to x is less than r. The set $N_r(x)$ is referred to as the ***open neighborhood*** or ***Euclidean ball*** of radius r centered at x. Note that in general, a ***neighborhood*** of x is an open set containing x.

Definition 7. *If d is a metric on the set X, then the collection of all open neighborhoods $N_r(x)$, for $x \in X$ and $r > 0$, is a basis for the **metric topology** induced by d.*

It can be easily verified that the metric topology satisfies all of the conditions of definition 1. If a topological space X has a metric d that induces the topology of X, it is called a ***metric space*** and is denoted (X, d). An example of a metric space is $(\mathbb{R}, d)$, which is the real line with the topology generated by the Euclidean metric. Its basic open sets are of the form (a, b), for $a, b \in \mathbb{R}$. This topology of this space is equivalent to the order topology on $\mathbb{R}$.

Note that it is possible that two metrics may generate the same topology on a particular space. The ***standard bounded metric*** $\overline{d}$ on $\mathbb{R}$ is given by

$$\overline{d}(x, y) = \min\{d(x, y), 1\}.$$

The topology induced by $\overline{d}$ is the same as that induced by the Euclidean metric d. For a detailed proof of this claim, see [4]. The next definition is the most subtle, yet perhaps the most important.

Definition 8. *Given a set X with topology $\mathcal{T}$, an **open covering** of X is a collection $\mathcal{A}$ of open sets of X such that $\bigcup_{A \in \mathcal{A}} A = X$. A set $Y \subseteq X$ is said to be **compact** if every open covering $\mathcal{A}$ of Y contains a finite subcollection that also covers Y.*

The open interval $(0, 1)$ in $\mathbb{R}$ is not compact. The set $\{(\frac{1}{n}, 1 - \frac{1}{n}) \mid n \in \mathbb{Z}_+\}$ is an open cover of $(0, 1)$ that does not have a finite subcover. By including the end points of the interval, we obtain the closed interval $[0, 1]$, which is compact. In fact, every closed and bounded interval in $\mathbb{R}$ is compact [3]. Note that a set $B \subseteq X$ is ***closed*** if and only if $(X - B)$ is open.

Heine-Borel Theorem. *A subset of Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Informally, this means that a set $X \subseteq \mathbb{R}^n$ is compact if and only if $(\mathbb{R}^n - X)$ is open and X can be enclosed in a large enough n-sphere.

There are only three more concepts to introduce and they are all closely related. If X is a topological space, then the ***interior*** of a set $A \subseteq X$ is the union of all open sets contained in A. This set is denoted A° or int A and it is the largest open set contained in A. Note that if A is open, then $A = A^\circ$ since A is contained in itself.

The ***closure*** of a set $A \subseteq X$ is the intersection of all closed sets containing A. The closure of A is the smallest closed set containing A and is denoted $\overline{A}$. If A is closed, then $A = \overline{A}$. For any $A \subseteq X$, one has the relationship

$$A^\circ \subseteq A \subseteq \overline{A}. \tag{1}$$

If $A \subseteq X$, then the ***boundary*** of A is

$$\partial A = \overline{A} \cap \overline{(X - A)}.$$

A useful result states that if $A \subset X$ and $x \in X$, then $x \in \partial A$ if and only if every neighborhood of x intersects both A and $(X - A)$ [4]. The next lemma gives two more well known topological properties related to the boundary of a set.

Lemma 1. *If $A \subseteq X$, then $A^\circ \cap \partial A = \emptyset$ and $\overline{A} = A^\circ \cup \partial A$.*

Informally, the boundary of a set can be thought of a the collection of all of its edges. For instance, the boundary of $N_1((0,0))$ in $\mathbb{R}^2$ is the unit circle. To illustrate these concepts, consider the set $A = (0,1]$ in $\mathbb{R}$. The following are true.

- $\overline{A} = [0,1]$
- $A^\circ = (0,1)$
- $\partial A = \{0,1\}$

Notice that $(0,1) \subseteq (0,1] \subseteq [0,1]$, illustrating (1).

All of these concepts will be used extensively in the discussions and results that follow. We begin by defining the Hausdorff metric, which generates a topology on the collection of all non-empty compact subsets of $\mathbb{R}^n$.

2.2 The Hausdorff Metric

Denote the collection of all non-empty compact subsets of $\mathbb{R}^n$ by $\mathcal{H}(\mathbb{R}^n)$. This collection forms a metric space $(\mathcal{H}(\mathbb{R}^n), h)$ where h is the Hausdorff metric. We define h in terms of the Euclidean metric d [1].

Definition 9. *Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$.*

- *If $x \in \mathbb{R}^n$, the distance from x to B is*

$$d(x,B) = \min_{b \in B}\{d(x,b)\}.$$

- *The "distance" from A to B is*

$$d(A,B) = \max_{x \in A}\{d(x,B)\}.$$

  Note that this "distance" does not satisfy all of the necessary conditions to be classified as a metric, since it is possible to have $d(A,B) \neq d(B,A)$. Figure 1 is an example of this situation. The set A is the disk centered at $(-\frac{5}{2},0)$ of radius $(\frac{1}{2},0)$ and B is the disk centered at 1 of radius 1. So $d(A,B) = 3$ and $d(B,A) = 4$, as shown by the bold segments.

- *The* **Hausdorff distance**, *$h(A,B)$, between A and B is*

$$h(A,B) = max\{d(A,B), d(B,A)\}.$$

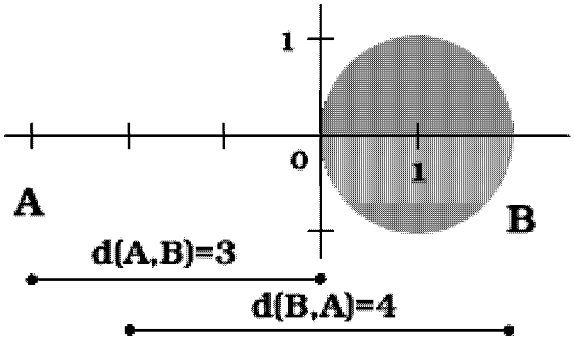

Figure 1: For the sets A and B, $d(A,B) \neq d(B,A)$.

When using the Hausdorff metric h on $\mathcal{H}(\mathbb{R}^n)$, we can be assured that for any non-empty compact sets A and B the Hausdorff distance $h(A,B)$ is defined. That is, for any $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $x \in A$, a minimum value is always achieved on the set $\{d(x,b) \mid b \in B\}$. This is equivalent to the following theorem from [4]:

Theorem 1. *Let X be a metric space with metric d; let $A \subseteq X$ be nonempty. If A is compact, $d(x,A) = d(x,a)$ for some $a \in A$.*

The proof of Theorem 1 depends on the Extreme Value Theorem of topology [4].

Extreme Value Theorem. *Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Proof of Theorem 1. Suppose $x \in X$ and A is compact. Define $f : A \to \mathbb{R}$ by $f(a) = d(a, \{x\})$.

It is straightforward to show that f is a continuous function on A. Let $a_1, a_2 \in A$ and $\epsilon > 0$. Then $f(a_1) = d(a_1, \{x\}) = d(a_1, x)$. Using the triangle inequality of d, we have $f(a_1) = d(a_1, x) \leq d(a_1, a_2) + d(a_2, x)$. Noting that $d(a_2, x) = f(a_2)$, we have

$$f(a_1) \leq d(a_1, a_2) + f(a_2). \tag{2}$$

Rearranging (2), we see that

$$f(a_1) - f(a_2) \leq d(a_1, a_2). \tag{3}$$

From 3 we can conclude that $|f(a_1) - f(a_2)| \leq d(a_1, a_2)$ since the relationship holds for all $a_1, a_2 \in A$. If $d(a_1, a_2) < \epsilon$ then $|f(a_1) - f(a_2)| < \epsilon$. Therefore, f is continuous on A.

Since A is compact, the Extreme Value Theorem guarantees that there is some $c \in A$ such that $f(c) \leq f(a)$ for every $a \in A$. Now, for any $y \in A$, $f(y) = d(y, \{x\}) = d(x, \{y\})$. Thus, $d(x, \{c\}) \leq d(x, \{a\})$ for any $a \in A$. Since $d(x, \{c\}) = d(x, c)$ and $d(x, \{a\}) = d(x, a)$ we can conclude that $d(x, c) \leq d(x, a)$ for every $a \in A$. Therefore $d(x, c) = \min\{d(x, a) \mid a \in A\} = d(x, A)$. ■

Thus, we achieve a minimum value in the set $\{d(x, b) \mid b \in B\}$ so that $d(x, B)$ is always defined. We now prove that h is a metric on $\mathcal{H}(\mathbb{R}^n)$.

Theorem 2. *The function h as defined above is a metric on $\mathcal{H}(\mathbb{R}^n)$.*

Proof. Let $A, B, C \in \mathcal{H}(\mathbb{R}^n)$.

1. $h(A, B) = \max\{d(A, B), d(B, A)\} = h(B, A)$.

2. We now show that $h(A, B) \geq 0$ and that $h(A, B) = 0$ if and only if $A = B$. For any $a \in A$ and $b \in B, d(a, b) \geq 0$. Thus, $d(a, B) = \min\{d(a, b) \mid b \in B\} \geq 0$ for any $a \in A$ which implies that $d(A, B) = \max\{d(a, B) \mid a \in A\} \geq 0$ also. Similarly, $d(B, A) \geq 0$. Therefore $h(A, B) = \max\{d(A, B), d(B, A)\} \geq 0$.

   Suppose $h(A, B) = 0$. Then $d(A, B) = d(B, A) = 0$ which implies that $d(a, B) = 0$ for every $a \in A$ and $d(b, A) = 0$ for every $b \in B$. Now, for every $a \in A$ there exists a $b \in B$ such that $d(a, b) = 0$. Thus, $a = b$ and $A \subseteq B$. Similarly, $B \subseteq A$. Now suppose $A = B$. Then $d(a, B) = 0$ for every $a \in A$ which implies $d(A, B) = 0$. Similarly, $d(B, A) = 0$ so that $h(A, B) = 0$.

3. (Triangle Inequality) We first show that $d(A, C) \leq d(A, B) + d(B, C)$ for arbitrary sets A, B, and C in $\mathcal{H}(\mathbb{R}^n)$. Choose $a \in A, b \in B$, and $c \in C$. Then the triangle inequality for d guarantees that

   $$d(a, c) \leq d(a, b) + d(b, c).$$

   Choose $c' \in C$ such that $d(b, c') = d(b, C)$. Then,

   $$d(a, c') \leq d(a, b) + d(b, C).$$

   Since $d(a, C) \leq d(a, c')$,

   $$d(a, C) \leq d(a, b) + d(b, C).$$

   Now, choose $b' \in B$ such that $d(a, b') = d(a, B)$. Then,

   $$d(a, C) \leq d(a, B) + d(b', C).$$

   Since $d(b', C) \leq d(B, C)$,

   $$d(a, C) \leq d(a, B) + d(B, C).$$

Choose $a' \in A$ such that $d(a', C) = d(A, C)$. Then,

$$d(A, C) \leq d(a', B) + d(B, C).$$

Since $d(a', B) \leq d(A, B)$, we have the final result that

$$d(A, C) \leq d(A, B) + d(B, C). \tag{4}$$

The Hausforff distance from A to C is $h(A, C) = \max\{d(A, C), d(C, A)\}$. Applying (4) we see that

$$\begin{aligned} h(A, C) &\leq \max\{d(A, B) + d(B, C), d(C, B) + d(B, A)\} \\ h(A, C) &\leq \max\{d(A, B), d(B, A)\} + \max\{d(B, C), d(C, B)\} \\ h(A, C) &\leq h(A, B) + h(B, C). \end{aligned}$$

■

3 Geometric Structures in $\mathcal{H}(\mathbb{R}^n)$

With a way to measure distance between the points of $\mathcal{H}(\mathbb{R}^n)$, we can define two basic geometric structures: circles and lines. We first give the definitions of lines and circles in $\mathcal{H}(\mathbb{R}^n)$, after which we will look at a specific example of a Hausdorff line. This example will motivate the rest of our discussion and will lead to the discovery of some surprising results.

Before beginning our discussion of geometric structures in $\mathcal{H}(\mathbb{R}^n)$, we give a definition.

Definition 10. *For $B \in \mathcal{H}(\mathbb{R}^n)$ and $r > 0$, define the set $N_r(B)$ as follows:*

$$N_r(B) = \bigcup_{b \in B} N_r(b).$$

Figure 3 illustrates this definition. The set B is the square shown in black and $N_r(B)$ is all of the shaded region. Note that $N_r(B)$ does *not* include the boundary of the shaded region because $N_r(B)$ is open.

3.1 Circles

If $B \in \mathcal{H}(\mathbb{R}^n)$, we denote by $C_r(B)$ the circle of radius $r \geq 0$ centered at the point B. Since $C_r(B)$ is simply the set of all points in $\mathcal{H}(\mathbb{R}^n)$ that are exactly a distance r from B:

$$C_r(B) = \{A \in \mathcal{H}(\mathbb{R}^n) \mid h(A, B) = r\}.$$

A general description for circles in $\mathcal{H}(\mathbb{R}^n)$ is given by a theorem from [2].

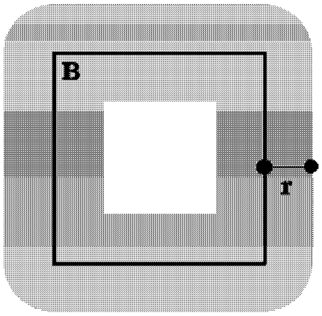

Figure 2: The shaded region is the set $N_r(B)$.

Theorem 3. (Circle Theorem) *Let B be a non-empty compact subset of $\mathbb{R}^n$. Then $A \in C_r(B)$ if and only if A is a non-empty compact subset of $\mathbb{R}^n$ and*

1. $A \subseteq \bigcup_{b \in B} \overline{N_r(b)}$.
2. $A \cap \overline{N_r(b)} \neq \emptyset$ *for each* $b \in B$.
3. *At least one of the following is satisfied:*
   - (a) $A \cap \partial N_r(B) \neq \emptyset$, *or*
   - (b) *there exist* $b \in B$ *and* $a \in A \cap \partial N_r(b)$ *with* $A \cap N_r(b) = \emptyset$.

We will now introduce a set that is important in the results of the sections to come. After defining the set we will state a theorem without proof. This theorem will give a better sense of the importance of the set. A formal proof can be found in [2].

Definition 11. *For $B \in \mathcal{H}(\mathbb{R}^n)$ and $r > 0$, define the set $B + r$ as follows:*

$$B + r = \{x \in \mathbb{R}^n \mid d(x, b) \leq r \text{ for some } b \in B\}.$$

In Figure 3, A is the circle of radius 2 centered at the origin and $A + \frac{1}{2}$ is all of the shaded region, including its boundary. If $x \in (A + \frac{1}{2})$, then there exists an element $a_x \in A$ so that $d(x, a_x) \leq \frac{1}{2}$. The following results will give the reader a better sense of the importance of sets of the form $B + r$. Theorem 4 and Lemma 2 come from [2] and are presented without proof.

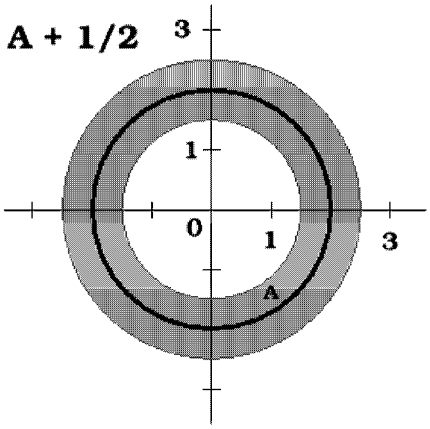

Figure 3: $A + \frac{1}{2}$

Theorem 4. *Let* $B \in \mathcal{H}(\mathbb{R}^n)$ *and* $r > 0$. *The set* $B + r$ *is the largest element in* $\mathcal{H}(\mathbb{R}^n)$ *that is a distance* r *from* B.

One might ask in what sense is $B + r$ the largest set that is a distance r from B. The answer may become more clear after we state a lemma.

Lemma 2. *Let* $B \in \mathcal{H}(\mathbb{R}^n)$ *and* $r > 0$. *Then* $B + r = \bigcup_{b \in B} \overline{N_r(b)}$.

Lemma 2, together with condition 1 of the Circle Theorem tell us that if $A, B \in \mathcal{H}(\mathbb{R}^n)$ and A is a distance r from B, then A must be a subset of $B + r$. It is in this sense that $B + r$ is the largest set a distance r from B.

Our next lemma is used implicitly throughout our discussion. It comes from [2] and is stated without proof.

Lemma 3. *If* $A \in \mathcal{H}(\mathbb{R}^n)$ *and* $r, s > 0$, *then* $(A + r) + s = A + (r + s)$.

3.2 Lines

Just as one can define circles with the Hausdorff metric, one can also define lines. The concept of a Hausdorff line is taken from that of Euclidean lines. If the Euclidean line $\overleftrightarrow{ab}$ is defined by two points $a, b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ so that $c \in \overleftrightarrow{ab}$, then one of the following is always true:

(i) $d(a, b) = d(a, c) + d(c, b)$

(ii) $d(a, c) = d(a, b) + d(b, c)$

(iii) $d(c,b) = d(c,a) + d(a,b)$

If (i) is true we say that c is "between" a and b. Similarly, (ii) implies that c lies to the "right" of b and (iii) implies that c lies to the "left" of a. This idea is used to define lines in $\mathcal{H}(\mathbb{R}^n)$ [2].

Definition 12. (Hausdorff Line) *Let* $A \neq B \in \mathcal{H}(\mathbb{R}^n)$. *The Hausdorff line* $\overleftrightarrow{AB}$ *defined by* A *and* B *is the set of all points* $C \in \mathcal{H}(\mathbb{R}^n)$ *that satisfy one of the triangle equalities:*

(I) $h(A,B) = h(A,C) + h(C,B)$,

(II) $h(A,C) = h(A,B) + h(B,C)$,

(III) $h(C,B) = h(C,A) + h(A,B)$.

We often use the same terminology described above to specify which triangle equality a point $C \in \overleftrightarrow{AB}$ satisfies. If C satisfies (I), then C lies "between" A and B, if C satisfies (II), then C lies to the "right" of B, and if C satisfies (III), then C lies to the "left" of A. We will now consider an example.

3.3 An Example of a Hausdorff Line

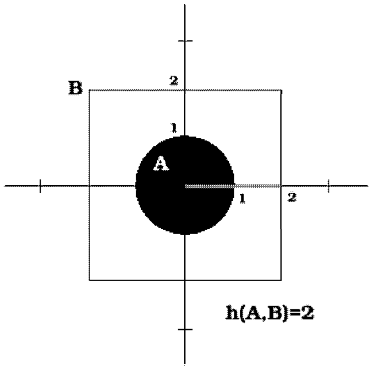

Figure 4: The set A is the unit disk and B is the square with vertices at (2,2), (-2,2), (-2,-2), and (2,-2). The Hausdorff distance from A to B is $h(A,B) = 2$ and can be measured along the shaded segment.

Euclidean lines are well understood, but Hausdorff lines present a new challenge since they can behave in very unintuitive ways. To illustrate this point we will consider an example. Let A be the unit disk and let B be the square with vertices at $(2,2), (-2,2), (-2,-2)$, and $(2,-2)$. These sets are shown in Figure 4. The Hausdorff distance from A to B is $h(A,B) = 2$ and is indicated by the shaded segment. Can we find a set C not equal to A or B for each of the

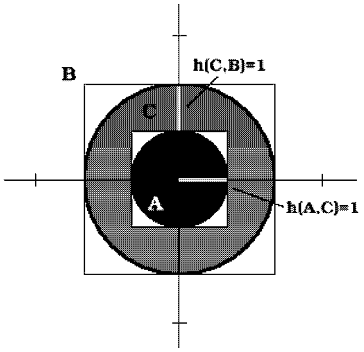

Figure 5: Since $h(A,C) + h(C,B) = h(A,B)$, C lies between A and B.

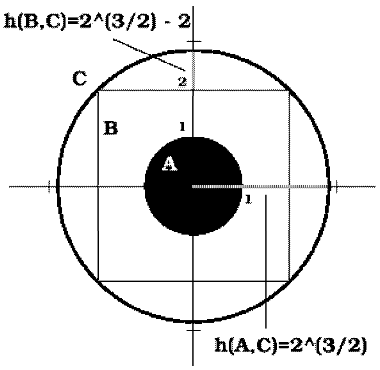

Figure 6: Since $h(A,B) + h(B,C) = h(A,C)$, C lies to the right of B.

cases in definition 12? As we will see, there are simple sets satisfying (I) and (II). Condition (III), however, presents a greater challenge.

Condition I. Let C be a disk centered at the origin with radius 2 that does not include the box with vertices at $(1,1), (1,-1), (-1,1)$, and $(-1,-1)$. In Figure 5, C is the black region. The distance $h(A,C)$ is measured from $(0,1)$ to the outer boundary of C at $(0,2)$, so $h(A,C) = 1$. The distance $h(C,B)$ is measured from any point on the inner boundary of C perpendicular to B, so $h(C,B) = 1$. In Figure 5, we have measured $h(C,B)$ from $(1,-1)$ to $(1,-2)$. In this case, C is between A and B on $\overleftrightarrow{AB}$ because $h(A,B) = h(A,C) + h(C,B)$.

Condition II. Let C be a circle centered at the origin with radius $2\sqrt{2}$ (see Figure 6). Then $h(A,C)$ is measured from the origin to any point on C. Thus,

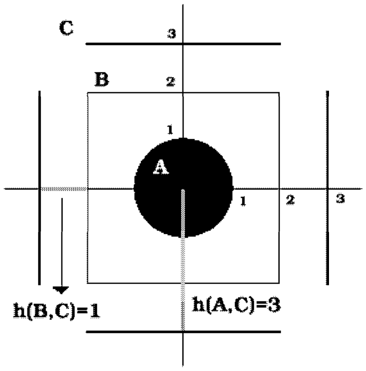

Figure 7: The set C formed by the union of all four line segments lies to the right of B.

$h(A,C) = 2\sqrt{2}$. We measure $h(B,C)$ from the center of any of the four sides of B perpendicular to C, so $h(B,C) = 2\sqrt{2}-2$. Since $h(A,B)+h(B,C) = h(A,C)$, C lies to the right of B on $\overleftrightarrow{AB}$.

Another point that lies to the right of B on $\overleftrightarrow{AB}$ is $C = \{(x,y) \mid x = 3 \text{ or } x = -3, y \in [-2,2]\} \cup \{(x,y) \mid y = 3 \text{ or } y = -3, x \in [-2,2]\}$. This set is simply the union of the four line segments shown in Figure 7. Here we have $h(A,C) = 3$, which can be measured from the origin to C along any axis. The distance $h(B,C)$ is measured from any point on C perpendicular to B, so that $h(B,C) = 1$. Condition (II) is once again satisfied.

Condition III. When trying to find a point C satisfying (III), one quickly realizes that this case is not as simple as the other two. In fact, there are no points $C \in \mathcal{H}(\mathbb{R}^2)$ that lie to the left of A on the Hausdorff line $\overleftrightarrow{AB}$.

Proposition 1. *Let A be the unit disk centered at the origin and let B be the square with vertices at $(2,2)$, $(2,-2)$, $(-2,2)$, and $(-2,-2)$. There does not exist a point $C \in \mathcal{H}(\mathbb{R}^2)$ that lies on the Hausdorff line $\overleftrightarrow{AB}$ such that $h(C,B) = h(C,A) + h(A,B)$.*

The proof makes use of the following lemma.

Lemma 4. *Let p_0 denote the origin. If B is the square with vertices at $(2,2)$, $(2,-2)$, $(-2,2)$, and $(-2,-2)$, then $N_{s+4}(p_0) \subset N_{s+2}(B)$.*

Proof. Let $x \in N_{s+4}(p_0)$. Let b' be the point at which the line $\overrightarrow{p_0x}$ intersects B. We show that the distance from x to B is always less than $s+2$, which implies that $x \in N_{s+2}(B)$.

If b' lies between p_0 and x on $\overrightarrow{p_0x}$, then

$$d(p_0,x) = d(p_0,b') + d(b',x).$$

Rearranging, we have

$$d(b', x) = d(p_0, x) - d(p_0, b').$$

Since $d(p_0, x) < s + 4$,

$$d(b', x) < s + 4 - d(p_0, b').$$

Furthermore, $d(p_0, b') \geq 2$. Thus,

$$d(b', x) < s + 2.$$

So $d(x, B) < s + 2$.

Now suppose that x lies between p_0 and b' on $\overrightarrow{p_0 x}$. The distance from x to b' will not necessarily be less than $s+2$ in this situation, but clearly it will always be true that $d(x, B) < 2$. Thus, $d(x, B) < s + 2$.

There are two more cases to consider. If $x \in B$, then $d(x, B) = 0$. If $x = p_0$, then $d(x, B) = 2$. Therefore, in all cases $d(x, B) < s + 2$. ■

Proof of Proposition 1. Suppose there exists a set C such that $h(C, B) = h(C, A) + h(A, B)$. Let $h(C, A) = s$. Since $h(A, B) = 2$, we know $h(C, B) = s+2$. It follows that $C \in C_s(A)$ and $C \in C_{s+2}(B)$. We will use the Circle Theorem to show $h(C, A) > s$, which contradicts the fact that $C \in C_s(A)$.

The last condition of Theorem 3 implies that either $C \cap \partial N_{s+2}(B) \neq \emptyset$ or there exist $b \in B$ and $c \in C \cap \partial\,(N_{s+2}(b))$ such that $C \cap N_{s+2}(b) = \emptyset$.

Suppose $c \in C \cap \partial N_{s+2}(B)$. The set $N_{s+2}(B)$ is open since it is the union of the open sets $N_{s+2}(b)$. Therefore, $N_{s+2}(B)$ equals its interior and is disjoint from its boundary. So, $c \notin N_{s+2}(B)$ which by Lemma 4 implies that $c \notin N_{s+4}(p_0)$. Thus,

$$d(p_0, c) \geq s + 4. \tag{5}$$

If $a \in A$, then

$$d(p_0, c) \leq d(p_0, a) + d(a, c).$$

From (5), we see that

$$s + 4 \leq d(p_0, a) + d(a, c) \tag{6}$$

Since A is the closed unit disk centered at p_0, $d(p_0, a) \leq 1$. So (6) becomes

$$s + 4 \leq 1 + d(a, c).$$

Simplifying, we have $d(a, c) \geq s + 3 > s$ for any $a \in A$. So $c \notin \bigcup_{a \in A} \overline{N_s(a)}$, which contradicts the second statement of Theorem 3.

Therefore, there exist $b_0 \in B$ and $c_0 \in C \cap \partial\,(N_{s+2}(b_0))$ such that $C \cap N_{s+2}(b_0) = \emptyset$. We will show that this implies that there is some $a_0 \in A$ such that $d(a_0, c) > s$ for every $c \in C$. Thus, $h(A, C) > s$.

Choose a_0 such that $d(b_0, a_0) = d(b_0, A)$. For any $c \in C$ the triangle inequality of the metric d gives

$$d(b_0, c) \leq d(b_0, a_0) + d(a_0, c).$$

Rearranging this expression gives

$$d(a_0, c) \geq d(b_0, c) - d(b_0, a_0).$$

Now, $c_0 \in \partial(N_{s+2}(b_0))$ so $d(b_0, c_0) = s + 2$. Since $C \cap N_{s+2}(b_0) = \emptyset$, $d(b_0, c) \geq s + 2$ for any other $c \in C$. We can then say that

$$d(a_0, c) \geq s + 2 - d(b_0, a_0).$$

The distance $d(B, A)$ is measured from one of the vertices of B, so $d(B, A) = 2\sqrt{2} - 1$. Thus, $d(b_0, A) = d(b_0, a_0) \leq 2\sqrt{2} - 1$. Therefore,

$$d(a_0, c) \geq s + 2 - (2\sqrt{2} - 1),$$

which simplifies to give

$$d(a_0, c) \geq s + (3 - 2\sqrt{2}).$$

Since $3 > 2\sqrt{2}$, $d(a_0, c) > s$ for any $c \in C$. Thus, $d(a_0, C) > s$. Since $d(A, C)$ is the maximum of the distances $d(a, C)$ for all $a \in A$, $d(A, C) \geq d(a_0, C) > s$. Furthermore, $h(A, C) \geq d(A, C) > s$, so $C \notin C_s(A)$, contrary to assumption. Thus, there is no set $C \in \mathcal{H}(\mathbb{R}^2)$ such that $h(C, B) = h(C, A) + h(A, B)$. ■

The example given in this section is surprising because it conflicts with our geometric intuition. That is, we do not expect that there may be places on a line where points do not exist. Lines in Euclidean space always have a full continuum of points. Given the Euclidean line $\overleftrightarrow{ab}$, we can find points $c \in \overleftrightarrow{ab}$ at any distance from a or b. This is not true for the Hausdorff line in the example we have just seen. This example opens up numerous questions concerning the existence of points on an arbitrary Hausdorff line and we will begin to answer some of these questions in Section 5. Before that, however, we will discuss another interesting property of the example given in this section.

4 Measuring Distance From Interior Points

As we will see in Lemma 6, if $A, B \in \mathcal{H}(\mathbb{R}^n)$, we know that there exist $a \in A$ and $b \in \partial B$ so that $d(A, B) = d(a, b)$. Note that it is always true that $b \in \partial B$, but it is not always the case that $a \in \partial A$. In the example in Section 3.3, $d(A, B)$ was measured from the origin. It would be interesting and possible useful to know when this occurs. Judging from this particular case, we might conclude that the distance is measured from an interior point because B "surrounds" A. However, it is not necessary that B completely enclose A. For example, if we took B to be the set $\{(2, 0), (0, 2), (-2, 0), (0, -2)\}$ and A to be the unit disk, then $d(A, B)$ would still be measured from the origin and yet B clearly does not enclose A. The main idea here is that all of the points on ∂A must be closer to B than the points of A°. Theorem 5 completely describes the circumstances under which we measure $d(A, B)$ from a point on the interior of A.

We will need the following result, which is Lemma 1 of [2]. The proof is straightforward and therefore has been omitted.

Lemma 5. *Let $B \in \mathcal{H}(\mathbb{R}^n)$ and $r > 0$. Then*

$$\partial N_r(B) = \left(\bigcup_{b \in B} \partial N_r(b) - N_r(B) \right).$$

Theorem 5. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and let $d(\partial A, B) = m$. Then $d(A, B) = d(a_0, B)$ for some $a_0 \in A^\circ$ if and only if*

(i) *$A^\circ - (B + m) \neq \emptyset$, or*

(ii) *$A^\circ - (B + m) = \emptyset$ and $A^\circ \cap \partial N_m(B) \neq \emptyset$.*

If $A^\circ - (B + m) \neq \emptyset$, then $d(A, B)$ can only be measured form a point in A°. If $A^\circ - (B + m) = \emptyset$ and $A^\circ \cap \partial N_m(B) \neq \emptyset$, then $d(A, B)$ can be measured from both a point in A° and a point in ∂A.

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and let $d(\partial A, B) = m$. We prove the forward implication first, so assume $d(A, B) = d(a_0, B)$ for some $a_0 \in A^\circ$. If $a' \in \partial A$, then $a' \in A$ because A is closed. For every $a' \in \partial A$, we have $d(a', B) \leq d(A, B)$. Since $d(A, B) = d(a_0, B)$, we know that $d(a_0, B) \geq d(a', B)$ for every $a' \in \partial A$. Furthermore, $d(\partial A, B) = m$ implies that there exists an element $a' \in \partial A$ such that $d(a', B) = m$. So $d(a_0, B) \geq m$. We consider two cases.

Case 1. If $d(a_0, B) > m$, then for every $b \in B$, $d(a_0, b) > m$. So $a \notin \overline{N_m(b)}$ for every $b \in B$. Thus, $a_0 \notin \bigcup_{b \in B} \overline{N_m(b)} = B + m$. Therefore, $a_0 \in A^\circ - (B + m)$. This proves (i).

Case 2. If $d(a_0, B) \not> m$, then $d(a_0, B) = m$. So for every $a \in A^\circ$, we have $d(a, B) \leq m$. This implies that for every $a \in A^\circ$ there exists $b \in B$ such that $a \in \overline{N_m(b)}$. So if $a \in A^\circ$, then $a \in \bigcup_{b \in B} \overline{N_m(b)}$. Thus, $A^\circ - (B + m) = \emptyset$. This proves the first half of (ii).

Now, $d(a_0, B) = m$ implies that for at least one $b \in B$, it holds that $d(a_0, b) = m$. So $a_0 \in \partial N_m(b)$ for at least one $b \in B$. Therefore, $a_0 \in \bigcup_{b \in B} \partial N_m(b)$. Since $d(a_0, B) = m$, we also know that for every $b \in B$, $d(a_0, b) \geq m$. Thus, for every $b \in B$, $a_0 \notin N_m(b)$. This implies $a_0 \notin N_m(B)$. By lemma 5, we have $a_0 \in \partial N_m(B)$. Hence, $a_0 \in A^\circ \cap \partial N_m(B)$. This completes the proof of (ii).

We now prove the reverse implication. The two cases are considered separately.

Case 1. Assume $A^\circ - (B + m) \neq \emptyset$. Notice that for every $a \in A^\circ - (B + m)$, it must be true that $d(a, B) > m$. This is so because $d(a, B) \leq m$ implies $a \in B + m$. Choose $a_0 \in A$ such that $d(a_0, B) = d(A, B)$. Then $d(a_0, B) > m$. We will show that $a_0 \in A^\circ$. This will also imply that if (i) is true, then $d(A, B)$ can only be measured from a point in A°.

Suppose $a_0 \notin A^\circ$. For an arbitrary set X, we know $\overline{X} = X^\circ \cup \partial X$ [4]. Since A is closed, $A = \overline{A}$. Thus, $A = A^\circ \cup \partial A$. Since $a_0 \notin A^\circ$ and $a_0 \in A$, it must be true that $a_0 \in \partial A$. Now, $d(a_0, B) \leq d(\partial A, B)$, so $d(a_0, B) \leq m$. This is a contradiction since $d(a_0, B) > m$. So, $a_0 \in A^\circ$ and $d(a_0, B) = d(A, B)$.

Case 2. Assume $A^\circ - (B + m) = \emptyset$ and $A^\circ \cap \partial N_m(B) \neq \emptyset$. Let $a_0 \in A^\circ \cap \partial N_m(B)$. Then $a_0 \in \partial N_m(b)$ for some $b \in B$ but $a_0 \notin N_m(b)$ for every $b \in B$. So $d(a_0, B) = m$. We now show that $d(A, B) = m$. Since $a_0 \in A^\circ$ and $d(a_0, B) = m$, this will prove the desired result.

As stated above, $A = A^\circ \cup \partial A$. So for all $a \in A$, we have either $a \in A^\circ$ or $a \in \partial A$. If $a \in \partial A$, then $d(a, B) \leq d(\partial A, B) = m$. If $a \in A^\circ$, then $a \in B + m$ because $A^\circ - (B + m) = \emptyset$. This again implies that $d(a, B) \leq m$. Thus, for every $a \in A$, $d(a, B) \leq m$. So $d(A, B) = m$ only if there is some $a \in A$ such that $d(a, B) = m$. But we know that $a_0 \in A$ and $d(a_0, B) = m$, so $d(A, B) = m$.

Now, $d(a_0, B) = d(A, B)$ with $a_0 \in A^\circ$. We show that $d(a', B) = d(A, B)$ for some $a' \in \partial A$. Choose $a' \in \partial A$ so that $d(a', B) = d(\partial A, B) = m$. Since $d(A, B) = m$, we know $d(a', B) = d(A, B)$. Thus, if (ii) is true, then $d(A, B)$ can be measured from both a point in A° and a point in ∂A. ■

The sets A and B in Figure 4 satisfy condition (i) of Theorem 5. Note that $m = d(\partial A, B) = 1$. In this case, $d(A, B) = d((0, 0), B)$ and $(0, 0) \in A^\circ - (B + m)$.

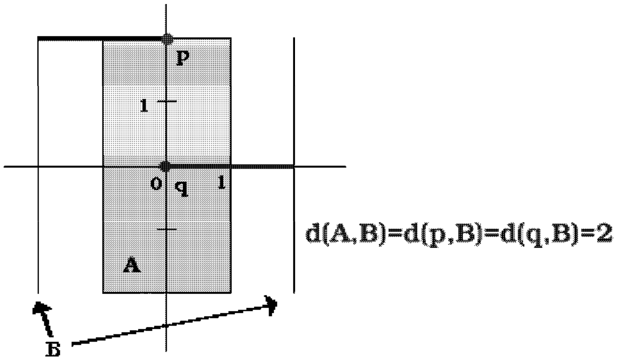

Figure 8: The distance $d(A, B)$ can be measured from either $p = (0, 2) \in \partial A$ or $q = (0, 0) \in A^\circ$.

Figure 8 shows two sets that satisfy condition (ii) of Theorem 5. The set A is the rectangle with vertices at $(1, 2), (1, -2), (-1, 2)$, and $(-1, -2)$ along with the area that it encloses. The set B is $\{(x, y) \mid x = 2 \text{ or } x = -2, y \in [-2, 2]\}$. Here, $d(A, B) = d(p, B) = d(q, B)$ where $q \in A^\circ$ and $p \in \partial A$. The point q is in $A^\circ \cap \partial N_m(B)$.

For $A, B \in \mathcal{H}(\mathbb{R}^n)$, Theorem 5 allows us to determine the location of the point from which we measure $d(A, B)$ and $d(B, A)$. We now return to the discussion of the existence of points on Hausdorff lines.

5 Existence of Points on Hausdorff Lines

The sets A and B from Section 3.3 form a line $\overleftrightarrow{AB}$ that is much different from what we might expect. This is because there are no sets $C \in \mathcal{H}(\mathbb{R}^n)$ satisfying condition (III) of definition 12. That is, there are no points of $\mathcal{H}(\mathbb{R}^n)$ on the Hausdorff line $\overleftrightarrow{AB}$ that lie to the "left" of A. This example leads us naturally to consider which properties of A and B determine that $h(C, B) = h(C, A) + h(A, B)$ can not be satisfied for $C \in \mathcal{H}(\mathbb{R}^n)$. This section contains results concerning the existence of such points on Hausdorff lines.

5.1 Points Satisfying $h(A, C) + h(C, B) = h(A, B)$

We begin our discussion of the existence of points on Hausdorff lines by looking at the segment $\overline{AB}$. Theorem 6 will show that for arbitrary points $A, B \in \mathcal{H}(\mathbb{R}^n)$, $\overline{AB}$ will contain a continuum of points. That is, there will not be any holes in $\overline{AB}$. The proof of Theorem 6 relies on Lemmas 6 and 7. Lemma 6 comes from [2] and is stated without proof.

Lemma 6. *Let A, and B be elements of $\mathcal{H}(\mathbb{R}^n)$. If $d(B, A) > 0$, then there exist $a_0 \in \partial A$ and $b_0 \in B$ so that $d(b_0, a_0) = d(B, A)$, $d(b_0, A) \geq d(b, A)$ for all $b \in B$, and $d(b_0, a_0) \leq d(b_0, a)$ for all $a \in A$.*

Lemma 7. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $d(B, A) = r > 0$. If $0 < s < r$, then $d(B, A + s) = r - s$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $d(B, A) = r > 0$. Let $s \in \mathbb{R}$ so that $0 < s < r = d(B, A)$. By Lemma 6 there are elements $b_0 \in B$ and $a_0 \in \partial A$ so that $d(b_0, a_0) = d(B, A) = r$, $d(b_0, A) \geq d(b, A)$ for all $b \in B$, and $d(b_0, a_0) \leq d(b_0, a)$ for all $a \in A$. Let $c_0 \in \overrightarrow{a_0 b_0}$ so that $d(a_0, c_0) = s$. Then $d(b_0, a_0) = d(b_0, c_0) + d(c_0, a_0)$ implies $d(b_0, c_0) = r - s$. We will now show that $d(B, A + s) = d(b_0, c_0)$.

First we show that $d(b_0, A + s) = \min\limits_{c \in (A+s)} \{d(b_0, c)\} = r - s$. Suppose there is a $c' \in (A + s)$ so that $d(b_0, c') < d(b_0, c_0) = r - s$. Since $c' \in (A + s)$, there is an element $a' \in A$ so that $d(c', a') \leq s$. Then

$$d(b_0, a') \leq d(b_0, c') + d(c', a') < (r - s) + s = r.$$

However, we know that $r = d(b_0, a_0) \leq d(b_0, a)$ for any $a \in A$. This is a contradiction. Therefore, $d(b_0, A + s) = d(b_0, c_0)$. To complete the proof, we need to show that $d(b_0, A + s)$ is the maximum such distance.

Now we show that if $b \in B$, then $d(b, A + s) \leq d(b_0, A + s)$. Let $b \in B$. Recall that $r = d(b_0, A) \geq d(b, A)$. By Lemma 6, there is an $a \in A$ so that $d(\{b\}, A) = d(b, A) = d(b, a) \leq r$. Let c be the point at which the ray $\overrightarrow{ab}$ intersects $N_s(a)$. Now, $d(c, a) = s$, so $c \in (A + s)$. Since a, c, and b lie on a Euclidean line with c between a and b, we have

$$d(b, c) = d(b, a) - d(a, c) \leq r - s.$$

This shows that $d(b, A+s) = \min_{c \in (A+s)} \{d(b,c)\} \leq r - s = d(b_0, A+s)$. ■

Finally, we need a lemma that tells us about the boundary of the intersection of elements of $\mathcal{H}(\mathbb{R}^n)$.

Lemma 8. *Let $X, Y \in \mathcal{H}(\mathbb{R}^n)$. Then $\partial(X \cap Y) = (\partial X \cap Y) \cup (X \cap \partial Y)$.*

Proof: Let $X, Y \in \mathcal{H}(\mathbb{R}^n)$. Let $z \in \partial(X \cap Y)$. Since $X \cap Y \in \mathcal{H}(\mathbb{R}^n)$, we know $z \in (X \cap Y)$. So $z \in X$ and $z \in Y$. Suppose $z \notin \partial X \cap Y$. Since $z \in Y$, we must have $z \notin \partial X$. Since $z \in X$, we must have $z \in X^\circ$. So there is a $\delta > 0$ so that $N_\delta(z) \subset X$. To show $z \in X \cap \partial Y$, we need to show $z \in \partial Y$. Let $\epsilon > 0$. Choose ϵ' so that $0 < \epsilon' < \min\{\epsilon, \delta\}$. Then $N_{\epsilon'}(z) \subset X$. Since $z \in \partial(X \cap Y)$, there is an element $w \in N_{\epsilon'}(z)$ so that $w \notin (X \cap Y)$. Since $w \in N_{\epsilon'}(z) \subset X$, we must have $w \notin Y$. So $N_{\epsilon'}(z) \subset N_\epsilon(z)$ contains an element in Y and an element not in Y. Thus, $z \in \partial Y$. This shows $z \in X \cap \partial Y$. A similar argument shows that if $z \notin X \cap \partial Y$, then $z \in \partial X \cap Y$. Therefore, $\partial(X \cap Y) \subseteq (\partial X \cap Y) \cup (X \cap \partial Y)$.

Now let $z \in (\partial X \cap Y) \cup (X \cap \partial Y)$. Suppose $z \in (\partial X \cap Y)$. Then $z \in X \cap Y$. Let $\epsilon > 0$. Since $z \in \partial X$, there is an element $w \in N_\epsilon(z)$ so that $w \notin X$. Thus, $w \notin X \cap Y$. Siilarly, if $z \in (X \cap \partial Y)$, every neighborhood of z will contain an element of $X \cap Y$ and an element not in $X \cap Y$. Thus, $z \in \partial(X \cap Y)$ and $\partial(X \cap Y) = (\partial X \cap Y) \cup (X \cap \partial Y)$ as desired. ■

The next theorem shows that there are points at any distance along $\overline{AB}$.

Theorem 6. *Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ with $d(B, A) \geq d(A, B)$. Let $r = h(A, B)$. Let $s \in \mathbb{R}$ with $0 < s < r$, and let $t = r - s$. If C is a compact subset of $(A+s) \cap (B+t)$ containing $\partial((A+s) \cap (B+t))$, then C satisfies*

$$h(A, B) = h(A, C) + h(C, B). \tag{7}$$

Proof: Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(B, A) \geq d(A, B)$. Let $s \in \mathbb{R}$ so that $0 < s < r$ and let $t = r - s$. First we will show $\partial((A+s) \cap (B+t)) \neq \emptyset$. To do this we consider two cases:

I $s \geq r - d(A, B)$

II $0 < s < r - d(A, B)$.

Case I: Assume $s \geq r - d(A, B)$. Let $a_1 \in A$ and $b_1 \in B$ so that $d(a_1, b_1) = d(A, B)$. Let $y_1 \in \overrightarrow{b_1 a_1}$ so that $d(b_1, y_1) = t$. Since $s \geq r - d(A, B)$, we have $d(A, B) \geq r - s = t$. Thus, $d(a_1, b_1) = d(A, B) \geq t$. So $y_1 \in \overline{b_1 a_1}$ and

$$d(y_1, a_1) = d(b_1, a_1) - d(b_1, y_1) = d(A, B) - t \leq r - t = s. \tag{8}$$

Thus, $y_1 \in (A+s)$. Now we will show $d(y_1, B) = t$. Suppose there exists $b' \in B$ so that $d(y_1, b') < t$. Then

$$d(a_1, b') \leq d(a_1, y_1) + d(y_1, b'). \tag{9}$$

Now (8) and (9) combine to show $d(a_1, b') < (d(A, B) - t) + t = d(A, B)$, a contradiction to $d(A, B) = d(a_1, b_1)$. Therefore, $d(y_1, B) = t$ and, consequently, $y_1 \in \partial(B + s)$. By lemma 8, we have $y_1 \in \partial\left((A + s) \cap (B + t)\right)$.

Case II: Assume $0 < s < r - d(A, B)$. We consider two subcases: $(A + s) \subseteq (B + t)$ and $(A + s) \nsubseteq (B + t)$. If $(A + s) \subseteq (B + t)$, then $C = (A + s)$. In this case, lemma 7 shows $d(B, C) = r - s = t$. Now assume $(A + s) \subseteq (B + t)$. Since $s < r - d(A, B)$, we have $s + d(A, B) < r = s + t$. Thus, $d(A, B) < t$. Let $a \in A$. Then there is a $b \in B$ so that $d(a, b) = d(A, B) \leq d(A, B) < t$. Let $\epsilon_a = t - d(a, b)$ If $z \in N_{\epsilon_a}(a)$, then $d(z, b) \leq d(z, a) + d(a, b) < (t - d(a, b)) + d(a, b) = t$. Therefore, $z \in (B + t)$ and $A \subset (B + t)^\circ$. Now let $x \in ((A + s) - (B + t))$. Then $x \in (A + s)$. So there is an element $a_x \in A$ with $d(x, a_x) = d(x, A) \leq s$. Let

$$S = \overline{xa_x} \cap (B + t).$$

Note that $N_{\epsilon_{a_x}}(a_x) \subset (B + t)$ so S is not empty. Since S is the intersection of two compact sets, S is compact. Let $y_x \in S$ so that $d(x, y_x) = d(x, S)$. Since $x \notin (B + t)$, we know that $y_x \neq x$. Now we show that $y_x \in \partial(B + t)$. Since $y_x \in S$, we have $y_x \in (B + t)$. Let $\epsilon > 0$. Now, for any $y \in S$, we have $d(x, y) \geq d(x, y_x)$. So no $y \neq y_x$ in $\overline{xy_x}$ can be in S. Therefore, if $y \in \overline{xy_x}$ with $0 < d(y_x, y) \leq \epsilon$, then $y \notin (B + t)$. Therefore, every neighborhood of y_x contains a point in $B + t$ and a point not in $B + t$. Thus, $y_x \in \partial(B + t)$. Also, $d(y_x, a_x) < d(x, a_x) \leq s$. So $y_x \in \partial(B + t) \cap (A + s) \subset \partial\left((A + s) \cap (B + t)\right)$. Therefore, $\partial\left((A + s) \cap (B + t)\right)$ is not empty.

Let $C \in \mathcal{H}(\mathbb{R}^n)$ so that $C \subset (A+s) \cap (B+t)$ and C contains $\partial\left((A + s) \cap (B + t)\right)$. We will show $h(A, C) = s$ and $h(B, C) = t$. Since $C \subset (A + s) \cap (B + t)$, we know $d(C, A) \leq s$ and $d(C, B) \leq t$. Now, C contains an element $c \in \partial\left((A + s) \cap (B + t)\right)$, and $d(c, A) = s$, $d(c, B) = t$. Thus, we have $d(C, A) = s$ and $d(C, B) = t$. It remains to show $d(A, C) \leq s$ and $d(B, C) \leq t$.

Let $a \in A$. Then there is an element $b \in B$ with $d(a, b) \leq d(A, B) \leq r$. If $d(a, b) \leq s$, then $b \in (A + s) \cap (B + t)$. If this is the case, let $x = b$. If $d(a, b) > s$, then let $x \in \overline{ab}$ so that $d(a, x) = s$. So $x \in (A + s)$. Also, $d(a, x) + d(x, b) = d(a, b)$, so $d(x, b) = d(a, b) - d(a, x) \leq r - s = t$. Thus, $x \in (A+s) \cap (B+t)$. By lemma 6, there is an element $x_a \in \partial\left((A + s) \cap (B + t)\right)$ so that $d(a, x_a) = d(a, (A + s) \cap (B + t))$. So $d(a, x_a) \leq d(a, x) \leq s$. Thus, we see that $d(A, \partial\left((A + s) \cap (B + t)\right)) \leq s$.

To show $d(B, C) \leq t$, we argue in a similar manner. Let $b \in B$. Then there is an element $a \in A$ with $d(b, a) \leq d(B, A) \leq r$. If $d(b, a) \leq t$, then $a \in C$. In this case, let $y = a$. If $d(b, a) > t$, then let $y \in \overline{ba}$ so that $d(b, y) = t$. So $y \in (B + t)$. Also, $d(b, y) + d(y, a) = d(b, a)$, so $d(y, a) = d(a, b) - d(b, y) \leq r - t = s$. Thus, $y \in (A+s) \cap (B+t)$. By lemma 6, there is an element $y_b \in \partial\left((A + s) \cap (B + t)\right)$ so that $d(b, y_b) = d(b, (A + s) \cap (B + t))$. So $d(b, y_b) \leq d(b, y) \leq t$. Thus, we see that $d(B, \partial\left((A + s) \cap (B + t)\right)) \leq t$, which completes our proof. ■

This result guarantees that for any $A, B \in \mathcal{H}(\mathbb{R}^n)$, there will never be holes in $\overline{AB}$.

5.2 Rays In $\mathcal{H}(\mathbb{R}^n)$

Theorem 6 states that for any two points $A, B \in \mathcal{H}(\mathbb{R}^n)$, there is a full continuum of points between A and B. However, we know that in general there is not always a full continuum of points satisfying to the left of A or to the right of B, as evidenced by the example in Section 3.3. In this section we will see that if $A, B \in \mathcal{H}(\mathbb{R}^n)$ and $\overleftrightarrow{AB}$ does not contain a continuum of points for each of the triangle equalities, then $\overleftrightarrow{AB}$ must be a ray. That is, $\overleftrightarrow{AB}$ will never be a segment.

Our first step in exploring rays in $\mathcal{H}(\mathbb{R}^n)$ is to generalize the result obtained in Section 3.3. We proved that if A is the unit disk centered at the origin and B is the square with vertices $(2,2), (2,-2), (-2,2)$, and $(-2,-2)$, then there does not exist $C \in \mathcal{H}(\mathbb{R}^n)$ such that

$$h(B,C) = h(B,A) + h(A,C). \tag{10}$$

The results of this section will show that there are two conditions that together imply that (10) can not be satisfied. Theorem 7 will show that one of these conditions is $d(A,B) > d(B,A)$. Before stating this theorem, we present two lemmas. The proof of Lemma 10 requires Lemma 9 from [2], which is stated here without proof.

Lemma 9. *Let* $A \in \mathcal{H}(\mathbb{R}^n)$ *and* $s > 0$. *If* $x \in \partial(A + s)$, *then* $d(x, A) = s$.

Lemma 10. *If* $B \in \mathcal{H}(\mathbb{R}^n)$ *and* $r > 0$, *then* $\partial(B + r) \subseteq \partial N_r(B)$.

Proof. Let $x \in \partial(B + r)$. We will show that $x \in \left(\bigcup_{b \in B} \partial N_r(b) - N_r(B) \right)$. By Lemma 5, this will show $x \in \partial N_r(B)$. Lemma 9 guarantees that $d(x, B) = r$. Therefore, there is a $b_x \in B$ such that $d(x, b_x) = r$. So $x \in \partial N_r(b_x)$, which implies $x \in \bigcup_{b \in B} \partial N_r(b)$.

We now show that $x \notin N_r(B)$. Suppose $x \in N_r(B)$. Then $x \in N_r(b)$ for some $b \in B$. Let $\epsilon = r - d(x, b)$. If $y \in N_\epsilon(x)$, then

$$d(b,y) \leq d(b,x) + d(x,y).$$

Since $d(x,y) < \epsilon$,

$$d(b,y) < d(b,x) + \epsilon.$$

Now,

$$d(b,x) + \epsilon = d(b,x) + (r - d(x,b)) = r.$$

Therefore,

$$d(b,y) < r.$$

So $y \in B + r$ which implies $N_\epsilon(x) \subseteq B + r$. Since $N_\epsilon(x)$ is a neighborhood of x which does not intersect $\mathbb{R}^n - (B + r)$, it follows that $x \notin \partial(B + r)$. This contradiction proves that $x \notin N_r(B)$.

Thus, $x \in \left(\bigcup_{b \in B} \partial N_r(b) - N_r(B) \right)$, so $x \in \partial N_r(B)$. ■

In general, it is not true that $\partial(B+r) = \partial N_r(B)$. That is, it will not always be the case that $\partial N_r(B) \subseteq \partial(B+r)$. For example, consider Figure 8. Notice that, $\partial N_2(B)$ contains the set $\{(x,y) \mid x = 0,\ y \in [-2,2]\}$. However, $\{(x,y) \mid x = 0,\ y \in (-2,2)\} \subset (B+2)^\circ$, so $\{(x,y) \mid x = 0,\ y \in (-2,2)\} \not\subset \partial(B+2)$.

Theorem 7. *Let* $A, B, C \in \mathcal{H}(\mathbb{R}^n)$ *such that* $h(C,B) = h(C,A) + h(A,B)$, $h(C,A) = s$, *and* $h(A,B) = r$. *If* $d(A,B) > d(B,A)$, *then* $C \cap N_{r+s}(b) \neq \emptyset$ *for each* $b \in B$.

Proof. Assume $d(A,B) > d(B,A)$. Let $h(A,B) = r$ and $h(C,A) = s$. Thus, $h(C,B) = r+s$. Let $b \in B$. For any $a \in A$ and any $c \in C$,

$$d(b,c) \leq d(b,a) + d(a,c). \tag{11}$$

Choose $a' \in A$ such that $d(b,a') = d(b,A)$. Since $d(b,A) \leq d(B,A)$ and $d(B,A) < r$, it follows that $d(b,a') < r$. Combing this fact with (11), we conclude

$$d(b,c) < r + d(a',c). \tag{12}$$

Since $h(A,C) = s$, we know $d(A,C) \leq s$. It follows that if $a \in A$, then $d(a,C) \leq s$. Choose $c' \in C$ such that $d(a',c') = d(a',C)$. Then $d(a',c') \leq s$. From this fact and (12) it follows that

$$d(b,c') < r + s. \tag{13}$$

Therefore, $c' \in N_{r+s}(b)$, which implies that $C \cap N_{r+s}(b) \neq \emptyset$. ■

This result is important because it implies that if $d(A,B) > d(B,A)$, then condition 3(b) of the circle theorem will not be satisfied. Therefore, if $d(A,B) > d(B,A)$ and there is a $C \in \mathcal{H}(\mathbb{R}^n)$ such that $C \in C_{r+s}(B)$, then it must be true that $C \cap \partial N_{r+s}(B) \neq \emptyset$. This condition is part 3(a) of the Circle Theorem. To determine if there are no points C on the Hausdorff line $\overleftrightarrow{AB}$ satisfying $h(C,B) = h(C,A) + h(A,B)$, we need to find the additional conditions under which $C \cap \partial N_{r+s}(B) = \emptyset$. The next theorem tells us when this occurs.

Theorem 8. *Let* $A, B, C \in \mathcal{H}(\mathbb{R}^n)$ *with* $r = d(A,B) > d(B,A)$. *If* $h(A,C) = s$ *and* $(A+s) \subseteq N_{r+s}(B)$, *then* $h(C,B) \neq r+s$.

Proof. Let $A, B, C \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(A,B) > d(B,A)$. Assume $h(A,C) = s$. Suppose $h(B,C) = r+s$, so $C \in C_{r+s}(B)$. Since $d(A,B) > d(B,A)$, Theorem 7 states that for all $b \in B$, $C \cap N_{r+s}(b) \neq \emptyset$. So 3(b) of the Circle Theorem is not satisfied for $C \in C_{r+s}(B)$. Therefore, we must have $C \cap \partial N_{r+s}(B) \neq \emptyset$. Let $c \in C \cap \partial N_{r+s}(B)$. Then Lemma 1 tells us $c \notin N_{r+s}(B)$. Since $(A+s) \subseteq N_{r+s}(B)$, we have $c \notin (A+s)$. However, $h(A,C) = s$ implies $C \in C_s(A)$. Therefore, condition 1 of the Circle Theorem states that $C \subseteq \bigcup_{a \in A} \overline{N_s(a)}$. By Lemma 2,

$C \subseteq (A+s)$. But $c \in C$ and $c \notin (A+s)$, which is a contradiction. Thus, $h(B,C) \neq r+s$. ■

Given a Hausdorff line $\overleftrightarrow{AB}$, the result of Theorem 8 tells us when a set $C \in \mathcal{H}(\mathbb{R}^n)$ does not satisfy $h(C,B) = h(C,A) + h(A,B)$. However, this result only holds when $h(A,C) = s$. We generalize this result in the following corollary.

Corollary 1. *Let $A, B, \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(A,B) > d(B,A)$. If $(A+s) \subseteq N_{r+s}(B)$ for all $s > 0$, then there are no sets $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(C,A) + h(A,B) = h(C,B)$.*

Corollary 1 applies to the example in Section 3.3, thus verifying the result of Proposition 1.

This is result is useful, but it is does not tell us when $(A+s) \subseteq N_{r+s}(B)$ will be true for all $s > 0$. Lemma 12 gives conditions under which this occurs. Before proving it, we state and prove the following lemma.

Lemma 11. *Let $B \in \mathcal{H}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, and $t > 0$. Suppose $x \in \partial(B+t)$. If $b \in B$ and $x \in \partial N_t(b)$, then $b \in \partial B$.*

Proof. Assume $x \in \partial(B+t)$. Lemma 10 and Lemma 5 together imply that for every $b \in B$, $x \notin N_t(b)$. Lemma 10 also guarantees that $x \in \partial N_t(b)$ for some $b \in B$. Suppose $b \notin \partial B$, or equivalently, $b \in B^\circ$. Since $b \in B^\circ$, there is an $\epsilon > 0$ such that $N_\epsilon(b) \subseteq B^\circ$. Choose $b' \in \overrightarrow{bx} \cap N_\epsilon(b)$ such that $b \neq b'$. Then:

$$d(b,x) = d(b,b') + d(b',x).$$

Rearranging, we have

$$d(b',x) = d(b,x) - d(b,b').$$

Since $x \in \partial N_t(b)$, $d(b,x) = t$. Thus,

$$d(b',x) = t - d(b,b'). \tag{14}$$

Now, $0 < d(b,b') < \epsilon$. Together with (14), this implies $d(b',x) < t$. Therefore, $x \in N_t(b')$. Thus, the assumption that $b \in B^\circ$ was false and $b \in \partial B$. ■

Lemma 12. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and $s > 0$. If $d(\partial A, B) < d(A,B) = r$ and $(A+s)^\circ = N_s(A)$, then $(A+s) \subseteq N_{r+s}(B)$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(A,B) > d(\partial A, B)$. Let $s > 0$ and $x \in A+s$. Then either $x \in \partial(A+s)$ or $x \in (A+s)^\circ$. Suppose $x \in \partial(A+s)$. Then $d(x,A) = s$ by Lemma 9. Thus, $d(x,a_x) = s$ for some $a_x \in A$, so $x \in \partial N_s(a_x)$. Lemma 11 then tells us that $a_x \in \partial A$. Therefore, $d(a_x, B) \leq d(\partial A, B) < r$. If we choose $b_{a_x} \in B$ such that $d(a_x, b_{a_x}) = d(a_x, B)$, then $d(x, b_{a_x}) \leq d(x, a_x) + d(a_x, b_{a_x}) < r+s$. Therefore, $x \in N_{r+s}(b_{a_x})$, which implies that $x \in N_{r+s}(B)$.

Now suppose $x \in (A+s)^\circ$. Then $x \in N_s(a_0)$ for some $a_0 \in A$, so $d(x,a_0) < s$. Let $b_{a_0} \in B$ such that $d(a_0, b_{a_0}) = d(a_0, B) \leq d(A,B) = r$. Then, $d(x, b_{a_0}) \leq d(x,a_0) + d(a_0, b_{a_0}) < r+s$, so $x \in N_{r+s}(b_{a_0})$. Thus, $x \in N_{r+s}(B)$. ■

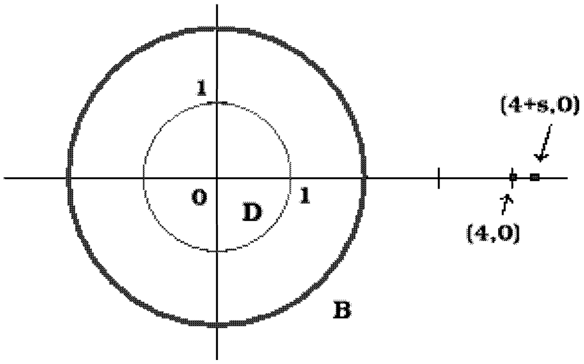

Figure 9: There exists $C \in \mathcal{H}(\mathbb{R}^n)$ satisfying(10) even though $d(A, B) > d(B, A)$ and $d(A, B) = d(a_0, B)$ for some $a_0 \in A^\circ$.

Corollary 2. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(A, B) > d(B, A)$. If $d(\partial A, B) < r$ and $(A+s)^\circ = N_s(A)$ for all $s > 0$, then there are no sets $C \in \mathcal{H}(\mathbb{R}^n)$ satisfying $h(B, C) = h(B, A) + h(A, C)$.*

Notice that Corollary 2 applies to the example in Section 3.3 where A was the unit disk centered at the origin and B was the square with vertices at $(2, 2)$, $(2, -2)$, $(-2, 2)$, and $(-2, -2)$.

The condition that $d(\partial A, B) < d(A, B)$ implies that if $d(a, B) = d(A, B)$ for some $a \in A$, then $a \in A^\circ$. In this situation $d(A, B)$ can *only* be measured from a point on the interior of A. The result of Corollary 2 does not hold if $d(A, B)$ is measured from both a point in ∂A and a point in A°. When this occurs, there may be sets $C \in \mathcal{H}(\mathbb{R}^n)$ satisfying (10).

Figure 9 will illustrate this point. The set D is the unit disk centered at the origin and B is the circle of radius 2 centered at the origin. Let $A = D \cup \{(4, 0)\}$ and $C = A \cup \{(4 + s, 0)\}$ for some $s > 0$. Then $d(A, B) = 2$ can be measured from either $(0, 0) \in A^\circ$ or $(4, 0) \in \partial A$. Since $d(B, A) = 1$, we have $h(A, B) = 2$. Now, $d(A, C) = 0$ since $A \subset C$ and $d(C, A) = d((4, 0), (4 + s, 0)) = s$. So $h(A, C) = s$. The distance $d(B, C)$ is measure from any point on B to the boundary of D, so $d(B, C) = 1$. We measure $d(C, B)$ from $(4 + s, 0)$ along the x-axis to B, so $d(C, B) = 2 + s$. Therefore, $h(B, C) = 2 + s$, which implies $h(C, B) = h(C, A) + h(A, B)$. The condition that $d(A, B) > d(B, A)$ was met here and $d(A, B)$ was measured from a point on the interior of A. However, we were able to find a set $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfied (10). This is because $d(A, B) = d(\partial A, B)$. Therefore, Corollary 2 will not hold unless we require that $d(\partial A, B) < d(A, B)$.

Theorem 8 gives conditions for when there are no sets at some distance $s > 0$ from A that lie on the Hausdorff line $\overleftrightarrow{AB}$ satisfying (10). In addition, Corollary 1 only describes situations in which there are *no* points on a Hausdorff line satisfying (10). The corollary tells us when a Hausdorff line "stops" at A.

It only applies to situations when

$$(A + s) \subseteq N_{r+s}(B) \tag{15}$$

for all $s > 0$. One may ask if it is possible for (15) to hold for some values of s and not for others. If so, we may be able to find points on $\overleftrightarrow{AB}$ satisfying (10) with $h(C, A) = s$ for some, but not all, values of s. The results below show that this is, in fact, the case. After stating a lemma, we will look at an example.

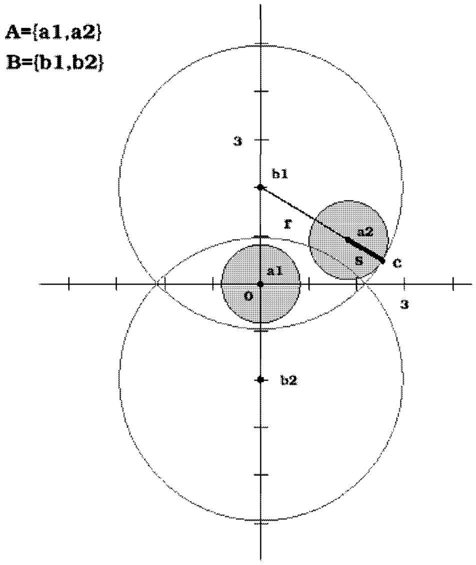

Figure 10: Shown here are the sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. The set $C = A + s$ satisfies (10).

Lemma 13. *If* $A \in \mathcal{H}(\mathbb{R}^n)$ *and* $s > 0$, *then* $h(A + s, A) = s$.

Proof. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $s > 0$. If $a \in A$, then $a \in A + s$. Therefore, $d(A, A + s) = 0$. If $x \in A + s$, then $d(x, a_x) \leq s$ for some $a_x \in A$. Thus, $d(x, A) \leq s$. Now, if $x \in \partial(A + s)$, then by Lemma 9 we know $d(x, A) = s$. Therefore, $d(A + s, A) = s$. It follows that $h(A + s, A) = s$. ■

Consider the sets shown in Figure 10. The two point sets A and B are $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Note that $b_1 = (0, 2)$, $b_2 = (0, -2)$, and $a_1 = (0, 0)$. The point a_2 is located so that $d(a_2, B) = d(a_2, b_2) > d(a_1, B) = 2$. Thus, $d(A, B) > 2$. Furthermore, $d(b_1, A) = d(b_2, A) = 2$, which implies $r = h(A, B) = d(A, B) > d(B, A)$. The distance r is shown by the thin segment

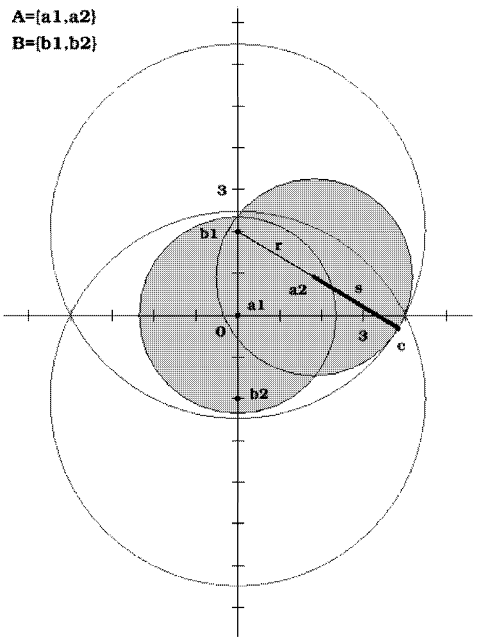

Figure 11: There are no sets $C \in \mathcal{H}(\mathbb{R}^n)$ satisfying (10).

$\overline{b_1a_2}$. The distance s is shown by the bold segment $\overline{a_2c}$, where $c \in \overrightarrow{b_1a_2}$ so that $d(b_1, c) = r+s$. The value of s was chosen so that $\overline{b_1c}$ would lie completely above the x-axis. We then have $c \in (A+s) \cap \partial N_{r+s}(B)$, so the results of Corollary 1 do not apply.

The set $(B+r+s)$ is the lighter shaded region along with its boundary, while $(A+s)$ is the darker shaded region with its boundary. Let $C = A+s$. By Lemma 13, $h(A,C) = s$. Now, $C = (A+s) \subseteq (B+r+s)$, so if $y \in C$, then $d(y,B) \leq r+s$. Since $d(c,B) = r+s$, we have $d(C,B) = r+s$. Furthermore, $N_s(a_2) \subset N_{r+s}(b_1)$ and $N_s(a_1) \subset N_{r+s}(b_2)$, so $d(B,C) < r+s$. We conclude that $h(C,B) = r+s$. Thus, $h(C,B) = h(C,A) + h(A,B)$.

Figure 11 illustrates the same sets A and B. However, this time we have chosen s to be much larger than in Figure 10. In particular, s was chosen so that $\overline{b_1c}$ intersects the x-axis and c is below the x-axis. Notice that in this case, $(A+s) \subseteq N_{r+s}(B)$. Therefore, Theorem 8 states that there are no sets $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy (10). In addition, taking $s' > s$ will give the same result. So we see that $\overleftrightarrow{AB}$ is really a ray even though $(A+s) \subseteq N_{r+s}(B)$ does not hold for all $s > 0$. The results that follow describe this situation.

Lemma 14. *If* $A, B \in \mathcal{H}(\mathbb{R}^n)$*,* $A \subseteq B$*, and* $0 < s \leq t$*, then* $(A+s) \subseteq (B+t)$*.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $A \subseteq B$. Let $s, t \in \mathbb{R}$ so that $0 < s \leq t$ and let $x \in (A + s)$. Then there exists $y \in A \subseteq B$ so that $d(x, y) \leq s \leq t$. Thus, $x \in (B + t)$ and $(A + s) \subseteq (B + t)$, as desired. ■

Lemma 15. *If $A, B \in \mathcal{H}(\mathbb{R}^n)$, $A \subseteq B$, and $0 < s \leq t$, then $N_s(A) \subseteq N_t(B)$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $A \subseteq B$. Let $s, t \in \mathbb{R}$ so that $0 < s \leq t$ and let $x \in N_s(A)$. Then there exists $a_x \in A \subseteq B$ such that $x \in N_s(a_x)$. Thus, $d(x, a_x) < s \leq t$ and $x \in N_t(a_x)$. Since $a_x \in B$, we have $N_s(a_x) \subseteq N_t(B)$. Therefore, $x \in N_s(B)$ and $N_s(A) \subseteq N_s(B)$. ■

Theorem 9. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $r = h(A, B)$. If $(A + s_0) \subseteq N_{r+s_0}(B)$ for some $s_0 \geq 0$, then $(A + s) \subseteq N_{r+s}(B)$ for all $s \geq s_0$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and assume $(A + s_0) \subseteq N_{r+s_0}(B)$ for some $s_0 \geq 0$. Let $r = h(A, B)$. Let $s \geq s_0$ and choose $x \in A + s$. Lemma 14 tells us that $(A + s_0) \subseteq (A + s)$. We also know $N_{r+s_0}(B) \subseteq N_{r+s}(B)$ by Lemma 15. If $x \in (A + s_0)$, then $x \in N_{r+s_0}(B) \subseteq N_{r+s}(B)$. Suppose $x \in (A + s) - (A + s_0)$. Then there exists $a_x \in A$ such that $s_0 < d(x, a_x) \leq s$. Let $y \in \overline{xa_x}$ so that $d(a_x, y) = s_0$. Note that such an element exists because $d(x, a_x) > s_0$. Now, $d(a_x, y) + d(y, x) = d(a_x, x) \leq s$ implies $d(x, y) \leq (s - d(a_x, y))$. Thus,

$$d(x, y) \leq (s - s_0). \tag{16}$$

Since $d(a_x, y) = s_0$, we know $y \in A + s_0$. Therefore, $y \in N_{r+s_0}(B)$. So there exists $b_y \in B$ such that $y \in N_{r+s_0}(b_y)$, which implies

$$d(y, b_y) < (r + s_0). \tag{17}$$

The triangle inequality tells us

$$d(x, b_y) \leq d(x, y) + d(y, b_y). \tag{18}$$

Together with (16) and (17), (18) implies that $d(x, b_y) < r + s$. Therefore, $x \in N_{r+s}(b_y)$, so $x \in N_{r+s}(B)$. Thus, we see that in all cases $x \in A + s$ implies $x \in N_{r+s}(B)$. ■

Corollary 1 gave conditions that implied a Hausdorff line was actually a ray. If $A, B \in \mathcal{H}(\mathbb{R}^n)$ so that $r = d(A, B) > d(B, A)$ and $(A + s) \subseteq N_{r+s}(B)$ for all $s > 0$, then the Hausdorff line $\overleftrightarrow{AB}$ is equivalent to the ray $\overrightarrow{AB}$. There are no points on $\overleftrightarrow{AB}$ to the left of A. Theorems 8 and 9 together show that another situation may occur when a Hausdorff line is a ray. They imply that if $A, B \in \mathcal{H}(\mathbb{R}^n)$ so that $d(A, B) > d(B, A)$ and $(A + s) \subseteq N_{r+s}(B)$ for some, but not all values of $s > 0$, then the line may extend to the left of A some distance before terminating. There will be a value $s_c = \inf\{s \in \mathbb{R} \mid (A + s) \subseteq N_{r+s}(B)\}$ such that there are no sets on $\overleftrightarrow{AB}$ a distance greater than s_c to the left of A.

Our focus thus far has been on determining when a Hausdorff line fails to have a full continuum of points. That is, we have described situations in which Hausdorff lines are incomplete in a sense. The next section approaches Hausdorff lines from the other direction and shows when such lines *are* complete.

5.3 When $h(C, B) = h(C, A) + h(A, B)$ Is Satisfied

Theorem 8 tells us that (10), with $h(A, C) = s$, is not be satisfied if

1. $d(A, B) > d(B, A)$, and
2. $(A + s) \subseteq N_{r+s}(B)$.

Theorem 10 will show that if 1 does not hold for $A, B \in \mathcal{H}(\mathbb{R}^n)$, then for each $s > 0$, we can find a set $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = s$ and (10) is satisfied. In addition, Theorem 11 proves that if 2 is not true for sets $A, B \in \mathcal{H}(\mathbb{R}^n)$, then for each $s > 0$ there are *infinitely* many sets $h(A, C) = s$ and (10) is satisfied.

Theorem 10. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(B, A) \geq d(A, B)$. For each $s > 0$ there is a set $C \in \mathcal{H}(\mathbb{R}^n)$ such that $h(A, C) = s$ and $h(C, B) = h(C, A) + h(A, B)$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(B, A) \geq d(A, B)$. Let $s > 0$. Choose $b_0 \in B$ and $a_0 \in A$ such that $d(b_0, a_0) = d(B, A) = r$. Let $C = (A + s) - N_{r+s}(b_0)$.

Claim 1. $C \in C_s(A)$.

We will show that this is true by proving that all three conditions of the Circle Theorem (Theorem 3) are satisfied.

1. Clearly, $C \subseteq (A + s)$. By Lemma 2 we have that $C \subseteq \bigcup_{a \in A} \overline{N_s(a)}$.
2. Let $a \in A$. If $a \notin N_{r+s}(b_0)$, then $a \in C$. So $C \cap \overline{N_s(a)} \neq \emptyset$. Suppose $a \in N_{r+s}(b_0)$. Choose $y \in \overrightarrow{b_0 a}$ such that $y \in \partial N_{r+s}(b_0)$. Then $d(b_0, y) = d(b_0, a) + d(a, y)$, so

$$d(b_0, a) = d(b_0, y) - d(a, y). \tag{19}$$

   We also have $d(b_0, a) \geq d(b_0, A) = d(B, A)$. Thus,

$$d(b_0, a) \geq r. \tag{20}$$

   Substituting (19) into (20), we find that

$$d(a, y) \leq d(b_0, y) - r.$$

   Since $y \in \partial N_{r+s}(b_0)$, we also know that $d(b_0, y) = r + s$. Therefore,

$$d(a, y) \leq (r + s) - r.$$

   Then $d(a, y) \leq s$, so $y \in \overline{N_s(a)}$. Note that by Lemma 2 this also implies $y \in A + s$. Since $y \in \partial N_{r+s}(b_0)$, we know $y \notin N_{r+s}(b_0)$. Therefore, $y \in ((A + s) - N_{r+s}(b_0)) = C$. We conclude that $C \cap \overline{N_s(a)} \neq \emptyset$.

3. We will now prove that 3(b) of the Circle Theorem is satisfied. Let $x_0 \in \overrightarrow{b_0a_0}$ such that $d(a_0, x_0) = s$ and $d(b_0, x_0) = r + s$. Then $x_0 \in \partial N_s(a_0) \subset \overline{N_s(a_0)} \subseteq A + s$, so $x_0 \in A + s$. Also, $x_0 \in \partial N_{r+s}(b_0)$ and $x_0 \notin N_{r+s}(b_0)$. This shows that $x_0 \in ((A + s) - N_{r+s}(b_0)) = C$. It follows that $x_0 \in C \cap \partial N_s(a_0)$.

   We prove that $N_s(a_0) \subseteq N_{r+s}(b_0)$, so that $C \cap N_s(a_0) = \emptyset$. Let $x \in N_s(a_0)$. Then $d(x, a_0) < s$. Furthermore, $d(x, b_0) \leq d(x, a_0) + d(a_0, b_0)$. Since $d(a_0, b_0) = r$, we have $d(x, b_0) < r + s$. Thus, $x \in N_{r+s}(b_0)$ and $N_s(a_0) \subseteq N_{r+s}(b_0)$. It follows that if $z \in N_s(a_0)$, then $z \in N_{r+s}(b_0)$. This implies $z \notin C$. Therefore, $C \cap N_s(a_0) = \emptyset$.

Since all three conditions of the Circle Theorem are satisfied, $C \in C_s(A)$. □

Claim 2. $C \in C_{r+s}(B)$.

Once again, we prove $C \in C_{r+s}(B)$ by showing that all of the conditions of the Circle Theorem hold.

1. From condition 1 of the Circle Theorem and Lemma 2, we know $A \subseteq B + r$. By Lemma 14 we then know that $(A + s) \subseteq (B + r + s)$. Therefore, $C \subseteq (B + r + s)$. Lemma 2 then guarantees that $C \subseteq \bigcup_{b \in B} \overline{N_{r+s}(b)}$.

2. Let $b \in B$. Then $d(b, A) \leq d(B, A) = r$. Choose $a_b \in A$ so that $d(b, A) = d(b, a_b)$. Then,
$$d(b, a_b) \leq r. \tag{21}$$
   By part 2 of Claim 1, we know $C \cap \overline{N_s(a_b)} \neq \emptyset$. Let $c \in C \cap \overline{N_s(a_b)}$. Then
$$d(c, a_b) \leq s. \tag{22}$$
   Using the triangle inequality, we have
$$d(b, c) \leq d(b, a_b) + d(a_b, c). \tag{23}$$
   From (21), (22), and (23) it follows that $d(b, c) \leq r + s$. Thus, $c \in C \cap \overline{N_{r+s}(b)}$, so $C \cap \overline{N_{r+s}(b)} \neq \emptyset$.

3. We will prove that 3(b) of the Circle Theorem holds. Notice that in part 3 of Claim 1 we chose $x_0 \in \overrightarrow{b_0a_0}$ such that $d(a_0, x_0) = s$ and $d(b_0, x_0) = r + s$. We also argued that $x_0 \in C$ and $x_0 \in \partial N_{r+s}(b_0)$. It follows that $x_0 \in C \cap \partial N_{r+s}(b_0)$. Since by definition $C \cap N_{r+s}(b_0) = \emptyset$, condition 3(b) is satisfied.

We conclude that $C \in C_{r+s}(B)$. □

Since $C \in C_s(A)$ and $C \in C_{r+s}(B)$, it follows that $h(C, A) = s$ and $h(C, B) = r + s$. Therefore, $h(C, B) = h(C, A) + h(A, B)$ as desired. ■

We will need the next two lemmas to prove Theorem 11.

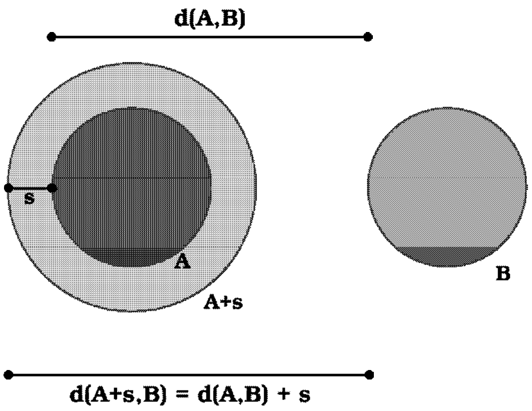

Figure 12: $d(A+s,B)=d(A,B)+s$

Lemma 16. *Let* $A,B\in\mathcal{H}(\mathbb{R}^n)$*. If* $s>0$*, then* $d(A+s,B)\le d(A,B)+s$*.*

Proof. Let $A,B\in\mathcal{H}(\mathbb{R}^n)$ with $d(A,B)=r$ and let $s\in\mathbb{R}$ with $s>0$. Let $x\in A{+}s$. Then there is an element $a\in A$ so that $d(x,a)\le s$. Since $d(A,B)=r$, there is an element $b\in B$ with $d(a,b)\le r$. Then

$$d(x,b)\le d(x,a)+d(a,b)\le r+s.$$

Therefore, $d(x,B)\le r+s$ for every $x\in A+s$. Thus, $d(A+s,B)\le r+s$. ■

If $A,B\in\mathcal{H}(\mathbb{R}^n)$ and $s>0$, then it is possible to have either $d(A+s,B)=d(A,B)+s$ or $d(A+s,B)<d(A,B)+s$. Figure 12 shows sets A and B for which $d(A+s,B)=d(A,B)+s$. For the sets C and D in Figure 13, however, $d(C+r,D)=d(C,D)$ because both distances are measured from the center of C. Therefore, $d(C+r,D)<d(C,D)+r$.

Lemma 17. *Let* $B\in\mathcal{H}(\mathbb{R}^n)$ *and* $r>0$*. If* $x\in\partial N_r(B)$*, then* $d(x,B)=r$*.*

Proof. Let $B\in\mathcal{H}(\mathbb{R}^n)$ and $r>0$. Let $x\in\partial N_r(B)$. By Lemma 5 we know that $x\in\left(\bigcup_{b\in B}\partial N_r(b)-N_r(B)\right)$. Since $x\notin N_r(B)$, we know that for every $b\in B$,

$$d(x,b)>r. \tag{24}$$

Furthermore, $x\in\bigcup_{b\in B}\partial N_r(b)$ implies that for some $b_x\in B$, it holds that $x\in\partial N_r(b_x)$. Thus,

$$d(x,b_x)=r. \tag{25}$$

It follows from (24) and (25) that $d(x,B)=r$. ■

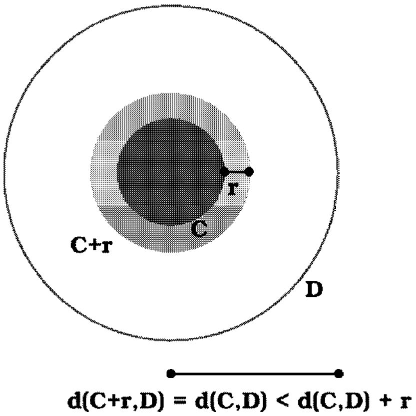

Figure 13: $d(C+r,D) < d(C,D)+r$

Theorem 11. *Let* $A,B \in \mathcal{H}(\mathbb{R}^n)$ *with* $h(A,B)=r$. *If* $(A+s)\cap \partial N_{r+s}(B) \neq \emptyset$ *for some* $s>0$, *then there are infinitely many sets* $C \in \mathcal{H}(\mathbb{R}^n)$ *such that:*

1. $h(A,C)=s$, *and*
2. $h(C,B)=h(C,A)+h(A,B)$.

Proof. Let $A,B \in \mathcal{H}(\mathbb{R}^n)$ with $h(A,B)=r$. Assume $(A+s)\cap \partial N_{r+s}(B) \neq \emptyset$ for some $s>0$. Let $x \in (A+s)\cap \partial N_{r+s}(B)$. We will first show $d(x,A)=s$. Since $x \in (A+s)$, we know $d(x,A)=d(x,a_x)\leq s$ for some $a_x \in A$. Suppose $d(x,a_x)<s$. Then for all $b\in B$, we have $d(x,b)\leq d(x,a_x)+d(a_x,b)$. Thus,

$$d(x,b) < s + d(a_x,b).$$

Now, $d(a_x,B)\leq d(A,B)\leq h(A,B)=r$ implies that there is an element $b_{a_x}\in B$ such that $d(a_x,b_{a_x})\leq r$. Therefore,

$$d(x,b_{a_x}) < r+s.$$

This implies that $d(x,B)<r+s$. However, Lemma 17 states that $d(x,B)=r+s$. We conclude that the assumption that $d(x,A)<s$ was false, so $d(x,A)=s$.

Let $t\in\mathbb{R}^n$ such that $0\leq t\leq s$. Let $C=(A+t)\cup\{x\}$. Since $A\subset C$, we know that $d(A,C)=0$. Let $c\in C$. If $c\in(A+t)$, then $d(c,A)\leq t\leq s$. If $c=x$, then $d(c,A)=s$. Therefore, $d(C,A)=h(C,A)=s$, which verifies 1.

We will now compute $h(B,C)$. If $c\in(A+t)$, then Lemma 16 tells us that $d(c,B)\leq r+t\leq r+s$. If $c=x$, then $c\in\partial N_{r+s}(B)$. This implies $d(c,B)=r+s$. So $d(C,B)=r+s$. Let $b\in B$. Since $d(B,A)\leq h(A,B)=r$, there is an element $a\in A\subset C$ such that $d(b,a)\leq r$. Therefore, $d(B,C)\leq r\leq r+s$, so $h(B,C)=r+s$. It is easily verified that 2 holds. Since there are infinitely many values that t may assume on the interval $[0,s]$, there are infinitely many sets $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy both 1 and 2. ■

Corollary 3. *Let* $A, B \in \mathcal{H}(\mathbb{R}^n)$ *with* $h(A,B) = r$. *If* $(A+s) \cap \partial N_{r+s}(B) \neq \emptyset$ *for all* $s > 0$, *then for each* $s > 0$ *there are infinitely many sets* $C \in \mathcal{H}(\mathbb{R}^n)$ *such that:*

1. $h(A,C) = s$, *and*
2. $h(C,B) = h(C,A) + h(A,B)$.

If we have sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfying the conditions of Corollary 3, then $\overleftrightarrow{AB}$ contains a full range of points to the left of A. That is, at any distance to the left of A, there are infinitely many points in $\mathcal{H}(\mathbb{R}^n)$ on $\overleftrightarrow{AB}$. The same is true of the sets A and B satisfy the conditions of Theorem 10.

6 Conclusion

A seemingly simple example of a Hausdorff line (that of Section 3.3) gave a surprising and counterintuitive result. We never expected that there would be situations in which the triangle equalities could not always be satisfied for lines in $\mathcal{H}(\mathbb{R}^n)$. The results we have given thus far give a good picture of the types of lines to be found in $\mathcal{H}(\mathbb{R}^n)$. Specifically, there are three situations that my occur for an arbitrary Hausdorff line, as opposed to the single case for Euclidean lines. For points $A, B \in \mathcal{H}(\mathbb{R}^n)$, it may be true that

1. $\overleftrightarrow{AB}$ contains a complete continuum of points.
2. $\overleftrightarrow{AB}$ is equivalent to either $\overrightarrow{AB}$ or $\overrightarrow{BA}$. That is, $\overleftrightarrow{AB}$ is actually a ray that stops at A or B.
3. $\overleftrightarrow{AB}$ is a ray, but extends beyond both A and B.

While nearly all questions have been answered concerning where points exist on an arbitrary Hausdorff line, there is still direction for further research. For example, the intersection properties of lines in $\mathcal{H}(\mathbb{R}^n)$ have not been completely determined. One may also try to formulate a concept of continuity for Hausdorff lines. Whatever direction one may take, it is clear that it will require creative thinking and hard work. For the geometry of $\mathcal{H}(\mathbb{R}^n)$ is unlike anything we had expected.

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