The Geometry of the Hausdorff Metric
GVSU REU 2008

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1 Introduction

The Hausdorff metric is familiar to fractal aficionados for describing the convergence of sequences of compact sets to their attractors in iterated function systems. In addition to giving rise to the fascinating geometry of fractals, the Hausdorff metric also imposes another geometry on the hyperspace $\mathcal{H}(\mathbb{R}^n)$ of all non-empty compact subsets of $\mathbb{R}^n$. This is a fascinating and unfamiliar non-Euclidean geometry that provides interesting links among Euclidean geometry, topology, number theory, and graph theory. There are many open questions that we could pursue in the geometry of the Hausdorff metric this summer in the GVSU REU program. In this short paper we will discuss some background material and then briefly introduce lines and segments in this geometry, connections to number theory, and notions of convexity as possible areas of investigation this summer. Please note, however, that I am open to working on any other questions related to this geometry that you might have.

2 Background

2.1 The Hausdorff Metric

In our geometry, points will be the non-empty compact subsets of n-dimensional real space $\mathbb{R}^n$. We will denote this hyperspace by $\mathcal{H}(\mathbb{R}^n)$ following the notation in [1], and use the Hausdorff metric to provide a measure of distance between elements in $\mathcal{H}(\mathbb{R}^n)$. Let d_E denote the standard Euclidean distance function.

Definition 2.1. *Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$.*

- *If $x \in \mathbb{R}^n$, the "distance" from x to B is*
$$d(x, B) = \min_{b \in B}\{d_E(x, b)\}.$$
A picture of this can be seen at left in Figure 1.
- *The "distance" from A to B is*
$$d(A, B) = \max_{x \in A}\{d(x, B)\}.$$
Note that $d(A, B) \neq d(B, A)$ in general as shown at right Figure 1.
- *The Hausdorff distance, $h(A, B)$, between A and B is*
$$h(A, B) = d(A, B) \vee d(B, A),$$
where $d(A, B) \vee d(B, A) = \max\{d(A, B), d(B, A)\}$.

The function h satisfies all of the properties of a distance function; namely

1. $h(A, B) = h(B, A)$,

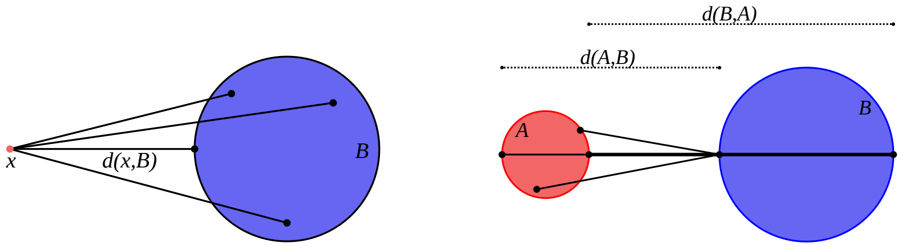

Figure 1: Left: $d(x, B)$. Right: $d(A, B)$ and $d(B, A)$.

2. $h(A, B) = 0$ if and only if $A = B$, and
3. $h(A, C) \leq h(A, B) + h(B, C)$

for all $A, B, C \in \mathcal{H}(\mathbb{R}^n)$. Any function satisfying these three properties is called a *metric*. An important point to note is that since A and B are compact sets, we can find specific elements $a \in A$ and $b \in B$ so that $d_E(a, b) = d(A, B)$ and thus achieve the maximum and minimum values in Definition 2.1.

This definition of the Hausdorff metric is somewhat counter-intuitive. For example, let $A = \{0\}$ and $B = \{0, 1\}$ in $\mathbb{R}$. Since $A \subset B$, we might think that $h(A, B)$ should be 0. Although $d(A, B) = 0$, it is the case that $d(B, A) = d_E(1, 0) = 1$, so $h(A, B) = 1$. Thus, A and B are actually a distance 1 apart under this metric. Note further that $d(A, B) \neq d(B, A)$ and so d itself is not a metric, which is why we need to take the maximum of $d(A, B)$ and $d(B, A)$ in Definition 2.1. In fact, ($\mathcal{H}(\mathbb{R}^n)$,$h$) is a complete metric space [1, 8].

As another example, in Figure 2 the set A is the circle with radius 100 and B the shaded disk with radius 40, both centered at the origin. Note that $d(A, B) = d_E((100, 0), (40, 0)) = 60$ while $d(B, A) = d_E((0, 0), (100, 0)) = 100$. So $h(A, B) = d(B, A) = 100$.

Throughout this paper we will need to differentiate between "points" in $\mathbb{R}^n$ and "points" in $\mathcal{H}(\mathbb{R}^n)$. We will use the word "point" and lowercase letters when referring to points in $\mathbb{R}^n$ and the word "element" and uppercase variables when referring to "points" in $\mathcal{H}(\mathbb{R}^n)$. Although we won't study the applications, the Hausdorff metric is well known and is used in image matching, in visual recognition by robots and in computer-aided surgery. The distance can be used to compare what is seen with pre-programmed or recognized patterns – the smaller the distance the better the match.

2.2 An Important Construction - The Dilation of a Set

One important construction in this geometry that is related to circles, segments, and lines is the dilation of a set - the collection of all points that are a distance less than or equal to r from A, where $A \in \mathcal{H}(\mathbb{R}^n)$ and $r > 0$.

Definition 2.2. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $r > 0$. The dilation of A by a ball of radius r is the set*

$$(A)_r = \{x \in \mathbb{R}^n : d_E(x, a) \leq r \text{ for some } a \in A\}.$$

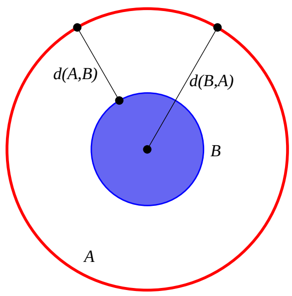

Figure 2: $h(A, B) = 100$

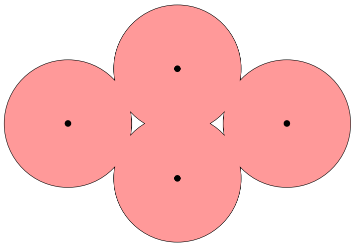

Figure 3: A dilation of a 4-point set.

In essence, the dilation of a set A by a ball of radius r is just the union of all closed Euclidean r-balls with centers in A. For example, if A is the set of 4 black points in Figure 3, then the dilation $(A)_r$ in $\mathbb{R}^2$ is the shaded region. With this idea in mind, we could alternatively define $h(A, B)$ to be the smallest r so that $(A)_r$ encloses B and $(B)_r$ encloses A (as in [12]).

The dilation $(A)_r$ has two important properties as stated in the next theorem (from [7]).

Theorem 2.1. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and let $r > 0$. Then $h(A, (A)_r) = r$ and if $B \in \mathcal{H}(\mathbb{R}^n)$ is such that $h(B, A) = r$, then $B \subseteq (A)_r$.*

Proof. Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $r > 0$. To show $(A)_r \in \mathcal{H}(\mathbb{R}^n)$, we show $(A)_r$ is a closed and bounded set and invoke the Heine-Borel Theorem[1] Since $A \in \mathcal{H}(\mathbb{R}^n)$, we know that A is closed and bounded. Let M be an upper bond for A, that is $|a| = d_E(a, 0) \le M$ for all $a \in A$. Let $x \in (A)_r$. Then there is an element $a_x \in A$ so that $d_E(x, a_x) \le r$. Thus

$$d_E(x, 0) \le d_E(x, a_x) + d_E(a_x, 0) \le r + M$$

and $(A)_r$ is bounded by $M + r$.

Next we demonstrate that $(A)_r$ is closed. Let x be a limit point of $(A)_r$. Then there is a sequence $\{x_m\}$ in $(A)_r$ that converges to x. For each x_m there is a point $a_m \in A$ so that $d_E(x_m, a_m) \le r$. Since the sequence $\{a_m\}$ is a bounded sequence (A is bounded), the Bolzano-Weierstrass Theorem[2] tells us that $\{a_m\}$ has a convergent subsequence $\{a_{m_k}\}$. Since A is closed, $a = \lim_{k\to\infty} a_{m_k}$ is an element of A. Now, let $\epsilon > 0$ and choose $N > 0$ so that $m, k > N$ imply $d_E(x, x_m) < \frac{\epsilon}{2}$ and $d_E(a, a_{m_k}) < \frac{\epsilon}{2}$. Then for $t > N$, we have

$$d_E(x, a) \le d_E(x, x_t) + d_E(x_t, a_{m_t}) + d_E(a_{m_t}, a) < \epsilon + r.$$

This shows that $d_E(x, a) \le r$ and $x \in (A)_r$. Therefore, $(A)_r$ is compact.

To show $h(A, (A)_r) = r$, note that $A \subseteq (A)_r$ and $d(A, (A)_r) = 0$. Now we show $d((A)_r, A) = r$. For each $x \in (A)_r$, there is an element $a_x \in A$ so that $d_E(x, a_x) \le r$. Therefore, $d(x, A) \le r$ for all $x \in A$. This shows $d((A)_r, A) \le r$ and, consequently, $h(A, (A)_r) \le r$. To obtain equality, we only need to find an element not in A that is a distance r from A. Let $a \in A$ with $|a|$ a maximum. Let $x = (1 + \frac{r}{|a|})a$ so that x is the point on the line through the origin and a that is r units farther from the origin than a. Then $d_E(x, a) = |x - a| = r$. Thus, $d(x, A) \le r$. To show $d(x, A) = r$, we need to know that $d_E(x, a') \ge r$ for all $a' \in A$. If there is an element $a' \in A$ so that $d_E(a', x) < r$, then

$$|a| + r = d_E(0, x) \le d_E(0, a') + d_E(a', x) < |a'| + r.$$

But this implies $|a'| > |a|$, a contradiction. We conclude $d(x, A) = r$ and $h(A, (A)_r) = r$.

Finally, we show that $(A)_r$ is the largest element (in the sense of containment) of $\mathcal{H}(\mathbb{R}^n)$ that is a Hausdorff distance r from A. Let $B \in \mathcal{H}(\mathbb{R}^n)$ with $h(A, B) = r$. Let $b \in B$. The fact that $h(A, B) = r$ implies that there is an element $a \in A$ so that $d_E(a, b) \le r$. Therefore, $b \in (A)_r$ and $B \subseteq (A)_r$. ☐

Theorem 2.1 shows that $(A)_r$ is the largest set (in the sense of containment) that is r units from A. In other words, every element in $\mathcal{H}(\mathbb{R}^n)$ that is r units from A is a subset of $(A)_r$.

[1]If you haven't seen the Heine-Borel Theorem before, you can look it up in most any standard topology or real analysis text. I encourage you to do so.

[2]Look this one up, too.

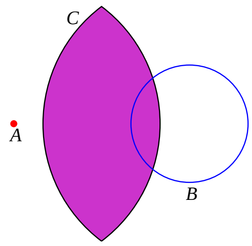

Figure 4: Both C and the boundary of C lie on this Hausdorff segment.

2.3 Betweenness in $\mathcal{H}(\mathbb{R}^n)$

We can extend the Euclidean notions of segment and line to $\mathcal{H}(\mathbb{R}^n)$. Recall that in Euclidean geometry, the point c lies on the line segment between the points a and b if $d_E(a,b) = d_E(a,c) + d_E(c,b)$. This idea depends only on the distance function d_E, so we can extend it to any set on which we have a metric. In particular, we define betweenness in $\mathcal{H}(\mathbb{R}^n)$ as follows.

Definition 2.3. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. The set $C \in \mathcal{H}(\mathbb{R}^n)$ lies between A and B if*

$$h(A,B) = h(A,C) + h(C,B). \tag{2.1}$$

As in [5], we will adopt the notation ACB to indicate that C lies between A and B. Note that this notion of betweenness truly is an extension of the notion of betweenness in Euclidean geometry. In fact, if $A = \{a\}$ and $B = \{b\}$ are single element sets in $\mathcal{H}(\mathbb{R}^n)$, then the elements in $\mathcal{H}(\mathbb{R}^n)$ that lie between A and B are exactly the single element sets $C = \{c\}$, where $c \in \overline{ab}$ [7]. However, the notion of betweenness in $\mathcal{H}(\mathbb{R}^n)$ is more general than its counterpart in Euclidean geometry. As an example, consider the sets $A = \{(-50,0)\}$ and B the circle centered at (100,0) with radius 50 as shown in Figure 4. In this case, we have $h(A,B) = 200$. Let $C = (A)_{125} \cap (B)_{75}$ be the shaded region in Figure 4. As Theorem 2.2 (from [2]) shows, both ACB and $A\partial CB$[3] are satisfied with $h(A,C) = h(A,\partial C)$. In fact, Theorem 2.2 shows that any compact subset of the shaded region together with the boundary of the shaded region is a set that lies at this location between A and B so that we have infinitely many different elements at a given location between A and B. The existence of many elements in $\mathcal{H}(\mathbb{R}^n)$ at a given location between sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ is well known. For example, in [6], the following lemma is proved.

Lemma 2.1. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$, $h(A,B) = r$ and let*

$$C_s = (A)_s \cap (B)_{(r-s)}$$

for each $s \in [0,r]$. Then $h(A,C_s) = s$ and $h(C_s,B) = r - s$.

Other examples can be found in [13]. In [2], we find a very important generalization of Lemma 2.1:

Theorem 2.2. *Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ with $d(B,A) \geq d(A,B)$, $r = h(A,B)$, $s \in \mathbb{R}$ with $0 < s < r$, and $t = r - s$. If C is a compact subset of $(A)_s \cap (B)_t$ containing $\partial\left((A)_s \cap (B)_t\right)$, then C satisfies BCA with $h(A,C) = s$.*

Proof. Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ with $r = d(B,A) \geq d(A,B)$. Let $s \in \mathbb{R}$ so that $0 < s < r$ and let $t = r - s$. Choose $C \subset (A)_s \cap (B)_t$. Then $h(A,C) \leq s$ and $h(B,C) \leq t$. In order to obtain the desired equalities, the key is to show $\partial\left((A)_s \cap (B)_t\right) \neq \emptyset$. We verify this in the case when $s \geq r - d(A,B)$ and leave the case $0 < s < r - d(A,B)$ to the reader.

Assume $s \geq r - d(A,B)$. Let $a \in A$ and $b \in B$ so that $d_E(a,b) = d(a,B) = d(A,B)$. Let $y \in \overrightarrow{ba}$ so that $d_E(b,y) = t$. Since $s \geq r - d(A,B)$, it follows that $d(A,B) \geq r - s = t$. Thus, $d_E(a,b) = d(A,B) \geq t$. So $y \in \overline{ba}$ and

$$d_E(y,a) = d_E(b,a) - d_E(b,y) = d(A,B) - t \leq r - t = s. \tag{2.2}$$

[3]The notation ∂B denotes the boundary of the set B.

Thus, $y \in (A)_s$. Now we will show $d(y, B) = t$. Suppose there exists $b' \in B$ so that $d_E(y, b') < t$. Then

$$d_E(a, b') \leq d_E(a, y) + d_E(y, b'). \tag{2.3}$$

Now (2.2) and (2.3) combine to show $d_E(a, b') < (d(A, B) - t) + t = d(A, B)$, a contradiction to $d(A, B) = d(a, B)$. Therefore, $d(y, B) = t$ and, consequently, $y \in \partial(B)_t$. Now use the fact that for any $X, Y \in \mathcal{H}(\mathbb{R}^n)$,

$$\partial(X \cap Y) = (\partial X \cap Y) \cup (X \cap \partial Y),$$

to show $y \in \partial\left((A)_s \cap (B)_t\right)$. □

Theorem 2.2 shows that for certain sets $A, B \in \mathcal{H}(\mathbb{R}^n)$, there can be more than one set (even infinitely many sets) C satisfying ACB at a fixed location between A and B. We formalize the notion of elements at the same location between sets A and B in the next definition. In the example of Figure 4, there are infinitely many compact subsets C^* of $\mathbb{R}^2$ contained in C that also contain ∂C. For each of these sets, Theorem 2.2 shows C^* satisfies AC^*B with $h(A, C^*) = 125$.

Definition 2.4. *Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$. The elements $C, C' \in \mathcal{H}(\mathbb{R}^n)$ are said to be at the same location between A and B if C and C' satisfy ACB and $AC'B$ with $h(A, C) = h(A, C') = s$ for some $0 < s < h(A, B)$.*

As we will see, this existence of multiple sets at a fixed location between elements A and B leads to many interesting results.

In the remainder of this paper, we will briefly introduce some context for problems we might investigate this summer. We will omit proofs and try to explain the big picture. If any of these topics interest you, we will go into more depth during the program.

3 Segments and Lines

In this section, we introduce segments and lines in $\mathcal{H}(\mathbb{R}^n)$.

3.1 Segments

We define the Hausdorff segment with end elements A and B to be the set of all $C \in \mathcal{H}(\mathbb{R}^n)$ satisfying ACB. Segments provide many problems to study and lead to a variety of surprising results. For example, familiar integer sequences such as the Fibonacci and Lucas numbers arise in a natural way in this context; there are also fascinating properties of the numbers 19 and 37 as illustrated by segments.

There are two types of Hausdorff segments. In one type, there are infinitely many different elements at each location between elements A and B (for a dynamic, continuous example, see
`http://faculty.gvsu.edu/schlicks/HausdorffGeometry/H10.htm`).

The more interesting (in my opinion) case is when there are only finitely many elements C at each location between A and B. The conditions under which this occurs motivate the following definition [10].

Definition 3.1. *A configuration with ends $A, B \in \mathcal{H}(\mathbb{R}^n)$ is a pair $[A, B]$ where*

$$h(A, B) = d(b, A) = d(a, B) = r \text{ for all } a \in A \text{ and } b \in B. \tag{3.1}$$

If $[A, B]$ is a configuration and both A and B are finite sets, then $[A, B]$ is a finite configuration. Otherwise, $[A, B]$ is an infinite configuration.

As an example, consider the elements A and B that are alternate vertices of a regular hexagon in $\mathbb{R}^2$ as shown at left in Figure 5. We will pictorially represent a configuration using red points to denote points in A and blue points to denote points in B. We draw line segments between points $a \in A$ and $b \in B$ satisfying $d_E(a, b) = h(A, B)$ and call such points *adjacent* points. The points $c_1, c_2, c_3, c_4, c_5, c_6$ are the points at the intersection of $(A)_s$ and $(B)_t$ for some $0 < s, t < h(A, B)$. In this example, the set $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ is one element at the indicated location between A and B. There are many other elements at this same location. For example, $C - \{c_1\}$, $C - \{c_1, c_3, c_6\}$ are two such elements. As an exercise, I encourage you to determine each of the 18 elements at this location between A and B.

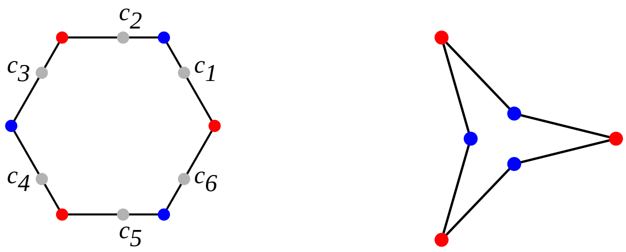

Figure 5: Left: The number of elements between A and B at each location is finite. Right: Another P_3.

While condition (3.1) is necessary for there to be finitely many sets at each location between elements A and B, it is not sufficient. If $[A, B]$ is a finite configuration, then there will be a finite number of elements C at each location between A and B. However, if $[A, B]$ is an infinite configuration, then there may still be an infinite number of elements at each location between A and B. For example, let A be the union of the origin and the circle of radius 100 centered at the origin and B the circle of radius 50 centered at the origin as shown in Figure 6. Clearly $[A, B]$ is a configuration, and $h(A, B) = 50$. For any $0 < s < 50$, the element $C_s = (A)_s \cap (B)_{50-s}$ is the union of two circles C_1 and C_2 as indicated in Figure 6. In this example, the elements $C^* = C_2 \cup \{c\}$, where c is any point in C_1, are elements satisfying AC^*B with $h(A, C^*) = s$, and thus infinitely many elements lie at each location between A and B.

In [3] the authors show that the number of elements between elements A and B in a configuration is the same at any location[4], so we let $\#([A, B])$ denote the number of elements at each location between A and B if that number is finite. Otherwise, $\#([A, B])$ is infinite.

3.2 Special Configurations: String and Polygonal Configurations

The simplest types of configurations are those obtained from selecting equally spaced points along a line. We will call these configurations *string configurations*. As an example, string configurations of length 6 (S_6) are shown in Figure 7. If we make adjacent the endpoints of a string configuration, we obtain another simple type of configuration we call a *polygonal configuration*. Two equivalent configurations for a polygonal configuration with 6 vertices (denoted as P_3) are shown in Figure 5.

String and polygonal configurations are, in some sense, the building blocks of all finite configurations. There are many closed and bounded sets in $\mathbb{R}^n$, so one might expect that, given any positive integer k, there is a configuration $[A_k, B_k]$ so that $\#([A_k, B_k]) = k$. Using string and polygonal configurations, we have found configurations $[A_k, B_k]$ so that $\#([A_k, B_k]) = k$ for infinitely many different values of k, including all k from 1 to 18. What was completely surprising to us was to prove that no configuration $[A, B]$ exists so that $\#([A, B] = 19$ [3]. We were later able to show that 19 isn't the only number with this property. In fact, no configuration $[A, B]$ exists so that $\#([A, B] = 37$ [4]. Equally interesting is the fact that the only integers that appear as $\#([A, B])$ for configurations in $\mathbb{R}$ are products of Fibonacci numbers. So there are infinitely

[4]Another way to say this is that $|C_s|$ is the same for every value of s with $0 < s < h(A, B)$.

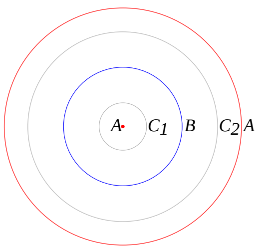

Figure 6: A configuration $[A, B]$ with infinite $\#([A, B])$.

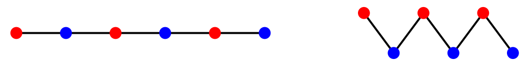

Figure 7: Two equivalent string configurations S_6.

many different numbers that are of the form $\#([A,B])$ for $A, B \in \mathcal{H}(R^2)$ that cannot be realized in $\mathcal{H}(R)$. We know of one integer (namely 57) so that $57 = \#([A,B])$ for some $A, B \in \mathcal{H}(\mathbb{R}^3)$ but for which there is no configuration X in $\mathcal{H}(\mathbb{R}^2)$ or $\mathcal{H}(\mathbb{R})$ with $\#(X) = 57$. To have some terminology to refer to this type of behavior, we say that a positive integer k is a SPACK-n integer if there is a configuration X in $\mathcal{H}(\mathbb{R}^n)$ with $\#(X) = k$, but there is no configuration $Y \in \mathcal{H}(\mathbb{R}^m)$ for $m < n$ with $\#(Y) = k$. If k has the property that there is no configuration X at all so that $\#(X) = k$, then we call k a SPACK-0 number.

3.3 POSSIBLE REU RESEARCH PROBLEMS

- We don't understand what special properties of 19 and 37 make them SPACK-0 and it would be fascinating to determine what it is about these numbers that differentiate them from the other primes that causes this behavior.
- The primes 19 and 37 are the smallest SPACK-0 integers. We suspect that there are many others. Can we find some more? We have some counting techniques that have helped us show that 19 and 37 are SPACK-0, but will need more to determine which other integers share this property.
- If someone is interested in programming, we have developed a computer program that acts like a sieve to help us identify the numbers like 19 and 37. However, the run time for this program makes it unwieldy for large examples. Can we improve the efficiency of the program to help find more candidates?
- Can we find some condition on the size of a finite configuration $[A,B]$ that will help us determine useful upper and lower bounds on the size of $\#([A,B])$? If so, this could turn our sieve into a practical method for actually finding SPACK-0 numbers.
- So far, we only know two prime SPACK-0 numbers. It would be fascinating to know if there are any composite SPACK-0 numbers (the smallest candidate is 82).
- Is the set of SPACK-0 numbers finite or infinite?
- The integer 57 is the only known SPACK-3 number. Are there others? If so, how can we find them? Can there be any SPACK-4 numbers or can we realize every realizable integer in $\mathcal{H}(\mathbb{R}^3)$?
- The ideal result would be to completely classify exactly which integers are SPACK-n for each n. Is this even possible?
- Can we completely classify which elements in $\mathcal{H}(\mathbb{R}^n)$ lie at each location between arbitrary elements A and B in $\mathcal{H}(\mathbb{R}^n)$?

3.4 Lines

The Hausdorff line defined by A and B is the set of elements C satisfying CAB, ACB, or ABC. provide many problems to study and lead to a variety of surprising results. We have some interesting results on Hausdorff lines, but we know less about lines than we do about segments. To understand the conditions under which our theorems about lines hold, we need to introduce the r-neighborhood of an element A.

Definition 3.2. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and let r be a positive real number. The r-neighborhood of A is the set*

$$N_r(A) = \bigcup_{a \in A} N_r(a).$$

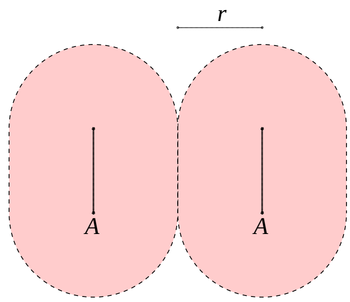

Figure 8: The r-neighborhood of A.

For example, if A is the union of the two line segments in Figure 8 and r is half the horizontal distance between the vertical segments, then $N_r(A)$ is the shaded region. Note that while $\overline{N_r(A)} = (A)_r$[5], it is not the case that the interior of $(A)_r$ is $N_r(A)$.

Here we will summarize our results about Hausdorff lines, omitting the details (found in [2]). There are three categories into which a Hausdorff line may fall.

Category 1. We will have elements at each location to the left of B, between B and A, and to the right of A if $(A)_s \cap \partial N_{h(A,B)+s}(B) \neq \emptyset$ for all s.

As an example of a Hausdorff line in this category, let $B = \{(-50,0),(0,0)\}$ and $A = \{(50,0)\}$ in $\mathcal{H}(\mathbb{R}^2)$. Then $h(A,B) = 100 = d(B,A) = d_E((-50,0),(50,0)) > d(A,B)$. Since $d(B,A) > d(A,B)$, we can find sets C satisfying BAC with $h(C,A) = s$ (one example is shown at right in Figure 9); the sets $C_s = (A)_s \cap (B)_t$ always satisfy BC_sA with $h(A,C_s) = s$ by Theorem 2.2 (one example is shown in the middle of Figure 9); and because $(B)_t \cap \partial N_{r+s}(A) \neq 0$ for every $t > 0$, we know that there are sets C satisfying CBA with $h(B,C) = t$ (an example is on the left in Figure 9).

Category 2. If we have elements $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfying $d(A,B) > d(B,A)$, and $A \subsetneq N_{h(A,B)}(B)$, then any Hausdorff line through A and B will be a halfline that stops at A.

An example is shown in Figure 10, where A is the union of the origin with the circle centered at the origin of radius 100 and B is the circle centered at the origin of radius 60. In this example, $h(A,B) = d(A,B) = 60$ and $(A)_s \subset N_{r+s}(B)$ for every $s > 0$. As a result, we can show that there are no elements $C \in \mathcal{H}(\mathbb{R}^n)$ that satisfy BAC with $h(A,C) = s$. Therefore, the Hausdorff line defined by A and B is a halfline with A as an end element. However, we can still find elements $C_s = (A)_s \cap (B)_{h(A,B)-s}$ satisfying BC_sA with $h(A,C_s) = s$ for each $0 < s < h(A,B)$ (an example is shown at left in Figure 10), and elements $C = (B)_s - N_{h(A,B)+s}(a_0)$ satisfying CBA and $h(B,C) = s$ for every $s > 0$ (an example is on the right in Figure 10).

Category 3. If $d(A,B) > d(B,A)$, and $(A)_s \subsetneq N_{h(A,B)+s}(B)$ for some $s > 0$, then there will be a value $s_{AB} = \inf\{s \in \mathbb{R}^+ \mid ((A)_s) \subseteq N_{h(A,B)+s}(B)\}$ such that there are no sets C satisfying BAC a distance greater than s_{AB} to the right of A. The completeness of the space $\mathcal{H}(\mathbb{R}^n)$ allows us to show that s_{AB} is the stopping point for this halfline. We will omit the details here.

[5] $\overline{S}$ is the closure of the set S.

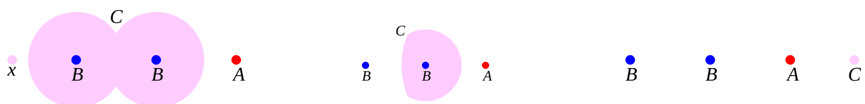

Figure 9: An element C satisfying Left: CBA, Middle: BCA, Right: BAC.

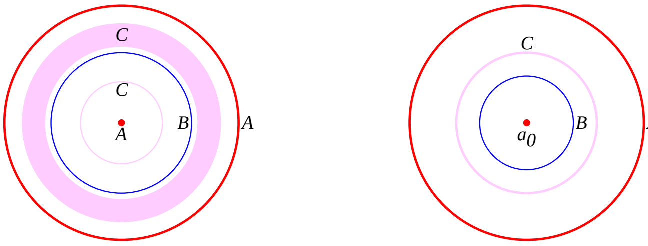

Figure 10: An element C satisfying Left: BAC, Right: CBA.

3.5 POSSIBLE REU RESEARCH PROBLEMS

There is much yet to learn about Hausdorff lines.

- We have only minimal theorems about which elements C satisfy CAB or ABC on a Hausdorff line. Can we prove some more theorems that provide additional information about elements on Hausdorff lines?
- As with segments, is it possible to find sets A and B so that there are only finitely many elements C satisfying ABC with $h(B, C) = s$? If so, can we completely characterize these situations?
- What is the cardinality of the collection of sets C satisfying ABC with $h(B, C) = s$. If infinite, it this cardinality always uncountable?

4 Connections to Number Theory

There are some interesting connections between the geometry of the Hausdorff metric and number theory. In [10] the authors show that $\#(S_m) = F_{m-1}$, where F_{m-1} is the $m-1^{\text{st}}$ Fibonacci number and $\#(P_m) = L_{2m}$, where L_{2m} is the $2m^{\text{th}}$ Lucas number. In addition, there are configurations X_k so that $\#(X_k)$ give us all of the Lucas numbers. We have found families of configurations that yield other known integer sequences. In 2006, we found a three parameter family of configurations $P^k_m(S_l)$ [11], called Polygonal Chains, so that the sequences

- $\{P^k_m(S_l)\}$ for fixed m and k,
- $\{P^k_m(S_l)\}$ for fixed m and l, and
- $\{P^k_m(S_l)\}$ for fixed k and l

provide previously unknown integer sequences (as determined by [14]). These configurations are built by connecting polygonal configurations with string configurations. Two examples are shown in Figure 11. The parameter m determines the size of the polygonal subconfigurations, the parameter l determines the length of the sting configuration connectors, and the parameter k controls the number of polygonal subconfigurations that are connected. Sample sequences are shown in Table 1.

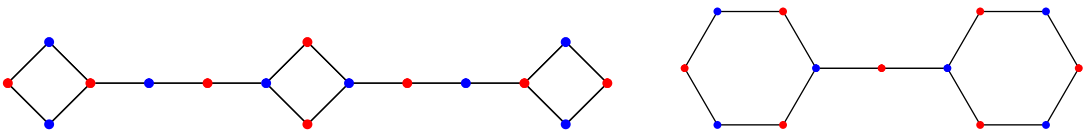

Figure 11: Left: $P^3_2(S_3)$, Right: $P^2_3(S_2)$.

i	1	2	3	4	5
$\#(P^i_{2,1})$	7	113	1815	29153	468263
$\#(P^2_{i,1})$		113	765	5234	35865
$\#(P^2_{2,i})$	113	176	289	465	754
$\#(P^i_{3,1})$	18	765	32733	1400634	59932665
$\#(P^3_{i,2})$		4393	80361	1425131	25671393
$\#(P^3_{3,i})$	32733	80361	215658	559305	1469565

Table 1: Some Polygonal Chain sequences

4.1 POSSIBLE REU RESEARCH PROBLEMS

- What other new families of configurations can we find that will provide more previously unknown integer sequences?

5 Convexity

Last summer we began to study notions of convexity in $\mathcal{H}(\mathbb{R}^n)$. As with segments and lines, we define convexity analogously to the definition in R^n.

Definition 5.1. *A subset X of $\mathbb{R}^n$ is convex if given any $a, b \in X$, the Euclidean line segment $\overline{ab}$ is entirely contained in X.*

Example of a convex and a non-convex set in $\mathbb{R}^2$ are shown in Figure 12.

Since there can be more than one element at each location between end elements in $\mathcal{H}(\mathbb{R}^n)$, convexity in $\mathcal{H}(\mathbb{R}^n)$ will depend on how we define segments. To date, we have begun studying three different types of convexity in $\mathcal{H}(\mathbb{R}^n)$. The strongest type of convexity is the one that forces every element between A and B to be in a convex subset.

Definition 5.2. *A set $\mathcal{X} \subseteq \mathcal{H}(\mathbb{R}^n)$ is* Complete Hausdorff Convex *(CHC) if, given $A, B \in \mathcal{X}$, the collection of all sets C satisfying ACB is a subset of $\mathcal{X}$.*

Examples of a CHC set is the collection of all concentric circles centered at the origin in $\mathbb{R}^2$.

Given A and B in $\mathcal{H}(\mathbb{R}^n)$, we know that the set $C_s = (A)_s \cap (B)_{h(A,B)-s}$ is the largest set that satisfies AC_sB for every s between 0 and $h(A, B)$. This leads us to a second type of convexity.

Definition 5.3. *A set $\mathcal{X} \subset \mathcal{H}(\mathbb{R}^n)$ is* Strong Hausdorff Convex *(SHC) if, given any $A, B \in \mathcal{X}$, then $(A)_s \cap (B)_{h(A,B)-s} \in \mathcal{X}$ for every $0 < s < h(A, B)$.*

An example of a SHC set is the collection of all disks centered at the origin in $\mathbb{R}^2$. Note that this set is not CHC.

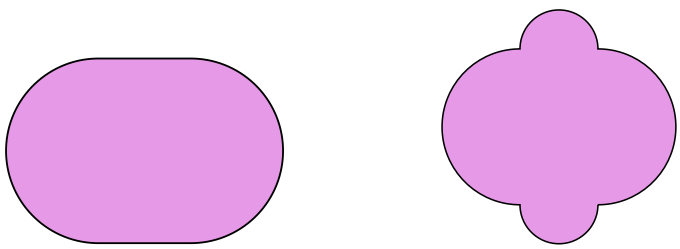

Figure 12: Left: A convex subset of $\mathbb{R}^n$. Right: A subset of $\mathbb{R}^n$ that is not convex.

Theorem 2.2 shows that the element $\partial C_s = \partial\left((A)_s \cap (B)_{h(A,B)-s}\right)$ always satisfies $A\partial C_s B$ for every $0 < s < h(A,B)$. If we consider a segment to contain only these sets, then we obtain a third type of convexity.

Definition 5.4. *A set $\mathcal{X} \subset \mathcal{H}(\mathbb{R}^n)$ is* Boundary Hausdorff Convex *(BHC) if, given $A, B \in \mathcal{X}$, then $\partial\left((A)_s \cap (B)_{h(A,B)-s}\right) \in \mathcal{X}$ for every $0 < s < h(A,B)$.*

For example, the collection of all finite unions of circles centered at the origin is BHC but not CHC or SHS.

5.1 POSSIBLE REU RESEARCH PROBLEMS

- Can we completely characterize the sets that are CHC, SHC, and BHC?
- Are there other interesting types of convexity to consider?

References

[1] Michael F. Barnsley, *Fractals Everywhere*, Second Edition, Academic Press Professional, 1998.

[2] C. Bay, A. Lembcke, and S. Schlicker, When Lines Go Bad In Hyperspace, *Demonstratio Mathematica*, Volume 38 : 689-701, (2005).

[3] C. Blackburn, K. Lund, S. Schlicker, P. Sigmon, and A. Zupan, A Missing Prime Configuration in the Hausdorff Metric Geometry, *submitted*.

[4] C. Blackburn, K. Honigs, S. Schlicker, and A. Zupan, Graph Theory Applications of the Hausdorff Metric Geometry, *in preparation*.

[5] L. M. Blumenthal, Theory and Applications of Distance Geometry, Oxford University Press, 1953.

[6] A. Bogdewicz, Some Properties of Hyperspaces, *Demonstratio Mathematica*, XXXIII: 135-149, 2000.

[7] D. Braun, J. Mayberry, A. Powers, and S. Schlicker, A Singular Introduction to the Hausdorff Metric Geometry, *Pi Mu Epsilon Journal*, Vol. 12, No. 3, p. 129-138, 2005.

[8] Gerald A. Edgar, *Measure, Topology, and Fractal Geometry*, Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1990.

[9] Chantel Blackburn, Katrina Honigs, Steven Schlicker, and Alexander Zupan, Graph Theory Applications of the Geometry of $\mathcal{H}(\mathbb{R}^n)$, *in preparation*.

[10] Kristine Lund, Patrick Sigmon, and Steven Schlicker, Fibonacci-Type Sequences in $\mathcal{H}(\mathbb{R}^n)$, submitted.

[11] Lisa Morales, Steven Schlicker, and Daniel Schultheis, Polygonal Chains, *in preparation*.

[12] James R. Munkres, *Topology*, Second Edition, Prentice Hall, 2000.

[13] R. Schneider, *Convex Bodies: the Brunn-Minkowsik Theory*, Cambridge University Press, 1993.

[14] N. J. A. Sloane, (2006), The On-Line Encyclopedia of Integer Sequences, published electronically at `www.research.att.com/~njas/sequences/`.