

Final Report

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1 The Basics

1.1 The Metric

We will be working with the geometry of the space of all non-empty compact subsets of \mathbb{R}^n , which we will denote as $\mathcal{H}(\mathbb{R}^n)$, under the Hausdorff metric. Later, we utilize this metric to define lines and circles. In order to begin, we define a metric:

Definition 1.1. Let X be a set. A function $d : X \times X \rightarrow \mathbb{R}$ is a metric on X if for all $x, y, z \in X$,

1. $d(x, y) = d(y, x)$,
2. $d(x, y) \geq 0$ with equality if and only if $x = y$,
3. $d(x, z) \leq d(x, y) + d(y, z)$.

A metric is a function which measures distance on a space. The Hausdorff metric, defined below, imposes a geometry on the space $\mathcal{H}(\mathbb{R}^n)$, which will be the subject of our study:

We will denote the Euclidean distance between x and y as $d_E(x, y)$.

Definition 1.2. Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$.

- If $x \in \mathbb{R}^n$, the “distance” from x to B is

$$d(x, B) = \min_{b \in B} \{d_E(x, b)\}.$$

A picture of this can be seen in Figure 1.

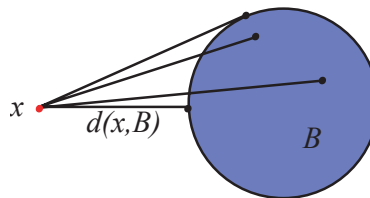


Figure 1: Distance from a point to a compact set.

- The “distance” from A to B is

$$d(A, B) = \max_{x \in A} \{d(x, B)\}.$$

Note that this is not a metric as described in definition 1.1, since $d(A, B)$ can be different than $d(B, A)$ as shown in figure 2.

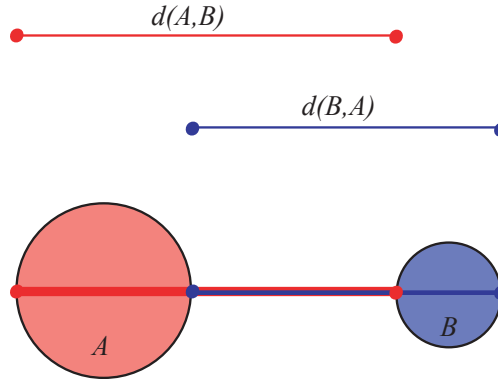


Figure 2: Distance from a compact set to a compact set.

- The Hausdorff distance, $h(A, B)$, between A and B is

$$h(A, B) = d(A, B) \vee d(B, A),$$

where $d(A, B) \vee d(B, A) = \max\{d(A, B), d(B, A)\}$.

1.2 The topology of $\mathcal{H}(\mathbb{R}^n)$

We will be looking at the properties of the space $\mathcal{H}(\mathbb{R}^n)$, and in this exploration we will utilize the idea of a topology on the space.

Definition 1.3. A topology on a set X is a collection τ of subsets of X having the following properties:

1. The empty set and X are in τ ,
2. The union of the elements of any subcollection of τ is in τ ,
3. The intersection of the elements of any finite subcollection of τ is in τ .

We will call the sets of τ open sets of X . A set X for which a topology τ has been specified is called a topological space.

Given the three point sets in Figures 3 and 4, the collections of circled sets in Figure 3 are examples of topologies whereas the collections of circled sets in Figure 4 are not.

The idea of a topology will be especially helpful in our study of compact sets of \mathbb{R}^n . The topological treatment of the idea of continuity will be particularly useful in showing that the metric in question is well-defined. In a topological space, continuity is defined as below, see [2]:



Figure 3: Topologies

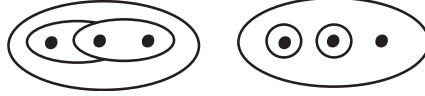


Figure 4: Not topologies

Definition 1.4. A metric space is a set X together with a metric d . [3]

We assert that $(\mathcal{H}(\mathbb{R}^n), h)$ is a metric space. To prove this, we will verify that the conditions of Definition 1.1 are met by $(\mathcal{H}(\mathbb{R}^n), h)$ in the following propositions.

Proposition 1.1. If A and B are in $\mathcal{H}(\mathbb{R}^n)$, then $h(A, B) = h(B, A)$

Proof. Let A and B be elements of $\mathcal{H}(\mathbb{R}^n)$. By the definition of the function h we see that $h(A, B) = \max\{d(A, B), d(B, A)\} = \max\{d(B, A), d(A, B)\} = h(B, A)$. \square

Proposition 1.2. Let A and B in $\mathcal{H}(\mathbb{R}^n)$. Then $h(A, B) \geq 0$ with equality if and only if $A = B$.

Proof. By definition we know $h(A, B) = \max\{d(A, B), d(B, A)\}$ and $d(A, B) = \max\{d(x, B) : x \in A\}$. But $d(x, B) = \min\{d_E(x, b) : b \in B\}$, and $d_E(x, y) \geq 0$, for all $x, y \in \mathbb{R}$. This means that for all $x \in A$, we have $d(x, B) \geq 0$, implying that $d(A, B) \geq 0$. A similar argument shows that $d(B, A) \geq 0$. Therefore, $h(A, B) \geq 0$.

To show we have equality if and only if the sets are equal we will begin by assuming $h(A, B) = 0$. It then follows that $d(A, B) = d(B, A)$, because by definition $h(A, B) = \max\{d(A, B), d(B, A)\} = 0$. We also know the distance from A to B and the distance from B to A are never negative. Thus $d(A, B) = 0$, which means that $d(x, B) = 0$ for all $x \in A$. By definition we then know $\min_{b \in B}\{d_E(x, b)\} = 0$ for all $x \in A$. Therefore, for every x in A there is a b in B such that $x = b$, which then implies that $A \subseteq B$. A similar argument shows that $B \subseteq A$. Thus we can conclude that $A = B$

We will now assume that $A = B$. We then know that for all $a \in A$ there exists $b \in B$ such that $a = b$. This then implies $\min_{b \in B}\{d_E(a, b)\} = 0$ for all $a \in A$. Thus we see that $d(x, B) = 0$ for all $x \in A$, from which it follows that $d(A, B) = 0$. Again, a similar argument proves that $d(B, A) = 0$. Therefore we know that $d(A, B) = d(B, A) = 0 = h(A, B)$. \square

To verify that the triangle inequality holds in the metric space $(\mathcal{H}(\mathbb{R}^n), h)$, we present a proof given in [6].

Proposition 1.3. Given A, B , and C in $\mathcal{H}(\mathbb{R}^n)$, $h(A, B) \leq h(A, C) + h(C, B)$

Proof. We first show that $d(A, C) \leq d(A, B) + d(B, C)$ for arbitrary sets A, B , and C in $\mathcal{H}(\mathbb{R}^n)$. For any $a \in A, b \in B$, and $c \in C$, the triangle inequality for d_E guarantees that

$$d_E(a, c) \leq d_E(a, b) + d_E(b, c).$$

For each $b \in B$ there is a $c' \in C$ such that $d_E(b, c') = d(b, C)$. Then,

$$d_E(a, c') \leq d_E(a, b) + d(b, C).$$

for all $a \in A, b \in B$ Since $d(a, C) \leq d_E(a, c')$,

$$d(a, C) \leq d_E(a, b) + d(b, C)$$

for all $a \in A, b \in B$ Now, choose $b' \in B$ such that $d(a, b') = d(a, B)$. Then,

$$d(a, C) \leq d(a, B) + d(b', C).$$

for all $a \in A$. Since $d(b', C) \leq d(B, C)$,

$$d(a, C) \leq d(a, B) + d(B, C)$$

for all $a \in A$. Choose $a' \in A$ such that $d(a', C) = d(A, C)$. Then,

$$d(A, C) \leq d(a', B) + d(B, C).$$

Since $d(a', B) \leq d(A, B)$, we have the final result that

$$d(A, C) \leq d(A, B) + d(B, C).$$

The Hausdorff distance from A to C is $h(A, C) = \max\{d(A, C), d(C, A)\}$. Applying the final result above, we see that

$$\begin{aligned} h(A, C) &\leq \max\{d(A, B) + d(B, C), d(C, B) + d(B, A)\} \\ h(A, C) &\leq \max\{d(A, B), d(B, A)\} + \max\{d(B, C), d(C, B)\} \\ h(A, C) &\leq h(A, B) + h(B, C). \end{aligned}$$

□

Further, we claim that h induces a topology on the space $\mathcal{H}(\mathbb{R}^n)$, in the manner described below.

Definition 1.5. Let (X, d) be a metric space and let $x \in X$. Then the ϵ -neighborhood of x , denoted $N_\epsilon(x)$, is defined as the set $\{y \in X : d(x, y) < \epsilon\}$.

Definition 1.6. Let (X, d) be a metric space. Let $\mathcal{C} = \{\bigcup_{\epsilon > 0, x \in X} N_\epsilon(x)\}$. Then, \mathcal{C} is called the metric topology induced by d .

To see \mathcal{C} is indeed a topology, see [2].

From the definitions above, we are able to see that the Hausdorff metric induces a topology on the space $\mathcal{H}(\mathbb{R}^n)$ and later we will see that it also induces a geometry on $\mathcal{H}(\mathbb{R}^n)$.

For the remainder of this paper, we assume all spaces are metric spaces. In a metric space X , open sets are arbitrary unions of ϵ -neighborhoods. Furthermore, a set $A \in X$ is closed if and only if $X - A$ is open. [2]

1.3 Compactness

We now present several distinct sets of conditions a subspace of a topological space can satisfy that are equivalent definitions of compactness.

Definition 1.7. • A collection \mathcal{A} of subsets of a space X is said to cover, or be a covering of X , if the union of the elements of \mathcal{A} is equal to X . Then \mathcal{A} is called an open covering of X if the elements of \mathcal{A} are open subsets of X .

- A space X is compact if every open covering \mathcal{A} of X contains a finite subcollection that also covers X .
- A subset B of X is said to be compact if for any open covering \mathcal{A} of X the set $\{A_i \cap B : A_i \in \mathcal{A}\}$ contains a finite subcollection which also covers B .

We note that any space containing only finitely many points is compact. An example of a compact subset of a metric space is $[0, 1] \subset \mathbb{R}$ under the metric topology induced by the Euclidean metric. On the other hand the open interval $(0, 1)$ is not a compact subset of \mathbb{R} , see [2]. Also note that \mathbb{R}^n is not compact under the Euclidean metric topology.

Definition 1.8. Given a set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, we say x is a limit point of A if for any $\epsilon > 0$, $N_\epsilon(x)$ contains a point $y \in A$ satisfying $x \neq y$.

Definition 1.9. A subset B of a space X is said to be limit-point compact if every infinite subset of B has a limit point.

Definition 1.10. A space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence. A subset A of X is said to be sequentially compact if every sequence of points in A has a convergent subsequence, which converges to some $x \in A$.

Definition 1.11. If X is a topological space, X is said to be metrizable if there exists a metric d on the set that induces the topology of X .

The following theorem comes from [2] and is helpful in understanding the space $\mathcal{H}(\mathbb{R}^n)$.

Theorem 1.1. Let A be a subset of a metrizable space. Then the following are equivalent:

1. A is compact
2. A is limit point compact
3. A is sequentially compact

1.4 Compact sets and Infimums

A question which arises when considering the definition of the Hausdorff metric is our level of certainty that the distance function is actually defined for any two points in the space. For instance, how can we be sure that $d(x, B)$ exists for any point $x \in \mathbb{R}^n$ and any set $B \in \mathcal{H}(\mathbb{R}^n)$? To be certain, we must prove Theorem 1.3. To do so, we will utilize the Heine-Borel Theorem, stated below, see [3].

Theorem 1.2. [Heine-Borel] A subset A of \mathbb{R}^n is compact if and only if A is both closed and bounded.

Definition 1.12. Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is said to be continuous if for each open set V of Y , the set $f^{-1}(V)$ is an open subset of X .

Theorem 1.3. Given any $x \in \mathbb{R}^n$ and any set $B \in \mathcal{H}(\mathbb{R}^n)$, the set $\{d(x, b) : b \in B\}$ contains a minimum value.

Proof. Let $x \in \mathbb{R}^n$, $B \in \mathcal{H}(\mathbb{R}^n)$. Let $d' : B \rightarrow \mathbb{R}$ be defined by $d'(b) = d(x, b)$. We note that B is a compact set by definition. Then we claim d' is a continuous function. To see that d' is continuous, let I be the open interval (u, v) with $(u, v) \subset \mathbb{R}^+$ and consider $d'^{-1}(I)$. Then $d'^{-1}(I) = \{b \in B : u < d(x, b) < v\}$. Then $d'^{-1}(I)$ is an open set, because $\{b \in B : u < d(x, b) < v\} = N_v(x) \cap \overline{N_u(x)}^c$. This is the intersection of two open sets, which we know to be open. Therefore, we are assured that d' is a continuous function. Now, the continuous image of a compact set is compact, see [2], implying that $d'(B) = \{d(x, b) : b \in B\}$ is compact. Since $d'(B)$ is a closed subset of the real line (Theorem 1.2), it must contain its infimum. Hence, d attains a minimum value for all $x \in \mathbb{R}^n$ and for all $B \in \mathcal{H}(\mathbb{R}^n)$. □

Therefore, $d(x, B)$ is defined for all $x \in \mathbb{R}^n$ and all $B \in \mathcal{H}(\mathbb{R}^n)$.

2 The Geometry of $\mathcal{H}(\mathbb{R}^n)$

2.1 Extensions

Given a set $B \in \mathcal{H}(\mathbb{R}^n)$ and an $r > 0$, the set $B + r = \{x \in \mathbb{R}^n : d(x, b) \leq r \text{ for some } b \in B\}$ is known as the extension of the point B by r .

Proposition 2.1. Let $B \in \mathcal{H}(\mathbb{R}^n)$ and let $r > 0$. Then $B + r$ is a compact set that is a distance r from B .

Proof. We will use Theorem 1.2 to show that the set $B + r$ is compact, by showing that it is both closed and bounded. Now consider the set $X - (B + r) = \{x \in X : d(x, b) > r \text{ for all } b \in B\}$, which is the complement of the set $(B + r)$. Let $x_0 \in X - (B + r)$. Then we have $d(x_0, B) = \epsilon > r$. Let $0 < \delta < \epsilon - r$ and consider the intersection of the δ -neighborhood of the point x_0 and the set $B + r$. We claim that $N_\delta(x_0) \cap (B + r) = \emptyset$. Assume to the contrary that $N_\delta(x_0) \cap (B + r) \neq \emptyset$. Let $x_1 \in N_\delta(x_0) \cap (B + r)$. This then implies that $d(x_0, x_1) < \delta$ and $d(x_1, B) \leq r$. We can then conclude that there exists a $b_1 \in B$ such that $d(x_1, b_1) \leq r$. We also know $d(x_1, B) \leq d(x_1, b_1)$ by definition 1.2. Now by the triangle inequality $d(x_0, B) \leq d(x_0, b_1) \leq d(x_1, b_1) + d(x_1, x_0) < r + \delta < \epsilon$ which is a contradiction since we assumed $d(x_0, B) = \epsilon$. Therefore we can conclude that $X - (B + r)$ is an open set, which implies that $B + r$ is closed.

We note that B is bounded by a ball of radius $R \in \mathbb{R}$, or $d(b_0, b_1) \leq R$, for all $b_0, b_1 \in B$. Also note that $d(x, B) \leq r$ for all $x \in B + r$. Let $x_0 \neq x_1 \in B + r$, then $d(x_0, B) \leq r$ and $d(x_1, B) \leq r$. So there exists $b_0, b_1 \in B$ such that $d(x_0, b_0) \leq r$ and $d(x_1, b_1) \leq r$. Thus $d(x_0, x_1) \leq d(x_0, b_0) + d(b_0, b_1) + d(b_1, x_1) \leq r + R + r = R + 2r$. This then implies $B + r$ is bounded, or that there is a ball of radius $R + 2r$ that contains the set $B + r$. We are now able to conclude that $B + r$ is compact. □

2.2 Circles and Lines in $\mathcal{H}(\mathbb{R}^n)$

To begin to discuss circles and lines in $\mathcal{H}(\mathbb{R}^n)$, we will need the following definitions and propositions.

Definition 2.1. Given a set A , the closure of A , denoted \overline{A} , is the intersection of all closed sets containing A .

Definition 2.2. The interior of a set A , denoted $\text{Int}A$, is the largest open set contained in A .

Definition 2.3. The boundary of a set A , written ∂A , is $\overline{A} - \text{Int}A$.

To illustrate, if we have a set defined to be the unit disk then the closure of the set would be the entire disk, the interior would be the open unit disk and the boundary would be the unit circle.

The following lemma is important when proving Theorem 2.1.

Lemma 2.1. Given the set B in $\mathcal{H}(\mathbb{R}^n)$ and $r \in \mathbb{R}^+$, we have $\partial(\bigcup_{b \in B} N_r(b)) = (\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b))$

Proof. We will begin by letting $x \in (\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b))$. This implies that there exists $b_0 \in B$ such that $x \in \partial N_r(b_0)$ and that there is no $b \in B$ which satisfies $x \in N_r(b)$. This implies the existence of b_0 implies that $d(x, B) \leq r$, while the lack of a suitable b means that $d(x, B) \geq r$. These facts together imply that $d(x, B) = r$.

Our claim is that x must be an element of $\partial(\bigcup_{b \in B} N_r(b))$. We will show this by contradiction.

Suppose $x \notin \partial(\bigcup_{b \in B} N_r(b))$. That would imply the existence of a $\delta > 0$ such that either $N_\delta(x) \cap (\bigcup_{b \in B} N_r(b)) = \emptyset$, or $N_\delta(x) \subset (\bigcup_{b \in B} N_r(b))$.

Now, if $N_\delta(x) \cap (\bigcup_{b \in B} N_r(b)) = \emptyset$, then for all $b \in B$, we have $d(x, b) > r$. This contradicts the fact that $d(x, B) = r$.

Furthermore, if $N_\delta(x) \subset (\bigcup_{b \in B} N_r(b))$ then there exists $b \in B$ such that $d(x, b) < r$. This implies that $d(x, B) < r$. Again, this is a contradiction, and x must indeed be a point in $\partial(\bigcup_{b \in B} N_r(b))$.

Resultingly, $(\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b)) \subset \partial(\bigcup_{b \in B} N_r(b))$.

Next we will show that $\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b) \supset \partial(\bigcup_{b \in B} N_r(b))$.

Let $x \in \partial(\bigcup_{b \in B} N_r(b))$. We now claim that $d(x, B) = r$. To prove our claim, we assume by contradiction that $d(x, B) > r$ or $d(x, B) < r$.

Now, if $d(x, B) > r$, then some $b_0 \in B$ exists such that $r < d(x, b_0) \leq d(x, b)$ for all $b \in B$. We can then find a δ satisfying $d(x, b_0) - r > \delta > 0$ such that for all $x_0 \in N_\delta(x)$ we have $d(x_0, B) > r$. Then, $N_\delta(x) \cap N_r(b) = \emptyset$, for all $b \in B$, and $N_\delta(x) \cap (\bigcup_{b \in B} N_r(b)) = \emptyset$. This contradicts the fact that $x \in \partial(\bigcup_{b \in B} N_r(b))$.

Next we assume $d(x, B) < r$. This implies that there exists $b_0 \in B$ such that $d(x, b_0) = r_0 < r$. Then we can find a $\delta_0 > 0$ where $r - r_0 > \delta_0$ such that $N_{\delta_0}(x) \subset N_r(b_0)$. This yields the contradiction, $N_{\delta_0}(x) \subset \bigcup_{b \in B} N_r(b)$.

So we see that $d(x, B)$ must indeed be r . As we have shown previously, this implies that there exists a $b_1 \in B$ such that $d(x, b_1) = r$. Thus, $x \in \partial N_r(b)$, which in turn implies that $x \in \bigcup_{b \in B} \partial N_r(b)$.

We now claim that $x \notin \bigcup_{b \in B} N_r(b)$. If this is not the case, then there must exist a $b_2 \in B$ such that $d(x, b_2) < r$. This contradicts the fact that $d(x, B) = r$. Thus $x \in \bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b)$. Therefore $\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b) \supset \partial(\bigcup_{b \in B} N_r(b))$.

Combining the above arguments, $\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b) = \partial(\bigcup_{b \in B} N_r(b))$, and the lemma is proved. \square

Definition 2.4 and Theorem 2.1 are essential for determining points on lines in $\mathcal{H}(\mathbb{R}^n)$.

Definition 2.4. Let $B \in \mathcal{H}(\mathbb{R}^n)$ and $r \in \mathbb{R}^+$. The Hausdorff circle of radius r centered at point B , denoted $\mathcal{C}_r(B)$, is the set $\{A \in \mathcal{H}(\mathbb{R}^n) : h(A, B) = r\}$.

Theorem 2.1, see [1], will aid in visualizing which sets lie a given distance from a fixed set, which will prove useful when determining points on lines in $\mathcal{H}(\mathbb{R}^n)$.

Theorem 2.1. Let B be a non-empty compact subset of \mathbb{R}^n . Then $A \in \mathcal{C}_r(B)$, the Hausdorff circle of radius r centered at B , if and only if A is a non-empty compact subset of \mathbb{R}^n and

- (a) $A \subseteq \bigcup_{b \in B} \overline{N_r(b)}$.
- (b) $A \cap \overline{N_r(b)} \neq \emptyset$ for each $b \in B$.
- (c) Either $A \cap \partial \left(\bigcup_{b \in B} N_r(b) \right) \neq \emptyset$ or there exists $b \in B$ and $a \in A \cap \partial N_r(b)$ with $A \cap N_r(b) = \emptyset$.

We now provide an example of a Hausdorff circle. Using the content of Theorem 2.1 we will completely describe the set of points on the circle centered at $B = \{(0, 0), (1, 0)\}$ with radius 1.

If $A \in \mathcal{H}(\mathbb{R}^n)$ is a point on $\mathcal{C}_1(B)$, then:

1. A must lie within the union of the closed neighborhoods of radius 1 about each $b \in B$,
2. A must have at least one point within each of the closed neighborhoods of radius 1 about each $b \in B$,
3. and either A has a point of the boundary of the union of all neighborhoods of radius 1 about each $b \in B$,

or there is a point in A that is on the boundary of the neighborhood of radius 1 about one point in B where no other point of A lies within that same neighborhood.

In Figure 5, $A \in \mathcal{C}_1(B)$ must be a closed subset of $B + 1$ behaving as described above.

2.3 Hausdorff Lines and Segments: Theorems and Definitions

The following list of theorems and lemmas are listed for easy reference and prove to be of use in understanding the space $\mathcal{H}(\mathbb{R}^n)$ and can be found in [1].

Definition 2.5. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$. The Hausdorff line defined by A and B is the set of all points $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy one of

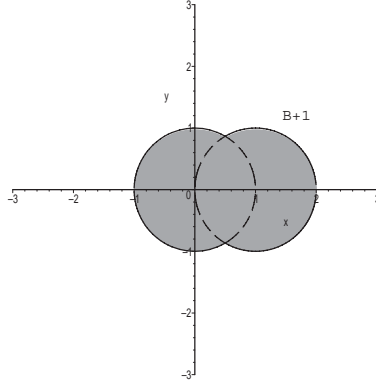


Figure 5: Hausdorff circle example

1. $h(A, B) = h(A, C) + h(C, B)$,
2. $h(A, C) = h(A, B) + h(B, C)$,
3. $h(C, B) = h(C, A) + h(A, B)$.

If $h(A, B) = h(A, C) + h(C, B)$, we say that C is between A and B , and denote this by ACB .

This definition tells us exactly how we classify all the points that lie on the Hausdorff line \overleftrightarrow{AB} , in that they lie between the points A and B , to the right of B , or to the left of A .

Definition 2.6. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$. The Hausdorff segment between A and B , denoted \overline{AB} , is the set of all points $C \in \mathcal{H}(\mathbb{R}^n)$ such that ACB .

Definition 2.7. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ and $C, C' \in \overline{AB}$. The sets C, C' are said to be at the same location on the Hausdorff segment between A and B if $h(B, C) = h(B, C') = s$ for some $0 < s < h(A, B)$

The following theorem tells us exactly how to find other points at a given location.

Theorem 2.2. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ with $d(B, A) \geq d(A, B)$. Let $r = h(A, B)$. Let $s \in \mathbb{R}$ with $0 < s < r$, and let $t = r - s$. If C is a compact subset of $(A + s) \cap (B + t)$ containing $\partial((A + s) \cap (B + t))$, then C satisfies ACB with $h(A, C) = s$ and $h(B, C) = t$.

We will see from the next theorem, see [1], that $C \in \mathcal{H}(\mathbb{R}^n)$ will lie on the Hausdorff line defined by $A, B \in \mathcal{H}(\mathbb{R}^n)$, given specific conditions.

Theorem 2.3. Let $a, b \in \mathbb{R}^n$. Let $A = \{a\}$, $B = \{b\}$, and $C \in \mathcal{H}(\mathbb{R}^n)$. Then C lies on the Hausdorff line defined by A and B if and only if there exist $r, s \in \mathbb{R}$ such that:

1. $C \subseteq (A + r) \cap (B + s)$ and
2. there exists $c_0 \in C$ such that $c_0 \in \partial(A + r) \cap \partial(B + s) \cap \overleftrightarrow{ab}$.

One interesting question arising in this study is: Is the geometry of $\mathcal{H}(\mathbb{R}^n)$ an incidence geometry like standard Euclidean geometry? In answering this question, we will begin by defining an incidence geometry.

Definition 2.8. An incidence geometry is a set of points, S , and a set of lines, L , which satisfy the following conditions:

1. for each pair of points P and Q in S , there exists a unique line $m \in L$ such that $P, Q \in m$.
2. for all $m \in L$, there exists at least 2 points $A, B \in m$.
3. there exists 3 distinct points $P, Q, R \in S$ such that there does not exist $m \in L$ with $P, Q, R \in m$.

We claim that the geometry of $\mathcal{H}(\mathbb{R}^n)$ is actually not an incidence geometry, because the first condition is not satisfied. That is, there exist two points P and Q in $\mathcal{H}(\mathbb{R}^n)$ that fail to uniquely characterize a Hausdorff line. For example, in Figure 6, the shaded region is on the lines $\overleftrightarrow{AB} \neq \overleftrightarrow{AC}$, with $A = \{a\}$, $B = \{b\}$, and $C = \{c\}$. Given the points $b_0 \in B$ and $c_0 \in C$ as seen in Figure 6, it follows from Theorem 2.3 that the point $Q = \{a, b_0, c_0\}$ is also on the lines \overleftrightarrow{AB} and \overleftrightarrow{AC} ! This means that there are at least two distinct lines in $\mathcal{H}(\mathbb{R}^n)$ which both contain these two distinct points, and the incidence conditions are not satisfied.

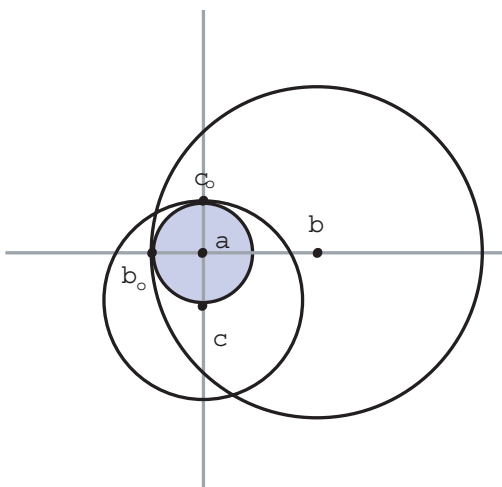


Figure 6: Two distinct points do not determine a unique line.

2.4 Finding points on a Hausdorff Segment

Example 2.1.

Given the points

$$A = \{(0.5, 0.5), (0.5, -0.5), (-0.5, 0.5), (-0.5, -0.5)\}$$

and

$$B = \{(2, 0), (0, 2), (-2, 0), (0, -2)\}$$

We see that for any $s, t \in \mathbb{R}^+$ with $r = s + t$, then $C = (A + s) \cap (B + t)$ is the collection of eight points and lies on the Hausdorff segment \overline{AB} as illustrated in Figure 7:

We will find all 47 points that lie on the Hausdorff segment between A and B . To begin, we note that the largest point between A and B is $C = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where the labeling system is shown in Figure 7. To find the other 46 points, we will be taking certain subsets of C as follows:

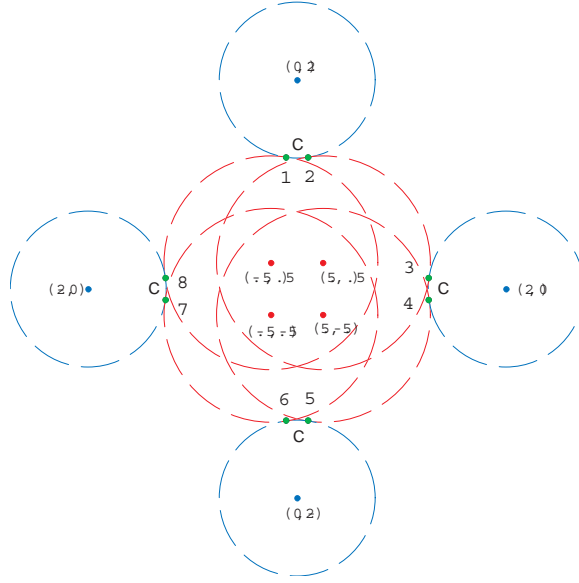


Figure 7: There are 47 points at each location on the Hausdorff segment between A and B .

1. $C - \{a\}$ where $a \in C$ (There will be 8 elements)
2. $C - \{a, b\}$ where $a \neq b \in C$ and a and b are not adjacent to each other (There will be 20 elements)
3. $C - \{a, b, c\}$ where $a \neq b \neq c \in C$ and a is not adjacent to b or c and b is not adjacent to c (There will be 16 elements)
4. $C - \{1, 3, 5, 7\}$ and $C - \{2, 4, 6, 8\}$

It is left as an exercise for the reader to verify that each of these sets lies at the same location as C on \overline{AB} . Thus we have found all 47 elements between A and B

3 Conditions for a finite number at each location on \overline{AB}

We will show that there are several conditions that must be met in order for two sets to have a finite number of points at each location between them.

Lemma 3.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. If all points $b \in B$ are equidistant from A and $h(A, B) = d(B, A) = r$, then every point $a \in A$ must satisfy $d(a, B) = r$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. Further, let all points $b \in B$ be equidistant from A and $h(A, B) = d(B, A) = r$. Note that these two conditions imply that $d(b, A) = r$ for all $b \in B$.

Clearly, no $a \in A$ exists such that $d(a, B) > r$, because then $h(A, B)$ is forced to be greater than r .

Assume then that there exists a point $a_0 \in A$ such that $d(a_0, B) < r$. Then there exists a $b_0 \in B$ such that $d(a_0, b_0) < r$. However, this implies that the distance from b_0 to A is also less than r . This is a contradiction of our condition that $d(b_0, A) = r$. Therefore, every $a \in A$ satisfies $d(a, B) = r$. \square

Lemma 3.2. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ and $h(A, B) = r$. Also, Let $0 < s < r$ and $t = r - s$. If two points, $a_0 \in A$ and $b_0 \in B$, exist such that $d(a_0, b_0) < r$, then $N_s(b_0) \cap N_t(a_0) \neq \emptyset$.*

Proof. Let $l(v) = \left(\frac{v}{d(a_0, b_0)}\right) \cdot b_0 + \left(\frac{d(a_0, b_0) - v}{d(a_0, b_0)}\right) \cdot a_0$ with $v \in [0, d(a_0, b_0)]$ be the Euclidean line segment $\overline{a_0 b_0}$. Note that this is an arc-length parameterization of the segment with $l(0) = b_0$ and $l(d(a_0, b_0)) = a_0$. That is, if $u, w \in [0, d(a_0, b_0)]$, then $|l(u) - l(w)| = |u - w|$.

Now, if $d(a_0, b_0) \leq s$, then a_0 and the open interval $(l(0), l(q)) \subseteq \overline{a_0 b_0}$, with $q = \min\{t, d(a_0, b_0)\}$, is contained in $N_s(b_0) \cap N_t(a_0)$. A similar open interval of $\overline{a_0 b_0}$ containing b_0 is in $N_s(b_0) \cap N_t(a_0)$ if $d(a_0, b_0) \leq t$. Therefore, we only need look at the case when $d(a_0, b_0) > s$ and $d(a_0, b_0) > t$.

In this case, let $0 < q < r - d(a_0, b_0) < r - t = s$ and consider the point $c = l(s - q)$. Since, $(s - q - 0) < (s - 0)$ and l is an arc-length parameterization, $d(b_0, c) = |l(s - q) - l(0)| < |l(s) - l(0)| = s$. Thus, $c \in N_s(b_0)$.

Now, since $d(a_0, c) + d(c, b_0) = d(a_0, b_0)$ and $d(b_0, c) = s - q$, by substitution $d(a_0, c) = d(a_0, b_0) - s + q$. By substitution, $d(a_0, c) < (d(a_0, b_0) - s + r - d(a_0, b_0)) = r - s = t$. Thus, $c \in N_t(a_0)$.

Therefore, $c \in N_s(b_0) \cap N_t(a_0)$ and our lemma is proved. \square

Theorem 3.1. *For A and B in $\mathcal{H}(\mathbb{R}^n)$, if the points of B are not all the same distance from A and if $r = h(A, B) = d(B, A)$ then there are infinitely many points at a given location on the Hausdorff segment between A and B .*

Proof. Let A and B be in $\mathcal{H}(\mathbb{R}^n)$. Assume all of the points of B are not equidistant from A and $r = h(A, B) = d(B, A)$. Then by assumption we know there exists $b_0 \neq b_1 \in B$ such that $d(b_0, A) < d(b_1, A)$. We also know that $r > d(b_0, A)$ by Definition 1.2. Let $a_0 \in A$ such that $d(b_0, a_0) = d(b_0, A)$. Let $0 < s < r$ and $t = r - s$. By Lemma 3.2, there exists $c \in N_s(b_0) \cap N_t(a_0)$.

Now, because $N_s(b_0) \cap N_t(a_0)$ is an open set, there exists a $\delta > 0$ such that $N_\delta(c) \subset (N_s(b_0) \cap N_t(a_0)) \subset ((B + s) \cap (A + t))$.

By Theorem 2.2, we can then choose any compact subset V of $N_\delta(c)$ and $V \cup \partial((B + s) \cap (A + t))$ will satisfy the conditions to be between A and B . Therefore there are infinitely many sets at each location on the Hausdorff segment between A and B . \square

Theorem 3.2. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. If all points b in B are equidistant from A and $r = h(A, B) = d(A, B) > d(B, A)$, then there are infinitely many points at every location between A and B on the Hausdorff segment \overline{AB} .*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$, with all points $b \in B$ equidistant from A and $r = h(A, B) = d(A, B) > d(B, A)$. Also, let $0 < s < r$ and $t = r - s$.

Because $d(b, A) = d(B, A) < r$ for all $b \in B$, we know that, for each $b_0 \in B$, there must exist an $a_0 \in A$ such that $d(b_0, a_0) < r$. But, by Lemma 3.2, this implies that $N_s(b_0) \cap N_t(a_0) \neq \emptyset$. This implies that for all $b_0 \in B$ the intersection of the closed s -neighborhood centered at b_0 and the t -extension of A , $\overline{N_s(b_0)} \cap (A + t)$, is non-empty with an infinite interior. Furthermore, since $(B + s) = \bigcup_{b \in B} \overline{N_s(b)}$, we know

that $(B + s) \cap (A + t)$ is also non-empty with an infinite interior. By Theorem 2.2, in order to construct an infinite number of points at this location on the Hausdorff line segment from A to B , we need only require that each point (sets in $\mathcal{H}(\mathbb{R}^n)$) be a compact subset of $(B + s) \cap (A + t)$ and contain $\partial((B + s) \cap (A + t))$ along with any combination of interior points. There are, of course, an infinite number of compact subsets of interior points. As a result, there are also an infinite number of sets on the Hausdorff segment \overline{AB} at the location s units from B , and our proof is complete

□

Theorem 3.3. *Let A, B be finite sets in $\mathcal{H}(\mathbb{R}^n)$. If all points $b \in B$ are equidistant from A and $r = h(A, B) = d(B, A) \geq d(A, B)$, then there is the same finite number of points at every location between A and B on the Hausdorff segment \overline{AB} .*

Theorem 3.4. *For A and B in $\mathcal{H}(\mathbb{R}^n)$, if the points of B are not all the same distance from A , then there are infinitely many points at a given location on the Hausdorff segment between A and B .*

Proof. Let A and B be in $\mathcal{H}(\mathbb{R}^n)$. Assume all of the points of B are not equidistant from A and $r = h(A, B) = d(B, A)$. Then by assumption we know there exists $b_0 \neq b_1 \in B$ such that $d(b_0, A) < d(b_1, A)$. We also know that $r > d(b_0, A)$ by Definition 1.2. Let $a_0 \in A$ such that $d(b_0, a_0) = d(b_0, A)$. Let $0 < s < r$ and $t = r - s$. By Lemma 3.2, there exists $c \in N_s(b_0) \cap N_t(a_0)$.

Now, because $N_s(b_0) \cap N_t(a_0)$ is an open set, there exists a $\delta > 0$ such that $N_\delta(c) \subset (N_s(b_0) \cap N_t(a_0)) \subset ((B + s) \cap (A + t))$.

By Theorem 2.2, we can then choose any compact subset V of $N_\delta(c)$ and $V \cup \partial((B + s) \cap (A + t))$ will satisfy the conditions to be between A and B . Therefore there are infinitely many sets at each location on the Hausdorff segment between A and B .

□

Theorem 3.5. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. If all points b in B are equidistant from A and $r = h(A, B) = d(A, B) > d(B, A)$, then there are infinitely many points at every location between A and B on the Hausdorff segment \overline{AB} .*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$, with all points $b \in B$ equidistant from A and $r = h(A, B) = d(A, B) > d(B, A)$. Also, let $0 < s < r$ and $t = r - s$.

Because $d(b, A) = d(B, A) < r$ for all $b \in B$, we know that, for each $b_0 \in B$, there must exist an $a_0 \in A$ such that $d(b_0, a_0) < r$. But, by Lemma 3.2, this implies that $N_s(b_0) \cap N_t(a_0) \neq \emptyset$. This implies that for all $b_0 \in B$ the intersection of the closed s -neighborhood centered at b_0 and the t -extension of A , $\overline{N_s(b_0)} \cap (A + t)$, is non-empty with an infinite interior. Furthermore, since $(B + s) = \bigcup_{b \in B} \overline{N_s(b)}$, we know that $(B + s) \cap (A + t)$ is also non-empty with an infinite interior. By Theorem 2.2, in order to construct an infinite number of points at this location on the Hausdorff line segment from A to B , we need only require that each point (sets in $\mathcal{H}(\mathbb{R}^n)$) be a compact subset of $(B + s) \cap (A + t)$ and contain $\partial((B + s) \cap (A + t))$ along with any combination of interior points. There are, of course, an infinite number of compact subsets of interior points. As a result, there are also an infinite number of sets on the Hausdorff segment \overline{AB} at the location s units from B , and our proof is complete

□

Theorem 3.6. *Let A, B be finite sets in $\mathcal{H}(\mathbb{R}^n)$. If all points $b \in B$ are equidistant from A and $r = h(A, B) = d(B, A) \geq d(A, B)$, then there is the same finite number of points at every location between A and B on the Hausdorff segment \overline{AB} .*

Proof. Let A, B be finite sets in $\mathcal{H}(\mathbb{R}^n)$ such that $|A| = m$ and $|B| = n$ for some $m, n \in \mathbb{N}$. Further, let $h(A, B) = d(b, A) = r$ for all $b \in B$. Finally, let $0 < s < r$ and $t = r - s$.

We want to show that there are a finite number of points at every location s away from B on the Hausdorff segment \overline{AB} . We claim that it is sufficient to prove that $|\overline{N_s(b_0)} \cap (A + t)| \leq m$ for all $b_0 \in B$.

For if $|\overline{N_s(b_0)} \cap (A + t)| \leq m$ for all $b_0 \in B$, then $|(B + s) \cap (A + t)| \leq n \cdot m$. Now, any set C which lies on the Hausdorff segment \overline{AB} must be some collection of the points contained in $(B + s) \cap (A + t)$. Therefore, an upper bound on the number of points at each location between A and B is the number of ways to choose fewer than $n \cdot m$ elements from a set of $n \cdot m$ elements, or $\sum_{i=0}^{n \cdot m} \binom{n \cdot m}{i}$. But, this is a finite sum, and so must be less than some $N \in \mathbb{N}$. Thus, the number of points at each location on the segment between A and B is finite.

We now prove that $|\overline{N_s(b_0)} \cap (A + t)| \leq m$, for all $b_0 \in B$.

Let $b_0 \in B$. Then $d(b_0, a) \geq r$ for all $a \in A$, because $r = \min\{d(b_0, a) : a \in A\}$. We construct a set $A'_0 = \{a \in A : d(b_0, a) = r\}$. The reason for this construction is that points in A for which $d(b_0, a) > r$ do not contribute points to the intersection $\overline{N_s(b_0)} \cap (A + t)$. That is, $\overline{N_s(b_0)} \cap \overline{N_t(a)} = \emptyset$ for all $a \in A \setminus A'_0$. Notice that $1 \leq |A'_0| \leq m$.

Let $a_0 \in A'_0$. We want to show that $\overline{N_s(b_0)} \cap \overline{N_t(a_0)} = \{c_0\}$ for some $c_0 \in \mathbb{R}^n$.

First, we show that such a c_0 exists. Let $\overline{a_0 b_0}$ denote the Euclidean line segment from a_0 to b_0 . Then, from basic Euclidean geometry, there exists exactly one point $c_0 \in \overline{a_0 b_0}$ with $d(a_0, c_0) = t$ and $d(c_0, b_0) = s$. By definition, c_0 is an element of $\overline{N_s(b_0)} \cap \overline{N_t(a_0)}$.

Next, we claim that no other point e_0 exists in \mathbb{R}^n such that $e_0 \in \overline{N_s(b_0)} \cap \overline{N_t(a_0)}$.

By contradiction, suppose that such an e_0 exists. Then $d(a_0, e_0) \leq t$ and $d(e_0, b_0) \leq s$, implying that $d(a_0, e_0) + d(e_0, b_0) \leq s + t = r$. Also, we know that $r = d(a_0, b_0) \leq d(a_0, e_0) + d(b_0, e_0)$ by the Euclidean triangle inequality. Thus $d(a_0, e_0) = t$, $d(b_0, e_0) = s$, and $e_0 \in \overline{a_0 b_0}$. But since there is only one such point on $\overline{a_0 b_0}$, it must be that $e_0 = c_0$.

Now, we know that

$$\overline{N_s(b_0)} \cap (A + t) = \overline{N_s(b_0)} \cap (A'_0 + t) = \overline{N_s(b_0)} \cap \left(\bigcup_{a_0 \in A'_0} \overline{N_t(a_0)} \right)$$

We also know that, for every $a_0 \in A'_0$, exactly one point will be added to $\overline{N_s(b_0)} \cap \left(\bigcup_{a_0 \in A'_0} \overline{N_t(a_0)} \right)$. Since there are a maximum of m points in A'_0 , there can be at most m points in $\overline{N_s(b_0)} \cap \left(\bigcup_{a_0 \in A'_0} \overline{N_t(a_0)} \right)$.

Notice finally that for each b_0 , the number of points in the intersection $\overline{N_s(b_0)} \cap \left(\bigcup_{a_0 \in A'_0} \overline{N_t(a_0)} \right)$ (and therefore the number of Hausdorff points at each location on the Hausdorff segment between A and B),

depends only on the cardinality of the set A'_0 . This cardinality is not altered by varying the values of s and t , implying that the Hausdorff segment from A to B has the same finite number of points at every location between A and B , and we have proved our theorem. □

Thus we can conclude that it is necessary for the sets A and B to satisfy the following two conditions to have a finite number of points between them:

1. all the points of B must be equidistant from A and
2. the Hausdorff distance between A and B must be the distance from B to A .

Notice that these two conditions also imply the following lemma.

Lemma 3.3. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. If all points of B are equidistant from A and $h(A, B) = d(B, A)$, then all points of A must also be equidistant from B and $h(A, B) = d(A, B) = d(B, A)$.*

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $d(b, A) = r$ for all $b \in B$ and $h(A, B) = d(B, A) = r$.

Suppose there exists a point $a \in A$ such that $d(a, B) > r$. This gives rise to an immediate contradiction, because it implies that $d(A, B) > r$ and $h(A, B) > r$.

Now, suppose there exists a point $a \in A$ such that $d(a, B) < r$. In that case, there must exist a point $b \in B$ such that $d_E(a, b) < r$. This again is a contradiction, because it implies that $d(b, A) < r$.

Therefore, $d(a, B) = r$ for all $a \in A$ and $d(A, B) = d(B, A)$. □

4 Examples of Hausdorff Segments

We now turn to some interesting examples of Hausdorff line segments between two sets $A, B \in \mathcal{H}(\mathbb{R}^n)$. For several of these examples, we will also demonstrate how to calculate the precise number of points at each location on the segment.

4.1 Line of alternating points

Let $r > 0$ and let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_m\}$, where a_i is the point $(2i - 2)r$ units along the positive x -axis, b_j is the point $(2j - 1)r$ units along the positive x -axis, and either $k = m$ or $k = m + 1$. Note that $d(a_i, B) = r = d(b_j, A)$ for each i and j . We will determine the number of points C in $\mathcal{H}(\mathbb{R}^n)$ satisfying ACB at each location between A and B . Recall that C will lie on both $\mathcal{C}_s(A)$ and $\mathcal{C}_t(B)$, so C will be a subset of $(A + s) \cap (B + t)$.

Let $n = k + m$, s a number between 0 and r , $t = r - s$, and let F_{n-1} denote the number of points C satisfying ACB with $h(A, C) = s$. We begin our investigation with a few special cases.

- I. The simplest case occurs when $k = m = 1$, which was considered in [1]. In this case we have $F_1 = 1$.

II. Now suppose $A = \{a_1, a_2\}$ and $B = \{b_1\}$. Note that $(A + s) \cap (B + t)$ is a two point set $X = \{x_1, x_2\}$, with $x_1 < x_2$. Each set C in question will be a subset of X . If $x_1 \notin C$, then $h(A, C) = d(a_1, x_2) > s$. A similar argument shows that C contains x_2 . Thus, $C = X$ and $F_2 = 1$.

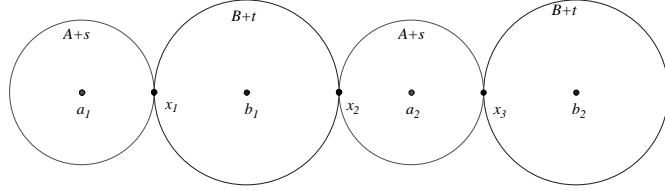


Figure 8: Line of intersecting points.

III. Consider $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Note that $(A + s) \cap (B + t)$ is a three point set $X = \{x_1, x_2, x_3\}$, with $x_1 < x_2 < x_3$ see the figure above. Again, each set C in question will be a subset of X . As above, if $x_1 \notin C$, then $h(A, C) \geq d(a_1, x_2) > s$. A similar argument shows that C contains x_3 . Notice that both $C = X$ and $C = \{x_1, x_3\}$ satisfy ACB with $h(A, C) = s$. Therefore, $F_3 = 2$.

You might see a pattern emerging. The next theorem makes this specific.

Theorem 4.1. F_n is the n^{th} Fibonacci number.

Proof. Let $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_m\}$ as above and $n = k + m$. The proof is by induction on n . We have already verified the cases $n=1, 2$, and 3 directly. Assume $n > 3$. Let s be a number between 0 and r , $t = r - s$. The set $(A + s) \cap (B + t)$ is an $n - 1$ point set $X = \{x_1, x_2, x_3, \dots, x_{n-1}\}$, with $x_1 < x_2 < x_3 < \dots < x_{n-1}$. Note that

$$a_i < x_{2i-1} < b_i \quad \text{and} \quad b_i < x_{2i} < a_{i+1}.$$

Each set C satisfying ACB with $h(A, C) = s$ and $h(B, C) = t$ will be a subset of X . If $x_1 \notin C$, then $h(A, C) \geq d(a_1, x_2) > s$. So $x_1 \in C$. Similarly, we can show $x_{n-1} \in C$. To find the remaining points in C , we argue cases: $x_2 \notin C$ and $x_2 \in C$.

Case I: $x_2 \notin C$

Assume $k > 1$. Let $A_1 = \{a_1\}$, $B_1 = \{b_1\}$ and let $A^* = A - A_1$, $B^* = B - B_1$. We will show that $C^* = C - \{x_1, x_2\}$ satisfies $h(A^*, C^*) = s$ and $h(B^*, C^*) = t$. Note that since $x_2 \notin C$, we must have $x_3 \in C$ and $x_3 \in C^*$. We will then have a one-to-one correspondence between the points C on the segment joining A and B and the points C^* on the segment joining A^* and B^* . The inductive hypothesis tells us that there are F_{n-2} such points C^* and, therefore, F_{n-2} points C .

First we show that C^* is not empty. If $k = m$, then since $h(B, C) = t$, it must be the case that C contains a point within t units of b_m . Recall that $C \subset X$, so the only such possible point is x_{n-1} . If $k = m + 1$, then since $h(A, C) = t$, it must be the case that C contains a point within t units of a_k . Again, the only such possible point is x_{n-1} . In either case, C must contain x_{n-1} . Since $x_{n-1} \notin C$, we must have $x_{n-1} \in C^*$.

Now we show $h(A^*, C^*) = s$. Let $a_i \in A^*$ ($i > 1$). Since $h(A, C) = s$, there is a point $c_i \in C$ so that

$$d(a_i, c_i) \leq s. \quad (4.1)$$

The only possible points in C satisfying (4.1) are x_{2i-2} and x_{2i-1} . Since $i > 1$, it follows that $2i - 2 > 2$, so neither x_{2i-2} nor x_{2i-1} is x_1 or x_2 . Therefore, $c_i \in C^*$ and $d(a_i, C^*) = s$. Thus, $d(A^*, C^*) = s$. A similar argument shows $d(B^*, C^*) = t$.

Now we show $d(C^*, A^*) \leq s$. Let $c \in C^*$. Then $c \in C$. So there is a point $a \in A$ so that $d(c, a) \leq s$. Since $C^* \subset X$, we must have $c = x_i$ for some $i > 2$. Again, the only points in A that can be within s units of x_i are $a_{\frac{i+1}{2}}$ if i is odd and $a_{\frac{i}{2}+1}$ if i is even. Since $i > 2$, neither of these points can equal a_1 . Therefore, $a \in A^*$ and $d(C^*, A^*) \leq s$. Therefore, $h(C^*, A^*) = s$. Similarly, $h(C^*, B^*) = t$.

Case II: $x_2 \in C$

In this case, we show that $C^* = C - \{x_1\}$ satisfies A^*C^*B with $h(A^*, C^*) = s$ and $h(C^*, B) = t$. Again, this will give us a one-to-one correspondence between the points C on the segment joining A and B and the points C^* on the segment joining A^* and B . There are exactly F_{n-1} such sets C^* by our inductive hypothesis, so there will be F_{n-1} sets C .

The same argument as above shows that C^* is not empty and that $d(A^*, C^*) = s$. To show $d(C^*, A^*) \leq s$, we argue as above. Let $c \in C^*$. Then $c \in C$. So there is a point $a \in A$ so that $d(c, a) \leq s$. Since $C \subseteq X$, we must have $c = x_i$ for some $i \geq 2$. Again, the only points in A that can be within s units of x_i are $a_{\frac{i+1}{2}}$ if i is odd and $a_{\frac{i}{2}+1}$ if i is even. Since $i \geq 2$, neither of these points can equal a_1 . Therefore, $a \in A^*$ and $d(C^*, A^*) \leq s$. Therefore, $h(C^*, A^*) = s$.

Now we show $h(C^*, B) = t$. Since $C \subset C^*$ and $d(C, B) = t$, we must have $d(C^*, B) \leq t$. Now we show $d(B, C^*) = t$. Let $b_j \in B$. If $j = 1$, we have $d(b_1, x_2) = t$, and $d(b_1, x_i) > t$ for all other $i > 2$. Therefore, $d(b_1, C^*) = t$. If $j > 1$, then there is a point $c \in C$ so that $d(b_j, c) \leq t$. Since $C \subset X$, we must have $c = x_i$ for some i . The only points in C that can be within t units of b_j are x_{2j-1} and x_{2j} . Since $j > 1$, neither of these points can equal x_1 . Therefore, $c \in C^*$ and $d(b_j, C^*) \leq t$. Thus, $h(C^*, B) = t$.

Cases I and II show us that there are exactly $F_{n-2} + F_{n-1} = F_n$ points at each location on the segment between A and B . □

4.2 n -gon

Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ be comprised of the vertices of a single square, as seen in Figure 9. We see that the distance $d(a, B) = d(b, A)$ for all a in A and all b in B . In fact, there are 47 sets that lie at each location on the Hausdorff segment between A and B , all such sets are exhaustively listed in Example 2.1.

Let A and B be vertices of regular n -gons with $n \in \mathbb{N}$ in which the n -gons share the same center point and initially are stacked such that the vertices correspond. Then B is rotated $\frac{\pi}{n}$ degrees with respect to A about the center point. The Hausdorff segment between A and B will have a finite number of points at each

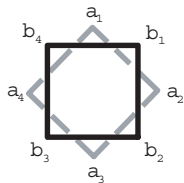


Figure 9: Example of two 4-gons with segment shown.

location between A and B . We now show that the precise number of points at each location s , denoted by $p(n)$, can be calculated using either of the following two formulas, where L_n denotes the Lucas numbers:

$$p(n) = F_{2n} + 2 \cdot F_{2n-1} = L_{2n}, \quad (4.2)$$

$$p(n) = \sum_{k=0}^n \binom{2n-k+1}{k} - \sum_{j=0}^n \binom{2n-4-j+1}{j}. \quad (4.3)$$

To verify (4.2), let $n \in \mathbb{N}$ and let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$, where a_1 is selected from the vertices of one n -gon and the point a_i is the $i + 1^{\text{st}}$ vertex from a_1 moving clockwise on the same n -gon. On the second n -gon, which was rotated $\frac{\pi}{n}$ about the center, b_1 is the first vertex that lies $\frac{\pi}{n}$ degrees clockwise from a_1 and the point b_j is the $j + 1^{\text{st}}$ vertex moving clockwise from b_1 on the same n -gon. Now let $d(a_i, B) = r = d(b_j, A)$ for each i and let $0 \leq s \leq r$ and $t = r - s$. We will determine the number of points C in $\mathcal{H}(\mathbb{R}^n)$ satisfying ACB at each location between A and B . Recall that C will lie on both $\mathcal{C}_t(A)$ and $\mathcal{C}_s(B)$, so C will be a subset of $(A + t) \cap (B + s)$.

Let $p(n)$ denote the number of points C satisfying ACB with $h(B, C) = s$. We begin our investigation with a few special cases, let 1-gon, where A and B are each comprised of a singular point, and the 2-gon, where both A and B are the end points of line segments of equal length and are perpendicular bisectors of each other.

- I. The simplest case occurs when $n = 1$, which was considered in [1]. In this case we have $p(1) = 1$.
- II. Now suppose $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Note that $(A + t) \cap (B + s)$ is a four point set $X = \{x_1, x_2, x_3, x_4\}$, with x_1 between a_1 and b_1 , x_2 between b_1 and a_2 , x_3 between a_2 and b_2 , and x_4 between b_2 and a_1 . All points C that will satisfy ACB will be subsets of X . If $x_1 \notin C$, then it must be that $x_2, x_4 \in C$. We notice the curve of alternating points, starting with b_1 and ending with a_1 while working clockwise, has the same restrictions for removing points from the set C as the Fibonacci line of four points as seen earlier. Thus we can conclude when $x_1 \notin C$ we will have $F_3 = 2$ points C which satisfy ACB . If $x_2 \notin C$ then a similar argument shows that there will be 2 points C which satisfy ACB . Now, if $x_1, x_2 \in C$ then we see that there will be another 3 points C that satisfy ACB and these points will be $\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_2, x_4\}$ and $\{x_1, x_2, x_3\}$. We can therefore conclude there are 7 points that satisfy ACB , thus $p(2) = 7$.

The next theorem generalizes $p(n)$ for $n \in \mathbb{N}$.

Theorem 4.2. *Let A and B be vertices of regular n -gons in which they share the same center point and B is rotated $\frac{\pi}{n}$ degrees with respect to A . If $p(n)$ is the number of points at each location on the Hausdorff segment between A and B , then $p(n) = 2F_{2n-1} + F_{2n}$ where F_n is the n^{th} Fibonacci number.*

Proof. Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ as above and let $a_{n+1} = a_1$. The proof is by induction on n . We have already verified the cases $n=1$ and 2 directly. Assume $n > 2$. Let s be a number between 0 and r and let $t = r - s$. The set $(A + t) \cap (B + s)$ is an $2n$ point set $X = \{x_1, x_2, x_3, \dots, x_{2n}\}$, where x_{2i-1} is the point of intersection of the t extension about a_i and the s extension about b_i and x_{2i} is the point of intersection of the s extension about b_i and the t extension about a_{i+1} for $i = \{1, 2, \dots, n\}$.

Each set C satisfying ACB with $h(A, C) = t$ and $h(B, C) = s$ will be a subset of X . To find the points C , we argue cases: $x_1 \notin C$, $x_2 \notin C$ and $x_1, x_2 \in C$.

Case I: $x_1 \notin C$

In order to have C satisfy ACB we must have $d(a_1, C) = t$ and $d(b_1, C) = s$. This implies that $x_2, x_{2n} \in C$. We now notice the curve of alternating points from A and B , starting with b_1 and ending with a_1 , is equivalent to a "Fibonacci line" of $2n$ points, which we have shown to have F_{2n-1} points which satisfy ACB by Theorem 4.1.

Case II: $x_2 \notin C$

This case can be argued in a similar manner as the previous case, thus we know that there will be an additional F_{2n-1} points which satisfy ACB .

Case III: $x_1, x_2 \in C$

We claim this case is similar to having a $2n + 1$ string of alternating points from A and B , which by Theorem 4.1 will have F_{2n} points that satisfy ACB . By assumption we have $C = \{x_1, x_2\} \cup C'$, where C' is a subset of $\{x_3, x_4, \dots, x_{2n}\}$ such that if $x_i \notin C'$ then x_{i-1} or $x_{i+1} \in C'$ for $i = \{4, 5, \dots, 2n - 1\}$. We can think of this as a string of alternating points starting with b_1 , working in the clockwise direction, and ending with a new point b_* , where $b_* = b_1$, such that x_1 lies between a_1 and b_* . Then we see this is exactly the case when there is a line of $2n + 1$ alternating points as desired. Therefore, by Theorem 4.1, we have F_{2n} points which satisfy ACB .

Cases I, II and III show us that there are exactly $2F_{2n-1} + F_{2n}$ points at each location on the segment between A and B . □

We now verify Formula 4.3:

Let A, B be two n -sets of points in \mathbb{R}^2 arranged as described above. That is, let A, B be the sets of vertices of two offset n -gons. Let the following be true: $h(A, B) = r$, $0 < s < r$, $t = r - s$, and $C = (A + t) \cap (B + s)$. Note that it is the case that $h(A, B) = h(A, C) + h(C, B)$.

Now, C is the set of vertices of a regular $2n$ -gon. We first choose one of the elements of C and number

it 1. We then proceed to number the remaining elements, up to $2n$, moving clockwise around the $2n$ -gon. Notice that any subset of C with no two consecutive elements omitted (Note: 1 and $2n$ are consecutive) remains distance s from B and distance t from A , and therefore at the desired location on the segment \overline{AB} . This is not the case when two consecutive elements are removed. Hence, the number of points at each distance s from B on the Hausdorff segment \overline{AB} is equal to the number of possible ways to select any number of elements from a string of $2n$ elements without selecting two consecutive elements (again, with 1 and $2n$ consecutive).

If we wanted to select k elements from a $2n$ element string (this time with the 1 and $2n$ not consecutive) without selecting two consecutive elements, we could count the number of ways to do so in the following manner:

Let there be $2n - k$ white pencils (representing the unselected elements) and k black pencils (representing the selected elements). Arrange the white pencils in a row. Note that there are $2n - k + 1$ spaces between the white pencils, including the space before the first and the space after the last. Now, we select k of these spaces and insert the black pencils into the selected spaces (one per selected space) to create a string of $2n$ pencils with no two black pencils consecutive. Thus, the number of ways to insert the black pencils is also the number of ways to select k elements from the $2n$ such that no two consecutive elements are chosen. There are $\binom{2n-k+1}{k}$ ways to accomplish this. Summing the ways to select k objects in such a manner for all $k \leq n$, we see that there are $\sum_{k=0}^n \binom{2n-k+1}{k}$ ways to select objects from a $2n$ string without selecting two consecutive objects. Note that if $k > n$ then two of the k selected objects must be consecutive. Therefore the cases where $k > n$ do not add to our sum.

Now, we find the number of ways to select k pairwise non-consecutive objects with 1 and $2n$ considered consecutive. Given the above formula for when 1 and $2n$ are not consecutive, we need only subtract the cases where, in the above process, we selected both the 1^{st} and $2n^{th}$ objects for removal. The resulting difference should then count exactly the number of ways to remove any number of objects from a $2n$ closed string without removing any consecutive objects. To count the number of cases where both the 1^{st} and $2n^{th}$ objects were selected for removal, we fix the 1^{st} and $2n^{th}$ object as being selected. This requires the second and $2n - 1^{th}$ objects to remain unselected. Otherwise, we would have consecutive objects selected in our open $2n$ -object string. However, the remaining $2n - 4$ "interior" objects can be selected for removal in any way such that there are no consecutive objects selected. But we already have a formula for the number of ways to select j non-consecutive elements of an open $2n - 4$ object string: $\binom{2n-4-j+1}{j}$.

Therefore, our formula $p(n) = \sum_{k=0}^n \binom{2n-k+1}{k} - \sum_{j=0}^n \binom{2n-4-j+1}{j}$ does in fact count the number of ways to select any number of non-consecutive objects from a $2n$ string, which is also the number of points at any location on the segment \overline{AB} .

4.3 n -armed star

Next we consider the " n -armed star" example. The title of this example follows from the shape of the resulting segment, \overline{AB} , as seen in Figure 11. The construction is fairly simple. First, select a point and

centered about that point, construct two concentric circles of radii r and $2r$. Then draw two circumscribed regular n -gons, one per circle, such that a line through the center point and one vertex of the inner n -gon also contains at least one vertex of the outer n -gon. Next, let A be the set of vertices of the inner n -gon, and let B be the set of vertices of the outer n -gon plus the center point. As you can see from Figure 10, this construction will yield an intersection $(A + t) \cap (B + s) = C$ which contains n points between the inner and outer n -gons and n points within the inner n -gon. Now, in looking for sets on the Hausdorff segment between A and B , none of the outer points can be removed from the set C , because then we would isolate one of the outer point in B from the set C . In other words, then no $c \in C$ would exist such that $d(c, B) = s$ and, consequently, $h(C, B) > s$. This in turn implies that $h(A, C) + h(C, B) > h(A, B)$, and C is no longer on the segment \overline{AB} . On the other hand, any combination of the inner points can be removed as long as they are not all removed. Therefore, the number of points at each location on the segment between A and B is given by $\sum_{k=0}^{n-1} \binom{n}{k} = 2^n - 1$.

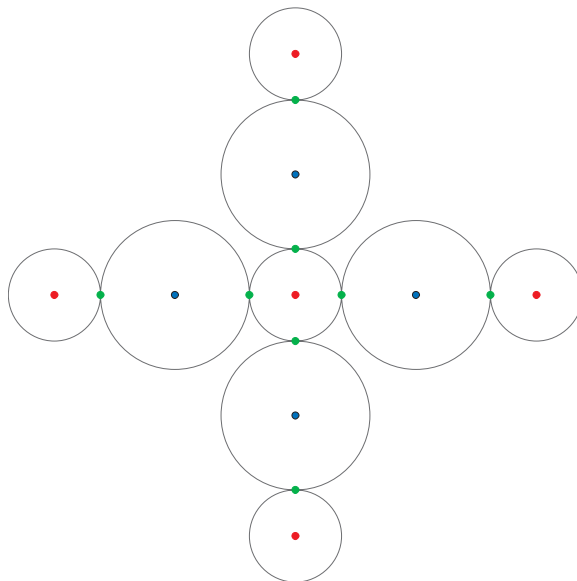


Figure 10: Example of 4-armed star.

4.4 Diamond Pattern

We have also begun investigating a method for constructing two finite sets A and B in $\mathcal{H}(\mathbb{R}^n)$, satisfying $|A| \leq 2|B|$, with a finite number of points at each location on the Hausdorff segment between them, which we will call the “Dickinson Diamond”, as seen in Figure 12. A few examples of the Dickinson Diamond are also shown below.

We have been able to calculate the number of points at each location between the points A and B given the following configurations.

- The single “diamond” where $|A| = 2$ and $|B| = 2$ has seven points at each location between A and B .

To find all seven of these points for a given location on the Hausdorff segment, we will consider the

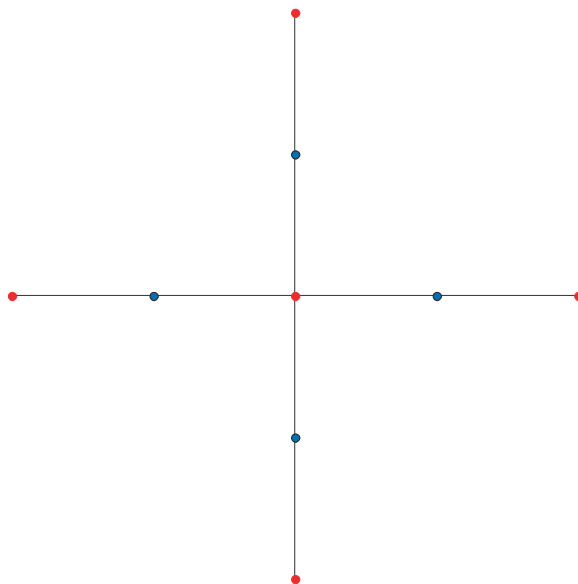


Figure 11: Example of Hausdorff segment generated from a 4-armed star.

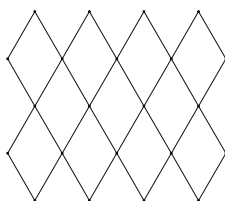


Figure 12: Segment shown for “Dickinson Diamond” pattern.

enumeration as seen in Figure 13. If $C = \{1, 2, 3, 4\}$ lies on the Hausdorff segment between A and B , then so do the following six points:

- Removing point 1 from the set C .
- Removing point 2 from the set C .
- Removing point 3 from the set C .
- Removing point 4 from the set C .
- Removing points 1 and 3 from the set C .
- Removing points 2 and 4 from the set C .

We are guaranteed that these are the only points at a given location on \overline{AB} since removing any other combination of points will result in points from the sets A or B that are isolated from the resulting set C' . It is noted that the number of points at each location between A and B on this Hausdorff segment can be calculated using the 2-gon formula as stated above.

- The vertical string of two “diamonds” where $|A| = 3$ and $|B| = 4$ has 63 points at each location. This value can be computed in much the same manner as seen in the single diamond case, where all possible

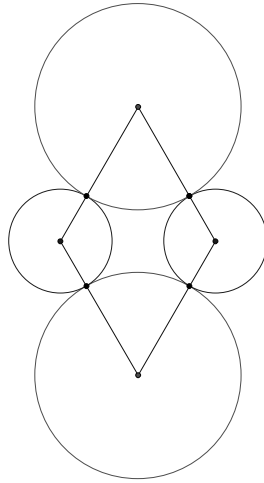


Figure 13: Hausdorff segment for a single diamond.

combinations of removing points from the set C are considered and those that do not isolate points of both A and B count as distinct points on the Hausdorff segment defined by A and B

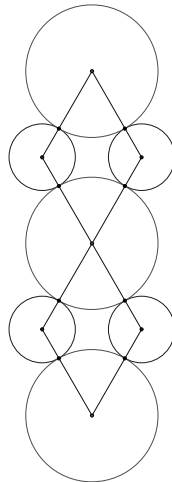


Figure 14: Hausdorff segment for a vertical string of two diamonds.

- The vertical string of three “diamonds” where $|A| = 4$ and $|B| = 6$ has 560 points at each location, found using combinatorial methods, omitted here.

4.5 Infinite Sets

We will not always have a finite number of points at each location between two sets, which is displayed in Figure 16. Here we have A as the center point and the outer bold circle and B is the inner bold circle. One point at location s on the Hausdorff segment \overline{AB} is the set consisting of the entire outer thin circle and any one point from the inner thin circle. Clearly, there is an infinite number of points of this variety at a location s on the segment \overline{AB}

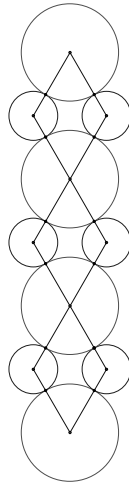


Figure 15: Hausdorff segment for a vertical string of three diamonds.

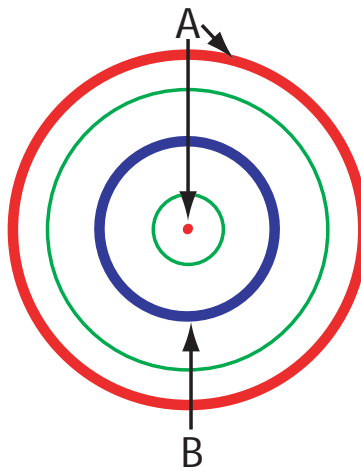


Figure 16: One point of intersection at each location between A and B .

We also note that if both sets are infinite and satisfy the above conditions, we can still have a finite number of points at each location between A and B as seen in Figure 17.

In the example shown in Figure 17 we have A as the set of three *'s and B as the large circle and pair of large points to the left. The smaller circle and set of 4 smaller points to the left again signify the points of intersection of the s and t extensions about B and A respectively. We see that there are 7 points at each location between the sets. As an aside, when sufficiently separated “components” as seen in Figure 17, then the number of points at each location on the Hausdorff segment between A and B is equal to the product of the numbers of points at each location of the Hausdorff segment for each component.

Here we have two infinite sets (Figure 18) where there is one point at each location between A and B .

We note that there can be two infinite sets A and B such that there are a finite number of points at each location between A and B on \overline{AB} . In Figure 4.5, we can remove the lone center point to give us a second

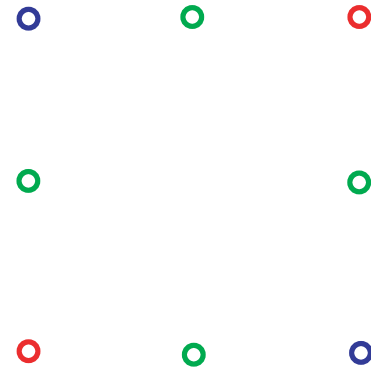
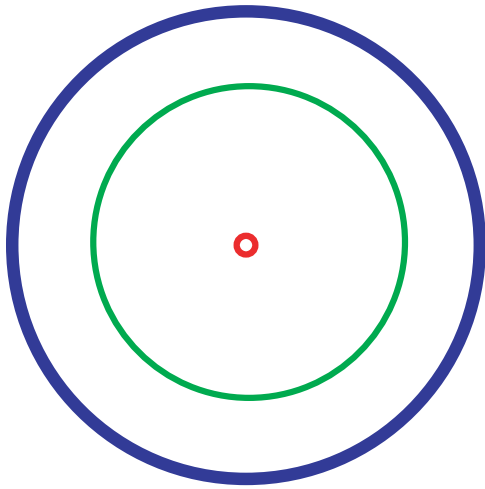


Figure 17: 7 points at each location between A and B .

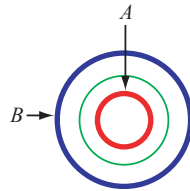
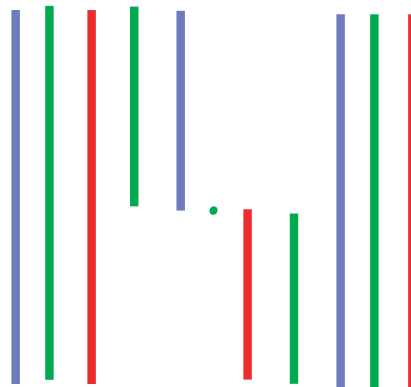


Figure 18: One point at each location between A and B .

point at each location on the Hausdorff segment between A and B .



5 Shell Components and a Characterization of Hausdorff Segments

In a previous section, we presented several necessary conditions for there to be a finite number of points at each location on a given Hausdorff line segment. In the following section, we will present and use several new definitions and theorems, with the goal of continuing to characterize those segments which do have a finite number of points at each location. By the end of this section, we will have succeeded in developing a

full characterization of such Hausdorff segments.

5.1 Definitions

Definition 5.1. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $y, z \in A$, with $d > 0$ in \mathbb{R} . If $N_d(y) \cap N_d(z) \neq \emptyset$, then $y \sim^d z$.

The relation \sim^d is illustrated in Figure 19. Note that this is a symmetric relation on the elements of A . It is left as an exercise to verify that this relation is symmetric.

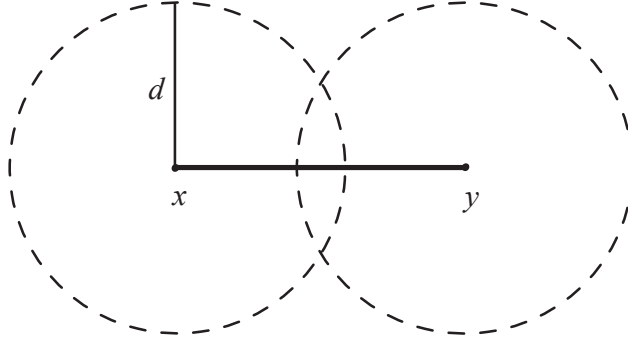


Figure 19: $x \sim^d y$.

Definition 5.2. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $y, z \in A$

$y \approx^d z$ if and only if there exists a finite sequence of the elements of A , denoted $\{b_1, b_2, \dots, b_m\}$ such that $y \in N_d(b_1)$ and $z \in N_d(b_m)$ and $b_i \sim^d b_{i+1}$ for $i \in \{1, 2, \dots, m-1\}$.

This relation is illustrated in Figure 20.

Lemma 5.1. Let $A \in \mathcal{H}(\mathbb{R}^n)$. Then, \approx^d is an equivalence relation on the elements of A .

Proof. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $x, y, z \in A$, with $x \approx^d y$ and $y \approx^d z$.

We first show that \approx^d is reflexive. Let $\{b^k\} = \{x\}$. Then $\{b_k\}$ satisfies $x \in N_d(b_1)$ and $x \in N_d(b_k)$. The sequence vacuously satisfies the condition that $b_i \sim^d b_{i+1}$ for all $i \in \{1, 2, \dots, k-1\}$, since there is no b_2 . Thus, $x \approx^d x$, and \approx^d is reflexive.

We next show that \approx^d is symmetric. We have assumed $x \approx^d y$. Thus, there exists a sequence $\{b_i\} = \{b_1, b_2, \dots, b_k\}$ satisfying $x \in N_d(b_1)$, and $y \in N_d(b_k)$, and $b_i \sim^d b_{i+1}$ for all $i \in \{1, 2, \dots, k-1\}$. The let $\{c_j\} = \{c_1, c_2, \dots, c_k\}$ with $c_j = b_{k+1-j}$. Then $y \in N_d(c_1) = N_d(b_k)$ and $x \in N_d(c_k) = N_d(b_1)$. Further, $c_j \sim^d c_{j+1}$ for all $j \in \{1, 2, \dots, k-1\}$ by the symmetry of the \sim^d relation. Thus, $y \approx^d x$, and \approx^d is symmetric.

Finally, we show that \approx^d is transitive. We have that $x \approx^d y$ and $y \approx^d z$. Then there exists sequences $\{b_i\} = \{b_1, b_2, \dots, b_k\}$ and $\{c_j\} = \{c_1, c_2, \dots, c_m\}$ satisfying $x \in N_d(b_1)$ and $y \in N_d(b_k)$ and $y \in N_d(c_1)$ and $z \in N_d(c_m)$. Also, $b_i \sim^d b_{i+1}$ and $c_j \sim^d c_{j+1}$ for all $i \in \{1, 2, \dots, k-1\}$ and all $j \in \{1, 2, \dots, m-1\}$. Let $\{a_s\} = \{a_1, a_2, \dots, a_{k+m}\}$ be the sequence with $a_p = b_p$ for $p \in \{1, 2, \dots, k\}$ and $a_p = c_{p-k}$ for $p \in \{k+1, k+2, \dots, k+m\}$. We know that $x \in N_d(a_1) = N_d(b_1)$ and $z \in N_d(a_{k+m}) = N_d(c_m)$. Further, we

know that $a_p \sim^d a_p + 1$ for all $p \in (\{1, 2, \dots, k + m - 1\} - \{k\})$. We only need to show that $a_k \sim^d a_k + 1$. However, we know that $y \in N_d(a_k) = N_d(b_k)$ and $y \in N_d(a_{k+1}) = N_d(c_1)$. So, $N_d(a_k) \cap N_d(a_{k+1}) \neq \emptyset$ and $a_k \sim^d a_{k+1}$. Thus, $x \approx^d z$, and \approx^d is transitive.

Thus, we see that \approx^d is indeed an equivalence relation on the elements of A . □

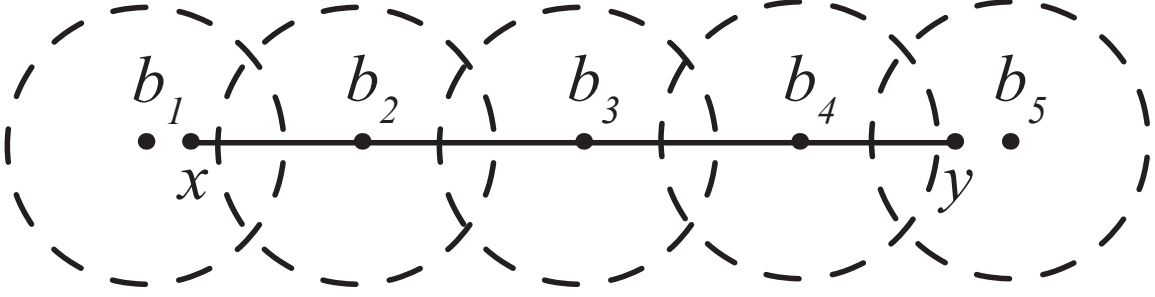


Figure 20: $x \approx^d z$.

Definition 5.3. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $x \in A$. The shell component of A containing x , denoted A_x , is defined as the set $\{y \in A : x \approx^d y, \text{ for all } d > 0\}$.

Definition 5.4. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ and let $x \in A, y \in B$. The shell A_x is said to be adjacent to the shell B_y if and only if $((A_x + h(A, B)) \cap B_y) \neq \emptyset$. Further, x is said to be adjacent to y , denoted $x \approx y$, if and only if $d_E(x, y) = h(A, B)$.

It is left to the reader to verify that both of these adjacency relations are symmetric. The relation \approx can be seen in Figure 21.

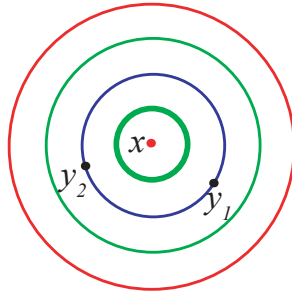


Figure 21: $x \approx y_1$ and $x \approx y_2$

We specify the conditions two points $A, B \in \mathcal{H}(\mathbb{R}^n)$ must satisfy in order for the Hausdorff segment \overline{AB} to contain finitely many points at each location between A and B . These conditions are necessary, but not sufficient for each location to have finitely many points.

1. $A \neq B \in \mathcal{H}(\mathbb{R}^n)$.

2. $h(A, B) = d(B, A) = d(A, B) = r$.

3. $d(b, A) = r$, for all $b \in B$.

Here we will set the convention that, if $0 < s < r$ and $s + t = r$, then $C = (A + t) \cap (B + s)$ and $h(A, C) = t$ and $h(B, C) = s$.

From this point forward, if these conditions are satisfied, we will say that A, B , and C satisfy the PFAEL (Possibly Finite At Each Location) conditions.

Definition 5.5. Let A, B, C satisfy the PFAEL conditions. Let $a \in A$. The adjacency set, $[a]_C$, is defined as the set $\{c \in C : c \approx a\}$.

Let $[A]_C$ denote the set $\{[a]_C : a \in A\}$.

Definition 5.6. Let (X, \mathcal{U}) be a topological space. Further, let $A \subset X$. The **subspace topology** of A , denoted \mathcal{U}_A , is defined as the set $\{U \cap A : U \in \mathcal{U}\}$.

The idea of a subspace topology will be of major importance in our characterization of Hausdorff line segments.

5.2 Preliminary Lemmas

We will now prove several new lemmas which will prove helpful in the following section.

Lemma 5.2. Let A, B, C satisfy the PFAEL conditions. Then $d(c, A) = t$ and $d(c, B) = s$ for all $c \in C$

Proof. Let A, B, C satisfy the PFAEL conditions. Let $c \in C$.

Then $c \in (A + t) \cap (B + s)$. Therefore, $d(c, A) \leq t$ and $d(c, B) \leq s$. This implies that there exists $a_0 \in A$ and $b_0 \in B$ such that $d_E(a, c) \leq t$ and $d_E(b, c) \leq s$. If $d(c, A) < t$, then there exists $a_1 \in A$ such that $d_E(a_1, c) < t$. Thus, $d(a_1, b_0) \leq d(a_1, c) + d(c, b_0) < s + t < r$. This is a contradiction of our conditions, and $d(a, c) \geq r$ for all $a \in A$. A similar argument ensures that $d(b, c) \geq s$ for all $b \in B$. The existence of a_0 and b_0 ensures that $d(c, A) = t$ and $d(c, B) = s$. □

Lemma 5.3. Let A, B, C satisfy the PFAEL conditions. Further, let $C' \subseteq C$.

$$h(A, C) \leq h(A, C')$$

Proof. Let A, B, C satisfy the PFAEL conditions. Further, let $C' \subseteq C$.

Since, by Lemma 5.2, $d(c, A) = t$ for all $c \in C$, we know that $d(A, C) = t$ by Lemma 3.3. Further, we know that $d(c', A) = t$ for all $c' \in C'$ because $C' \subseteq C$. This fact implies that $d(C', A) = t$. It only remains to show that $d(A, C') \geq t$. Suppose, by way of contradiction, that $d(A, C') < t$. Then, for all $a_0 \in A$, there exists $c_0 \in C'$ such that $d_E(a_0, c_0) < t$. However, consider $a_\star \in A$ such that $d(a_\star, C) = t$ (which is guaranteed to exist, because $d(A, C) = t$). Thus, we now have the situation that there exists $c_\star \in C' \subseteq C$ such that $d_E(c_\star, a_\star) < t$, but no such $c \in C$ can exist. This is an obvious contradiction, and $d(A, C') \geq t \implies h(A, C') \geq t$. □

Lemma 5.4. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ with $U \in \mathcal{U}_A$, the subspace topology of A . If $|U| = \infty$, then there exist infinitely many subsets $V \subset U$ with $V \in \mathcal{U}_A$.*

Proof. Let $A \in \mathcal{H}(\mathbb{R}^n)$ with $U \in \mathcal{U}_A$. Further, let $|U| = \infty$.

Suppose, by way of contradiction, that there does not exist an infinite number of sets $V \subset U$ with $V \in \mathcal{U}_A$. We claim that no point $x \in A$ exists with x a limit point of U . Notice that if $x \in A$ is a limit point of U , then there does not exist a $\delta_\star > 0$ such that $(N_{\delta_\star}(x) \cap A) \cap (U - \{x\}) = \emptyset$. Furthermore, $N_\epsilon(x) \cap U$ is necessarily an element of \mathcal{U}_A , for all $\epsilon > 0$. Therefore, we can construct infinitely many distinct open sets of the form $N_{\delta_i}(x) \cap U$ contained in U . We need only choose an original δ_0 and choose some point $a_0 \neq x$ in $N_{\delta_0}(x)$. We then choose $\delta_1 = d_E(x, a_0)$, forcing $N_{\delta_1}(x) \subsetneq N_{\delta_0}(x)$, and so on. Thus, we see that U cannot have any limit points. However, since U is an infinite subset of A and A is Limit Point Compact, U must have a limit point, and we arrive at a contradiction. Thus, there are infinitely many distinct subsets of U which are also elements of \mathcal{U}_A . \square

Lemma 5.5. *Let A, B, C satisfy the PFAEL conditions. Further, let C' satisfy $AC'B$ with $C' \subseteq (A + t) \cap (B + s)$. Given any $a_0 \in A$, the adjacency set $[a_0]_{C'} \neq \emptyset$.*

Proof. Let A, B, C satisfy the PFAEL conditions. Further, let C' satisfy $AC'B$ with $C' \subseteq (A + t) \cap (B + s)$. Let $a_0 \in A$.

Because $AC'B$ and, by Lemma 5.3 $h(A, C') \geq t$ and $h(B, C') \geq s$, we know $h(A, C') = t$. If $h(A, C') > t$, then $h(A, C') + h(B, C') > r = h(A, B)$ and C' does not satisfy $AC'B$. Further, due to Lemma 5.2 and the fact that $C' \subseteq (A + t) \cap (B + s)$, it is true that $d_E(c, A) = t$ for all $c \in C'$. In other words, for all $c \in C'$, $d_E(c, a) \geq t$ for all $a \in A$. Finally, because the Hausdorff distance between A and C' is t , $d(a_0, C') \leq t$ for our $a_0 \in A$, implying that there exists $c_0 \in C'$ such that $d_E(a_0, c_0) \leq t$. But we have seen that $d_E(c_0, a)$ is greater than or equal to t for all $a \in A$. Therefore, $d_E(a_0, c_0) = t$ and $c_0 \in [a_0]_{C'}$. Thus, $[a_0]_{C'} \neq \emptyset$. \square

Notice that a similar proof shows that for any $b_0 \in B$, the adjacency set $[b_0]_C \neq \emptyset$.

Lemma 5.6. *Let A, B, C satisfy the PFAEL conditions. Given $c \in C$, there exists exactly one $[a]_C \in [A]_C$ such that $c \in [a]_C$ and exactly one $[b]_C \in [B]_C$ such that $c \in [b]_C$.*

Proof. Let A, B, C satisfy the PFAEL conditions. Let $c \in C$.

Suppose no such $[a]_C$ (or $[b]_C$, substituting B for A and s for t) exists. This implies that there does not exist $a \in A$ such that $d_E(a, c) = t$. We already know, by Lemma 5.2, that $d(c, A) = t$, implying that there does not exist $a \in A$ such that $d_E(a, c) < t$. Then $(\{c\} + t) \cap A = \emptyset$. But this implies that $d(c, A) > t$ and $c \notin (A + t)$, contradicting the fact that $C = (A + t) \cap (B + s)$. Therefore, there must exist at least one $a \in A$ and one $b \in B$ such that $c \in [a]_C$ and $c \in [b]_C$.

We now show that each $c \in C$ is a member of the adjacency set of only one element in A and only one element of B . Suppose that there exists $a_1 \neq a_2 \in A$ such that $c \in [a_1]_C$ and $c \in [a_2]_C$. We know that there exists at least one $b \in B$ such that $c \in [b]_C$. This implies that all three of the following are true: $d_E(c, a_1) = t$ and $d_E(c, a_2) = t$ and $d_E(c, b) = s$. Then by the Euclidean triangle inequality, $d_E(a_1, b) \leq$

$d_E(a_1, c) + d_E(c, b) = t + s = r$ and $d_E(a_2, b) \leq d_E(a_2, c) + d_E(c, b) = t + s = r$. Now, either $d_E(a_1, b) < r$ or $d_E(a_2, b) < r$ or $d_E(a_1, b) = d_E(a_2, b) = r$. The first two possibilities are automatic contradictions, since $d_E(b, A) = r$ for all $b \in B$. The remaining possibility (that $d_E(a_1, b) = d_E(a_2, b) = r$) also yields a contradiction, because it implies that a_1 and a_2 are on the Euclidean line \overleftrightarrow{cb} at the same location. The properties of Euclidean lines then ensure that $a_1 = a_2$. Thus, our lemma is proved. \square

Definition 5.7. Let A, B, C satisfy the PFAEL conditions.

1. Let $q_A : C \rightarrow [A]_C$ be defined by $q_A(c) = [a]_C$ where $c \in [a]_C$.
2. Let $q_B : C \rightarrow [B]_C$ be defined by $q_B(c) = [b]_C$ where $c \in [b]_C$.

Both q_A and q_B are well-defined and onto functions by Lemmas 5.5 and 5.6.

5.3 Some Theorems

Theorem 5.1. Let A, B, C satisfy the PFAEL conditions. If

1. $q_A : C \rightarrow [A]_C$ is 1-1 or
2. $q_B : C \rightarrow [B]_C$ is 1-1,

Then C is the only point at location s on \overline{AB} . Equivalently, there is exactly one point at each location on \overline{AB} .

Proof. Let A, B, C satisfy the PFAEL conditions. Further, let q_A be 1-1.

For each $[a]_C \in [A]_C$, we know $q_A^{-1}([a]_C) = \{c\}$ for some $c \in C$. So for each $a_0 \in A$, there exists exactly one $c_0 \in C$ such that $d_E(a_0, c_0) = t$. Further, we know that any point, C' , satisfying $AC'B$ also satisfies $C' \subseteq C$. However, suppose we remove even a single point c_* from C .

Now, we have already seen, by Lemma 5.6, that there exists an a_* such that $q_A^{-1}([a_*]_C) = \{c_*\}$. Consider $C' = (C - c_*)$. More specifically, consider the function $q_A|_{C'}$. (q_A restricted to $C - c_*$.) We see that $q_A|_{C'}^{-1}([a_*]_{C'}) = \emptyset$. This implies that $[a_*]_{C'} = \emptyset$, which contradicts Lemma 5.5.

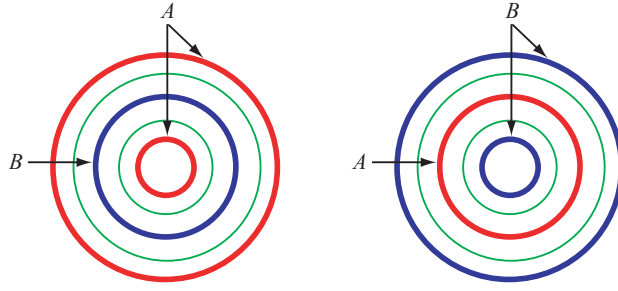
Therefore, no such C' exists, and C is the only point at this particular location on \overline{AB} .

Note: To prove the theorem, it is sufficient to show that if q_A is 1-1, then C is the only point at location on \overline{AB} . A similar argument will hold for the case when q_B is 1-1. \square

Further, notice that the reverse implication is not true.

Theorem 5.2. Let A, B, C satisfy the PFAEL conditions. If there exists exactly one point at each location on \overline{AB} , then there does not exist an unordered pair, (a, b) with $a \in A, b \in B$ such that:

1. $a \equiv b$,
2. $|q_A^{-1}([a]_C)| \geq 2$ and $|q_B^{-1}([b]_C)| \geq 2$, and



3. $c_0 \in q_A^{-1}([a]_C) \cap q_B^{-1}([b]_C)$, where c_0 is not a limit point of C .

Proof. Let A, B, C satisfy the PFAEL conditions. Further, let \overline{AB} contain only one point at each location.

By contradiction, suppose that there exists an unordered pair, (a, b) with $a \in A, b \in B$ such that:

1. $a_0 \equiv b_0$,
2. $|q_A^{-1}([a_0]_C)| \geq 2$ and $|q_B^{-1}([b_0]_C)| \geq 2$, and
3. $c_0 \in q_A^{-1}([a_0]_C) \cap q_B^{-1}([b_0]_C)$ where c_0 is not a limit point of C .

Consider $C' = C - \{c_0\}$.

Claim: C' satisfies $AC'B$.

We will show:

C' is compact, $h(A, C') = t$, and $h(B, C') = s$. Thus, C is not the only point at this location on \overline{AB} .

To show that C' is compact, we only need to show that c_0 is not a limit point of C' . This is sufficient, because we know C' is bounded since it is a subset of a bounded set. Further, we know that since C is closed, it contains all its limit points. This implies that C' contains all its limit points, unless c_0 is a limit point of C' . However, we know that there does not exist a sequence not containing c_0 in C that converges to c_0 (because c_0 is not a limit point of C). But since we are only removing the point c_0 from C to create C' , there cannot be any such sequence in C' either, and c_0 is not a limit point of C' .

Therefore we see that C' is indeed compact.

For the last two conditions, it is sufficient to show that $h(A, C') = t$, because a similar argument will hold to show that $h(B, C') = s$.

We know that $d_E(c, A) = t$ for all $c \in C'$ because $C' \subset C$ and, by Lemma 5.2, the distance from any c in C to the set A is equal to t . It is also true that, except for our specified point $a_0 \in A$, the distance $d(a, C') = t$ for all $a \in A$, because c_0 can only be in one adjacency set of A , by Lemma 5.6. Since it is true, by Lemma 5.2, that $d_E(a_0, c) \geq t$ for all c in C and hence in C' , we only need to show that there exists $c_\star \in C'$ such that $d_E(a_0, c_\star) = t$. But this is apparent because $|q_A^{-1}([a]_C)| \geq 2$, which implies that there exists a $c' \neq c_\star$ in C (which means $c' \in C'$) such that $c' \approx a$. This implies that $d_E(a, c') = t$, and therefore $d(a, C') = t$, since by Lemma 5.2, all $c \in C'$ are at exactly t units from A . Thus, $d(A, C') = t$ and $h(A, C') = t$.

□

We now present several theorems describing fully the conditions under which a Hausdorff segment \overline{AB} can have finitely many points at each location.

Theorem 5.3. *Let A, B, C satisfy the PFAEL conditions.*

There exists exactly one point at each location on \overline{AB}

\iff

There does not exist a set $U \subset C$ with $U \in \mathcal{U}_C$ such that

1. *for all $[a]_C \in q_A(U)$, it is true that $|[a]_C| \geq 2$ and there exists $c \in [a]_C$ such that $c \notin U$,*
2. *for all $[b]_C \in q_B(U)$, it is true that $|[b]_C| \geq 2$ and there exists $c \in [b]_C$ such that $c \notin U$.*

Proof. .

\implies

Let A, B, C satisfy the PFAEL conditions. Let there exist exactly one point at each location on \overline{AB} .

By way of contradiction, suppose there exists a subset, U , of C in \mathcal{U}_C satisfying 1 and 2. Then consider $C' = C - U = C \cap U^c$. In this situation, both C and U^c are closed sets, implying that C' is closed under the subspace topology \mathcal{U}_C . Because C' is closed under the subspace topology, C' is necessarily closed under the topology of $(\mathcal{H}(\mathbb{R}^n), h)$ [2]. We know C' is bounded, because it is a subset of a bounded set C . Therefore, C' is compact.

Now, if $h(A, C') = t$ and $h(B, C') = s$ then $AC'B$ is satisfied with C' and C at the same location, and we have a contradiction. We must therefore show that $h(A, C') = t$. (Note that a similar argument then holds for $h(B, C') = s$.)

To show that $h(A, C') = t$, we first note that $d(C', A) = t$ because $C' \subset C$ and $d(c, A) = t$ for all $c \in C$. Next, we show that $d(A, C') = t$. We know that for all $[a]_C \in q_A(U)$, and hence all $[a]_C \in [A]_C$, there exists a c in C but not in U such that $c \in [a]_C$. Thus, for all $a \in A$, there exists $c \in C'$ such that $c \approx a$. By definition, this implies that for all $a \in A$, there exists $c \in C'$ such that $d_E(c, a) = t$. Therefore, $d(a, C') \leq t$, for all $a \in A$, and $d(A, C') \leq t$. However, by Lemma 5.3, we know that $d(A, C') \geq t$. So $d(A, C') = t$ and $h(A, C') = t$, giving us the desired contradiction, $AC'B$.

Thus, no subset of U exists which is open in the subspace topology of C and satisfies 1 and 2.

\impliedby

Let A, B, C satisfy the PFAEL conditions. Let there exist no $U \subset C$, with U an element of \mathcal{U}_C satisfying 1 and 2.

Suppose, by contradiction, that there exists a $C' \subset C$ such that $AC'B$ is satisfied.

$C' = C - V$, for some $V \subset C$.

Then, either

$V \in \mathcal{U}_C$ and there exists some $[a_0]_C \in q_A(V)$ (or, similarly, some $[b_0]_C \in q_B(V)$) for which $q_A^{-1}([a_0]_C) \subseteq V$,

or

$V \notin \mathcal{U}_C$.

If the first possibility is true, then we know that for $a_0 \in A$, there does not exist a $c_0 \in C' = C - V$ such that $c_0 \approx a_0$. This contradicts Lemma 5.5.

On the other hand, if the second possibility is true, then there must exist $c \in V$ such that there does not exist $\delta > 0$ with $(N_\delta(c) \cap C) \subset V$. This further implies that for all $\delta > 0$, there exists c_\star with $c_\star \in (N_\delta(c) \cap C')$ (where $C' = C - V$). We can then define an infinite sequence $\{\delta_k\}$ (in a manner similar to that described in the proof of Lemma 5.4) such that $(N_{\delta_{i+1}}(c) \cap (C - V)) \subsetneq (N_{\delta_i}(c) \cap (C - V))$. To do so, we need only choose an original δ_0 and choose some point $c_1 \neq c$ in $N_{\delta_0}(c) \cap C'$. We then choose $\delta_1 = \min\{d_E(c, c_1), 1\}$, forcing $(N_{\delta_1}(c) \cap C') \subsetneq (N_{\delta_0}(c) \cap C')$, and so on. For each i , we choose $\delta_i = \min\{d(c_i, c), \frac{1}{2^i}\}$ to ensure that $\{\delta_k\} \rightarrow 0$. Now, for any infinite sequence $\{x_m\}$ satisfying $x_j \in (N_{\delta_j}(c) \cap C')$ and $x_j \notin (N_{\delta_{j+1}}(c) \cap C')$, we know that $\{x_m\}$ converges to $c \in V$. An example of such a sequence is the sequence $\{c_z\}$ of c 's used in constructing $\{\delta_k\}$. This implies that $c \notin C'$ with c a limit point of C' . Hence, C' is neither closed nor compact. This is also a contradiction, and our theorem is proved. □

Theorem 5.4. *Let A, B, C satisfy the PFAEL conditions.*

There exist infinitely many points at every location on \overline{AB} (excepting the end points A, B)

\iff

There exists a set $U \in \mathcal{U}_C$ such that $|U| = \infty$ and

- 1. for all $[a]_C \in q_A(U)$, it is true that $|[a]_C| \geq 2$ and there exists $c \in [a]_C$ such that $c \notin U$,*
- 2. for all $[b]_C \in q_B(U)$, it is true that $|[b]_C| \geq 2$ and there exists $c \in [b]_C$ such that $c \notin U$.*

Proof. .

\implies

Let A, B, C satisfy the PFAEL conditions. Further, let there exist infinitely many points at each location between A and B on \overline{AB} .

Suppose no set $U \in \mathcal{U}_C$ such that $|U| = \infty$ exists which satisfies 1 and 2. If this is case, one of the following two possibilities must be true:

Case I: No set $U \in \mathcal{U}_C$ exists which satisfies 1 and 2.

Suppose that no set U exists in \mathcal{U}_C which satisfies 1 and 2. We know this yields a contradiction, because Theorem 5.3 ensures that there is exactly one point at each location on \overline{AB} .

Case II: There exist $k \in \mathbb{N}$ finite sets $V_i \in \mathcal{U}_C$ satisfying 1 and 2, with $|V_i| = m_i \in \mathbb{N}$.

Note:

There can only be finitely many V_i 's, because the union of all V_i 's is also open and satisfies 1 and 2. If there were an infinite number of V_i 's, then there would be an infinite set $U \in \mathcal{U}_C$ satisfying 1 and 2, and we are excluding that possibility.

Now, suppose that there exist $k \in \mathbb{N}$ finite sets $V_i \in \mathcal{U}_C$ satisfying 1 and 2, with $|V_i| = m_i \in \mathbb{N}$. Then let $V^* = \bigcup_{i=1}^k V_i$. We see that V^* is the largest open set in \mathcal{U}_C which satisfies 1 and 2. Any set C' satisfying $AC'B$ at the same location as C must be a subset of C and of the form $C - V$ with $V \subseteq V^*$. Further, $|V^*| = M \in \mathbb{N}$. This implies that there are $\sum_{j=0}^M \binom{M}{j} = 2^M$ distinct subsets of V^* . But this means that there are at most 2^M points C' at the same location as C on \overline{AB} , and this is clearly a contradiction of our assumption that there are infinitely many such points.

\Leftarrow

Let A, B, C satisfy the PFAEL conditions. Suppose there exists a set $U \in \mathcal{U}_C$, such that $|U| = \infty$, satisfying 1 and 2.

We've seen in the proof of Theorem 5.3 that if there exists an open set $U \in \mathcal{U}_C$ satisfying 1 and 2, then $C - U = C'$ is on \overline{AB} at the same location as C . Further, every subset U' of U satisfies 1 and 2 by inheritance. We have seen in Lemma 5.4 that if U is infinite, then there exist infinitely many subsets U' of U in \mathcal{U}_C . Therefore, we know that there are infinitely many sets $U' \in \mathcal{U}_C$ satisfying 1 and 2. As a result, there also exist infinitely many points $C' = C - U'$ at the same location as C on \overline{AB} . Since C is at an arbitrary location on \overline{AB} , there must be an infinite number of points at every location between A and B on \overline{AB} . \square

Theorem 5.5. *Let A, B, C satisfy the PFAEL conditions.*

There exist $m \in \mathbb{N}$ ($m \geq 2$) points at each location between A and B on \overline{AB}

\iff

There exists $U \in \mathcal{U}_C$ with $|U| = k$ for some $k \in \mathbb{N}$ such that

1. *for all $[a]_C \in q_A(U)$, it is true that $|[a]_C| \geq 2$ and there exists $c \in [a]_C$ such that $c \notin U$,*
2. *for all $[b]_C \in q_B(U)$, it is true that $|[b]_C| \geq 2$ and there exists $c \in [b]_C$ such that $c \notin U$.*

and

There does not exist an infinite set $V \in \mathcal{U}_C$ satisfying 1 and 2.

Proof. .

\implies

Let A, B, C satisfy the PFAEL conditions. Let there be $m \geq 2$ points at each location between A and B on \overline{AB} .

We have seen in Theorem 5.3 that if no set $U \in \mathcal{U}_C$ exists which satisfies 1 and 2, then there is exactly one point at each location on \overline{AB} . Further, we know from Theorem 5.4 that if such a U exists and it is infinite, then there are infinitely many points at each location between A and B on \overline{AB} . Both of those situations lead to contradictions. Thus, we know that such a U must exist and that it cannot be of infinite cardinality.

\Leftarrow

Let A, B, C satisfy the PFAEL conditions. Let there exist a finite set $U \in \mathcal{U}_C$ satisfying 1 and 2. Further, let it be true that there does not exist a set $V \in \mathcal{U}_C$, such that $|V| = \infty$, satisfying 1 and 2.

By Theorem 5.4, we know that there cannot be infinitely many points at each location between A and B on \overline{AB} . However, by Theorem 5.3, there must be more than one point at each location. The only remaining possibility is that, for some $m \in \mathbb{N}$, there exist m points at each location between A and B on \overline{AB} , and so this must be the case. □

Notice that the argument used in the \Leftarrow portion of the proof of Theorem 5.4 implies that if the conditions of Theorem 5.5 hold, then $k = |U| \geq \log_2 m$. If $k < \log_2 m$, then the maximum number of points at each location on \overline{AB} is less than $2^{\log_2 m} < m$.

6 The Geometry of $\mathcal{H}(\mathbb{S}^2)$

We will now consider the space of all non-empty compact subsets of \mathbb{S}^2 , the surface of a sphere within \mathbb{R}^3 . Without loss of generality, we will consider only the unit sphere, or the sphere of radius 1. As done in the case for $\mathcal{H}(\mathbb{R}^n)$, we will define the Hausdorff metric on $\mathcal{H}(\mathbb{S}^2)$ and begin to look at the geometry induced by this metric.

6.1 Spherical Geometry: Background

First, we will define lines in \mathbb{S}^2 then we will define the distance between two points of \mathbb{S}^2 , see [4].

Definition 6.1. *A line on \mathbb{S}^2 , or great circle, is the intersection of the sphere and a plane that passes through the center of the sphere.*

Notationally, the line defined by $x, y \in \mathbb{S}^2$ is denoted \overleftrightarrow{xy} . As seen in [4] and [5], the length of a segment in \mathbb{S}^2 defined by x and y is equal to the measure of $\angle xOy$, where O is the center of the sphere.

Definition 6.2. *The distance between $x, y \in \mathbb{S}^2$, denoted $d_{arc}(x, y)$ is the length of the shortest segment, with x and y as end points, that lies on \overleftrightarrow{xy} .*

The segment whose length is $d_{arc}(x, y)$ for $x, y \in \mathbb{S}^2$ will be written as \overline{xy} . We also note that the length of any segment in \mathbb{S}^2 will be less than or equal to π .

6.2 The Hausdorff Metric

Definition 6.3. *Let A and B be elements in $\mathcal{H}(\mathbb{S}^2)$.*

- If $x \in \mathbb{S}^2$, the “distance” from x to B is

$$d(x, B) = \min_{b \in B} \{d_{arc}(x, b)\}.$$

- The “distance” from A to B is

$$d(A, B) = \max_{x \in A} \{d(x, B)\}.$$

Note that this is not a metric as described in definition 1.1, since $d(A, B)$ can be different than $d(B, A)$.

- The Hausdorff distance, $h(A, B)$, between A and B is

$$h(A, B) = d(A, B) \vee d(B, A),$$

where $d(A, B) \vee d(B, A) = \max\{d(A, B), d(B, A)\}$.

We assert that this function h is indeed a metric, the proof of this follows a similar argument as done in Section 1.2. It appears that the Hausdorff metric on $\mathcal{H}(\mathbb{S}^2)$ maintains many of the same properties as on $\mathcal{H}(\mathbb{R}^2)$, but as we will see there are some surprising differences.

Theorem 6.1. For all $A, B \in \mathcal{H}(\mathbb{R}^2)$, $h(A, B) \leq \pi$.

Proof. Let $A, B \in \mathcal{H}(\mathbb{R}^2)$. Assume, by contradiction, that $h(A, B) > \pi$. This then implies one of the following:

$$d(A, B) > \pi \tag{6.1}$$

$$d(B, A) > \pi \tag{6.2}$$

Without loss of generality, we will assume the case that (6.1) is true. It now follows by Definition 6.3 that there exists $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) = d(A, B) > \pi$. This is a contradiction to the fact that all segments of \mathbb{S}^2 are of length less than or equal to π . Thus, for $A, B \in \mathcal{H}(\mathbb{R}^2)$, we have $h(A, B) \leq \pi$. \square

Another notable difference between segments in $\mathcal{H}(\mathbb{R}^2)$ and $\mathcal{H}(\mathbb{S}^2)$ is the possible number of points of the Hausdorff segment between $A = \{a\}$ and $B = \{b\}$. For $\mathcal{H}(\mathbb{R}^2)$, it was shown in [1] that there would be only one point at each location on the Hausdorff segment defined by points A and B . This is not always the case in $\mathcal{H}(\mathbb{S}^2)$. If the points A and B are antipodal, meaning a and b are antipodal in \mathbb{S}^2 , then we claim there will be infinitely many points at each location on the Hausdorff segment. Let $s, t \in \mathbb{R}^+$ such that $s + t = \pi$. If we consider the intersection of $A + t$ and $B + s$, we see for antipodal points in \mathbb{S}^2 this will be a circle when $s \neq t$ and a great circle for $s = t$, as seen in Figure 22. Any non-empty compact subset of the intersection will satisfy ACB . Thus we see that there will be an infinite number of such sets C since there is an infinite number of ways to take a non-empty compact subset of a circle. If we looked at the trace of the largest set C , which would be the entire circle, we would cover the surface of the sphere. Using Figure 22, one can see this as the circle shown was traced from the point at the north pole of the sphere to the south pole.

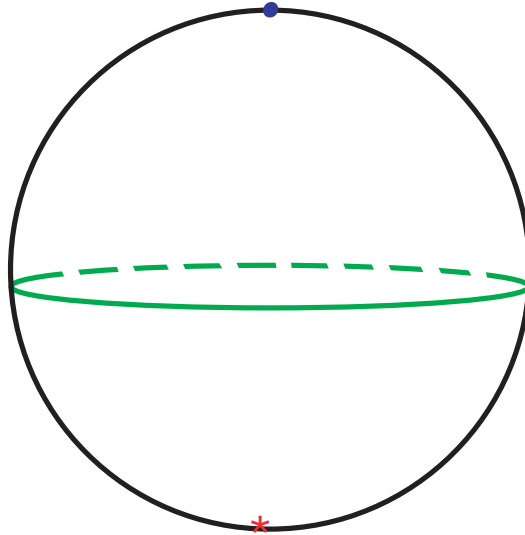


Figure 22: Infinite number of points between the points.

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