The Geometry of the Hausdorff Metric

Dominic Braun[*], University of Virginia
John Mayberry[†], California State University of Fullerton
Audrey Powers[‡], Agnes Scott College
Steven Schlicker[§], Grand Valley State University

August 25, 2003

1 Introduction

If X is a complete metric space, the collection of all non-empty compact subsets of X forms a complete metric space, $(\mathcal{H}(X),h)$, where h is the Hausdorff metric introduced by Felix Hausdorff in the early 20th century. as a way to measure the distance between compact sets. The Hausdorff metric has many applications. The metric is used in image matching and visual recognition by robots [4] and in computer-aided surgery. In these situations the Hausdorff distance is used to compare what is seen with pre-programmed or recognized patterns – the smaller the distance the better the match. The Hausdorff metric is also used to find the roots of derivatives of polynomials [8], perform approximations with polynomials [7], and to develop algorithms to compute the distance between polygons [2]. The United States military has used the Hausdorff distance in target recognition procedures [6] and it plays an important role in the study of fractal geometry as well [1].

---

[*] Supported by NSF REU grant DMS-9820221
[†] Supported by NSF REU grant DMS-0137264
[‡] Supported by NSF REU grant DMS-0137264
[§] Supported by NSF REU grants DMS-9820221 and DMS-0137264

Although the Hausdorff metric has many applications, the geometry of the space $\mathcal{H}(\mathbb{R}^n)$ is less well known. In this paper we discuss the structure of fundamental geometric shapes in $\mathcal{H}(\mathbb{R}^n)$ using the Hausdorff metric as our measure of distance between sets.

2 The Hausdorff Metric

Let (X, d) be a complete metric space. Let $\mathcal{H}(X)$ be the space of all non-empty compact subsets of X. A distance function, called the *Hausdorff metric*, on $\mathcal{H}(X)$ is defined as follows:

Definition 1. *Let (X, d) be a complete metric space. Let A and B be elements in $\mathcal{H}(X)$.*

- *If $x \in X$, the "distance" from x to B is*

$$d(x, B) = \min_{b \in B}\{d(x, b)\}.$$

- *The "distance" from A to B is*

$$d(A, B) = \max_{x \in A}\{d(x, B)\}.$$

- *The Hausdorff distance, $h(A, B)$, between A and B is*

$$h(A, B) = d(A, B) \vee d(B, A),$$

  where $d(A, B) \vee d(B, A) = \max\{d(A, B), d(B, A)\}$.

As an example of the above definition, let A be the circle of radius 3 centered at the origin and B the disk of radius 1 centered at the origin in $\mathbb{R}^2$ (see figure 1). In this example, $d(A, B)$ can be measured from the point a to the point b in figure 1. To find $d(B, A)$, however, we must measure from the origin to a point on A. Thus, $d(A, B) = 2$ while $d(B, A) = 3$. We see that the function d in definition 1 is not a distance function, since $dA, B)$ can be different than

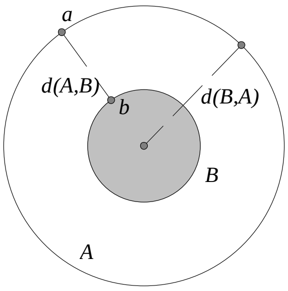

Figure 1: Left: $d(A,B)$ is not equal to $d(B,A)$.

$d(B,A)$. This is why we need to use the maximum of $d(A<B)$ and $d(B,A)$ to define a metric on $\mathcal{H}(X)$.

It is well known that h defines a metric on $\mathcal{H}(X)$. In fact, ($\mathcal{H}(X)$,h) is a complete metric space [1, 3]. This space ($\mathcal{H}(X)$,h) is an important one in the sense that it is the space in which fractals live. In the sections that follow we describe some properties of the Hausdorff metric and discuss some of the geometry of the space $\mathcal{H}(\mathbb{R}^n)$.

3 Circles and Neighborhoods in $\mathcal{H}(\mathbb{R}^n)$

We begin our tour of the geometry of $\mathcal{H}(\mathbb{R}^n)$ with a discussion of circles in the space. First, we need to review some relevant topological definitions and notation. To differentiate between points in $\mathbb{R}^n$ and points in $\mathcal{H}(\mathbb{R}^n)$, we will use lowercase letters to denote the former and uppercase letters for the latter.

Definition 2. *Let (X,d) be a metric space. Let $r>0$.*

1. *The open r-neighborhood around an element $b\in X$ is the set*

$$N_r(b)=\{x\in X : d(x,b)<r\}.$$

2. *The boundary of the r-neighborhood around an element $b\in X$ is the set*

$$\partial N_r(b)=\{x\in X : d(x,b)=r\}.$$

Recall from elementary analysis that in $\mathcal{H}(\mathbb{R}^n)$, the closure of $N_r(b)$ is the closed r-neighborhood of B, i.e.

$$\overline{N_r(b)} = \{x \in \mathbb{R}^n : d(x,b) \leq r\}.$$

We will use a different notation for neighborhoods in $\mathcal{H}(\mathbb{R}^n)$. Let r be a positive real number and let B be a element of $\mathcal{H}(\mathbb{R}^n)$. Let $\mathcal{C}_r(B)$ denote the circle of radius r centered at the point B and $\mathcal{D}_r(B)$ denote the open disk of radius r centered at the point B. In other words,

$$\mathcal{C}_r(B) = \{A \in \mathcal{H}(\mathbb{R}^n) : h(A,B) = r\}$$

and

$$\mathcal{D}_r(B) = \{A \in \mathcal{H}(\mathbb{R}^n) : h(A,B) < r\}.$$

Our goal in this section is to describe circles and disks centered at points B in $\mathcal{H}(\mathbb{R}^n)$.

The simplest case is when $B = \{b\}$, a single element set. Suppose A is in $\mathcal{C}_r(B)$. Let $a \in A$. If $d(a,b) > r$, then $d(A,B) = \max_{x\in A}\{d(x,B)\} > r$. Thus, $h(A,B) > r$. Therefore, for each $a \in A$, $d(a,b) \leq r$. In addition, to have $h(A,B) = r$, there must be a point $\tilde{a} \in A$ so that $d(\tilde{a},b) = r$. Therefore, $\mathcal{C}_r(B)$ is the collection of compact subsets of $\mathbb{R}^n$ that are entirely contained within the Euclidean ball, $N_r(b)$, of radius r centered at b and that intersect this ball at one or more points. Also, $\mathcal{D}_r(B)$ is the collection of compact subsets of $\mathbb{R}^n$ that are entirely contained within $N_r(b)$ that do not intersect the boundary of this ball.

The general case describing elements in $\mathcal{C}_r(B)$ is given by the following theorem.

Theorem 1. *Let B be a non-empty compact subset of $\mathbb{R}^n$. Then $A \in \mathcal{C}_r(B)$ if and only if A is a non-empty compact subset of $\mathbb{R}^n$ and*

(a) $A \subseteq \bigcup_{b\in B} \overline{N_r(b)}$.

(b) $A \cap \overline{N_r(b)} \neq \emptyset$ *for each* $b \in B$.

(c) Either $A \cap \partial\left(\bigcup_{b\in B} N_r(b)\right) \neq \emptyset$ *or there exists* $b \in B$ *with* $A \cap \partial N_r(b) \neq \emptyset$ *and* $A \cap N_r(b) = \emptyset$*.*

To prove theorem 1 we will use the following lemma.

Lemma 1. $\partial\left(\bigcup_{b\in B} N_r(b)\right) = \left(\bigcup_{b\in B} \partial N_r(b) - \bigcup_{b\in B} N_r(b)\right)$.

Proof of Lemma 1: Let $x \in \partial\left(\bigcup_{b\in B} N_r(b)\right)$. Then $x \in \partial\left(\bigcup_{b\in B} N_r(b)\right)$. So for any $\epsilon > 0$, the open neighborhood $N_\epsilon(x)$ contains a point in $\bigcup_{b\in B} N_r(b)$ and a point not in $\bigcup_{b\in B} N_r(b)$. The second condition shows that $x \notin \bigcup_{b\in B} N_r(b)$. Since B is compact, the open cover $\{N_r(b) : binB\}$ has a finite subcover $\{N_r(b_1), N_r(b_2), \ldots, N_r(b_m)\}$. For each $k \in \mathbb{Z}^+$, we know that $N_{\frac{1}{k}}(x)$ intersects at least one of $N_r(b_1), N_r(b_2), \ldots, N_r(b_m)$. There must be at least one neighborhood, $N_r(b_j)$ that intersects infinitely many of the $N_{\frac{1}{k}}(x)$. Then $x \in \partial N_r(b_j)$, which implies $x \in \bigcup_{b\in B} \partial N_r(b)$.

Now suppose $x \in \left(\bigcup_{b\in B} \partial N_r(b) - \bigcup_{b\in B} N_r(b)\right)$. Then there is a $\tilde{b} \in B$ so that $x \in \partial N_r(\tilde{b})$. So for any $\epsilon > 0$, the neighborhood $N_\epsilon(x)$ contains a point in $N_r(\tilde{b}) \subseteq \bigcup_{b\in B} N_r(b)$ and a point not in $N_r(\tilde{b})$. More specifically, each $N_\epsilon(x)$ must contain a point not in $\bigcup_{b\in B} N_r(b)$. If not, then $N_\epsilon(x) \subseteq \bigcup_{b\in B} N_r(b)$. However, this implies $x \in \bigcup_{b\in B} N_r(b)$, which contradicts our assumption. Therefore, for any $\epsilon > 0$, we have $N_\epsilon(x)$ contains a point in $\bigcup_{b\in B} N_r(b)$ and a point not in $\bigcup_{b\in B} N_r(b)$. So $x \in \partial\left(\bigcup_{b\in B} N_r(b)\right)$. ■

Proof of theorem 1: We prove the forward implication first. Suppose $A \in \mathcal{C}_r(B)$. By definition, A is a compact subset of $\mathbb{R}^n$. Since we are assuming $h(A, B) = r$, we know that $d(A, B) \leq r$ and $d(B, A) \leq r$, with equality in at least one of the cases.

Now we prove condition (a). Let $a \in A$. Then

$$r = h(A, B) \geq d(A, B) \geq d(a, B)$$

implies that there is a $b \in B$ so that $d(a, b) \leq r$. Therefore, $a \in \overline{N_r(b)} \subseteq \bigcup_{b \in B} \overline{N_r(b)}$.

To prove condition (b), suppose there is a $b \in B$ so that $A \cap \overline{N_r(b)} = \emptyset$. Then, for any $a \in A$, $d(a, b) = d(b, a) > r$. Since A is compact, there is an $a' \in A$ so that $d(b, a') = \min_{a \in A}\{d(b, a)\}$. So $d(b, A) > r$. Therefore, $h(A, B) \geq d(B, A) \geq d(b, A) > r$, a contradiction. This proves (b).

Next we show that $A \cap \left(\bigcup_{b \in B} \partial N_r(b) \right) \neq \emptyset$. Suppose $d(A, B) = r$. Then there is an $a \in A$ so that $d(a, B) = r$. It follows that there is a $b_a \in B$ so that $d(a, b_a) = r$. Therefore, $a \in \partial N_r(b_a)$. If $d(B, A) = r$. Then there is a $b \in B$ so that $d(b, A) = r$. It follows that there is an element $a_b \in A$ so that $d(b, a_b) = r$. Therefore, $a_b \in \partial N_r(b)$.

Now assume

$$A \cap \left(\bigcup_{b \in B} \partial N_r(b) - \bigcup_{b \in B} N_r(b) \right) = \emptyset.$$

In other words, for every $x \in A \cap \bigcup_{b \in B} \partial N_r(b)$, there is a $y_x \in B$ so that $x \in N_r(y_x)$. We will show that these conditions imply that $d(A, B) < r$. Suppose $a \in A$ so that there is a $b_a \in B$ with $d(a, b_a) = r$. Then $a \in \partial N_r(b_a)$. But then $a \in N_r(y_a)$ and $d(a, y_a) < r$. So for each $a \in A$, there is a $b \in B$ so that $d(a, b) < r$. This implies $d(a, B) < r$ for each $a \in A$. Thus $d(A, B) < r$. Since $h(A, B) = r$, it must therefore be the case that $d(B, A) = r$. Now we want to show that there exists $b \in B$ and $a \in A \cap \partial N_r(b)$ so that $A \cap N_r(b) = \emptyset$. Assume to the contrary that whenever $a \in A \cap \partial N_r(b)$, it is also true that $A \cap N_r(b) \neq \emptyset$. We will show that this assumption implies $d(B, A) < r$. Suppose $b \in B$ so that there is an element $a_b \in A$ with $d(b, a_b) = r$. Then $a_b \in \partial N_r(b)$. If $A \cap N_r(b) \neq \emptyset$, then there is an element $a' \in A \cap N_r(b)$ with $d(b, a') < r$. Again, this shows that for every $b \in B$, there is an $a \in A$ so that $d(b, a) < r$, or that $d(B, A) < r$. This contradiction verifies (c).

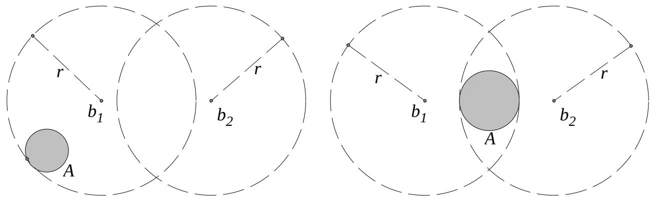

Figure 2: Left: $d(B, A) > r$. Right: $d(A, B) < r$ and $d(B, A) < r$.

Now we prove the reverse implication. Assume $A \in \mathcal{H}(\mathbb{R}^n)$ so that A satisfies conditions (a), (b), and (c). Let $a \in A$. Condition (a) implies that there is a $b \in B$ so that $d(a, b) \leq r$. Therefore, $d(a, B) \leq r$ for each $a \in A$. Consequently, $d(A, B) \leq r$. Now let $b \in B$. Condition (b) implies that there is an $a \in A$ so that $d(b, a) \leq r$. Therefore, $d(B, A) \leq r$. Since $d(A, B) \leq r$ and $d(B, A) \leq r$, we have $h(A, B) \leq r$.

To obtain equality, we use the conditions in (c). In either case in (c), we have that $A \cap \left(\bigcup\limits_{b \in B} \partial N_r(b) \right) \neq \emptyset$. Suppose $a \in A \cap \left(\bigcup\limits_{b \in B} \partial N_r(b) - \bigcup\limits_{b \in B} N_r(b) \right)$. Then there is an element $b_a \in B$ so that $a \in \partial N_r(b_a)$ or $d(a, b_a) = r$. In addition, $a \notin \bigcup\limits_{b \in B} N_r(b)$. In this case, $d(a, b) \geq r$ for all $b \in B$ with equality if $b = b_a$. Therefore, $d(A, B) = r$ and $h(A, B) = r$. In the other case, there is an element $b \in B$ so that $a \in \partial N_r(b)$ and $A \cap N_r(b) = \emptyset$. We then have $d(b, x) \geq r$ for all $x \in A$, with equality if $x = a$. So $d(B, A) = r$ and $h(B, A) = r$. ■

It will be helpful to look at a few examples to illustrate the conditions in theorem 1. Condition (a) is certainly reasonable, since each point of A must be within r units of some point in B. To illustrate the necessity of conditions (b) and (c), consider the case where $B = \{b_1, b_2\}$ is a two element set in $\mathbb{R}^2$. In figure 2, Euclidean circles of radius r are drawn around the points in B. Sets A are shaded. Note that condition (b) of theorem 1 is not satisfied in the example on the left. In this case, $d(b_2, A) > r$, so $h(A, B) \geq d(B, A) > r$. In the example on the right, the second part of condition (c) is not satisfied. In this case, both $d(A, B)$ and $d(B, A)$ are less than r, which means $h(A, B) < r$.

Now consider the examples in figure 3. On the left, the first part of condition

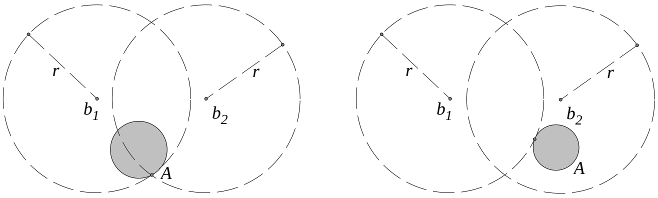

Figure 3: The shaded sets are a distance r from B.

(c) is satisfied. While $d(B, A) < r$ in this example, we do have $d(A, B) = r$. In the example on the right, the second part of condition (c) is satisfied. In this situation, $d(B, A) = r$, while $d(A, B) < r$. In both cases, $h(A, B) = r$.

Informally, this characterization indicates that, to be a distance r from a non-empty compact set B, a set A must contain points that are within r units of any point in B. If we collect all such points together, we should obtain the largest set that is r units from B, as the next theorem demonstrates.

Definition 3. *For $B \in \mathcal{H}(\mathbb{R}^n)$ and $s > 0$, define the set $B + s$ as*

$$B + s = \{x \in \mathbb{R}^n : d(x, b) \leq s \text{ for some } b \in B\}.$$

Theorem 2. *Let $B \in \mathcal{H}(\mathbb{R}^n)$ and let s be a positive real number. The set $B + s$ is the largest element in $\mathcal{H}(\mathbb{R}^n)$ that is a distance s from B.*

Proof: First we show that $B + s$ is in $\mathcal{H}(\mathbb{R}^n)$.

Since $B \in \mathcal{H}(\mathbb{R}^n)$, we know that B is closed and bounded. Let M be a bound for B. Thus, $|b| = d(b, 0) \leq M$ for all $b \in B$. Let $x \in B + s$. Then there is an element $b_x \in B$ so that $d(x, b_x) \leq s$. So

$$d(x, 0) \leq d(x, b_x) + d(b_x, 0) \leq s + M.$$

Thus, $B + s$ is bounded by $M + s$.

Next we demonstrate that $B + s$ is closed. Let x be a limit point of $B + s$. So there is a sequence $\{x_m\}$ in $B + s$ that converges to x. For each x_m there is a point $b_m \in B$ so that $d(x_m, b_m) \leq s$. Since the sequence $\{b_m\}$ is a bounded sequence (B is bounded), it has a convergent subsequence $\{a_l\}$. Since B is closed, $b = \lim_{l\to\infty} a_l$ is an element of B. Now, let $\epsilon > 0$ and choose $N > 0$ so that $m, l > N$ implies $d(x, x_m) < \frac{\epsilon}{2}$ and $d(b, a_l) < \frac{\epsilon}{2}$. Then for $t > N$, we have

$$d(x, b) \leq d(x, x_t) + d(x_t, a_t) + d(a_t, b) < \epsilon + s.$$

This shows that $d(x, b) \leq s$ and $x \in B + s$. Therefore, $B + s$ is compact.

Now we show that $h(B, B+s) = s$. Note that $B \subseteq B+s$, so $d(B, B+s) = 0$. For each $x \in B+s$, there is a $b_x \in B$ so that $d(x, b_x) \leq s$. Therefore, $d(x, B) \leq s$ for all $x \in B$. This shows $d(B + s, B) \leq s$ and, consequently, $h(B, B + s) \leq s$. To obtain equality, we only need to find an element not in B that is exactly a distance s from B. Let $b \in B$ with $|b|$ a maximum. Let $x = (1 + \frac{s}{|b|})b$. Then $d(x, b) = |x - b| = s$, so x is the point on the line through the origin and b that is s units farther from the origin than b. Since $|x| > |b|$, we see that $x \notin B$. To show that $d(x, B) = s$, we must demonstrate that there is no element in B closer to x than b. Let $c \in B$. A variation of the triangle inequality gives us

$$|x - b| = |x| - |b| \leq |x| - |c| \leq |x - c|.$$

Since $d(x, c) \geq d(x, b) = s$ for all $c \in B$, we must have $h(B, B + s) = s$.

Finally, we show that $B+s$ is the largest element of $\mathcal{H}(\mathbb{R}^n)$ that is a Hausdorff distance s from B. Let $C \in \mathcal{H}(\mathbb{R}^n)$ with $h(B, C) = s$. Let $c \in C$. The fact that $h(B, C) = s$ implies that there is an element $b \in B$ so that $d(b, s) \leq s$. Therefore, $c \in B + s$. So $C \subseteq B + s$. ■

Now that we have found the largest set a distance s from a given point B in $\mathcal{H}(\mathbb{R}^n)$, can we find a smallest set C that is a distance s from B in $\mathcal{H}(\mathbb{R}^n)$? Let $B \in \mathcal{H}(\mathbb{R}^n)$ and let s be greater than 0. Cover $B + s$ with open neighborhoods $N_{\frac{s}{2}}(b)$, for each $b \in B + s$. By theorem 2, $B + s$ is compact, so there is a finite subcover $\{N_{\frac{r}{2}}(b_1), N_{\frac{r}{2}}(b_2), \ldots, N_{\frac{r}{2}}(b_k)\}$ of $\{N_r(b) : b \in B\}$ that covers $B + s$. For each i, choose a point $c_i \in N_{\frac{r}{2}}(b_i) \cap (B + s)$. If none of these points c_i lies on $\partial(B + s)$, choose another point c_{k+1} on this boundary. Then, by theorem 1, the set $\{c_1, c_2, \ldots, c_{k+1}\}$ is a distance s from B. Therefore, for any non-empty compact set B and any $s > 0$, there is a finite set C with $h(B, C) = s$. By the

Well Ordering Principle there will be a set with the smallest number of elements that is a distance s from B. Note, however, that this set will not necessarily be unique, since we may be able to choose our sets in other ways.

As an example, let B be the unit disk in $\mathbb{R}^2$ and let $s = 2$. The set $\{N_1((\frac{1}{2},0)), N_1((-\frac{1}{2},0)), N_1((0,\frac{1}{2})), N_1((0,-\frac{1}{2}))\}$ is an open cover of B. Choose points ($\frac{3}{2}$,0), ($-\frac{3}{2}$,0), (0,$\frac{3}{2}$), (0,-$\frac{3}{2}$) from these balls. Since none of these points intersects $B+2$, add the point (3,0) to this list. Then the set

$$C = \{(\frac{3}{2},0),(-\frac{3}{2},0),(0,\frac{3}{2}),(0,-\frac{3}{2}),(3,0)\}$$

lies on $\mathcal{C}_2(B)$. Note that the set $\{(0,0),(3,0)\}$ also lies on $\mathcal{C}_2(B)$.

4 Miscellaneous Results about the Hausdorff Distance

In this section we present a few important results that describe the relationships between the Hausdorff distance in $\mathcal{H}(\mathbb{R}^n)$ and the Euclidean distance in $\mathbb{R}^n$. Many subsequent results will rely on the lemmas in this section.

Lemma 2. *Let A, and B be elements of $\mathcal{H}(\mathbb{R}^n)$. If $d(B,A) > 0$, then there exist $a_0 \in \partial A$ and $b_0 \in B$ so that $d(b_0,a_0) = d(B,A)$, $d(b_0,A) \geq d(b,A)$ for all $a \in A$, and $d(b_0,a_0) \leq d(b_0,a)$ for all $a \in A$.*

Proof: Assume $r = d(B,A) > 0$. Since $d(B,A) = \max_{b\in A}\{d(b,A)\}$, there is an element $b_0 \in B$ so that $d(b_0,A) = d(B,A)$ and $d(b_0,A) \geq d(b,A)$ for any $b \in B$. Also, $d(b_0,A) = \min_{a\in A}\{d(b_0,a)\}$ implies that there is an $a_0 \in A$ so that $d(B,A) = d(b_0,A) = d(b_0,a_0) \leq d(b_0,a)$ for any $a \in A$. Now we will show that $a_0 \in \partial A$. Note that since A is closed, $\partial A \subseteq A$.

We proceed by contradiction. Suppose $a_0 \notin \partial A$. Then there is an $\epsilon > 0$ so that $N_\epsilon(a_0) \subset A$. Let $x = a_0 + \frac{\epsilon}{2}\frac{b_0-a_0}{|b_0-a_0|}$, that is x is the point on the Euclidean line segment between a_0 and b_0 a distance $\frac{\epsilon}{2}$ from a_0. Notice that $x \in N_\epsilon(a_0) \subset A$ and

$$d(b_0,x) = \left|b_0 - \left(a_0 + \frac{\epsilon}{2}\frac{b_0-a_0}{|b_0-a_0|}\right)\right| = |b_0-a_0|\left(1-\frac{\epsilon}{2|b_0-a_0|}\right) < r.$$

This implies that $d(b_0, A) < r$, which is a contradiction. ■

Note that we cannot conclude that $b_0 \in \partial B$ in lemma 2 as the example in figure 1 illustrates.

Lemma 3. *Let* $A, B \in \mathcal{H}(\mathbb{R}^n)$ *with* $d(B, A) = r > 0$. *If* $0 < s < r$, *then* $d(B, A + s) = r - s$.

Proof: By lemma 2 there are elements $b_0 \in B$ and $a_0 \in \partial A$ so that $d(b_0, a_0) = d(B, A) = r$, $d(b_0, A) \geq d(b, A)$ for all $b \in B$, and $d(b_0, a_0) \leq d(b_0, a)$ for all $a \in A$.

Consider the case where $0 < s < r = d(B, A)$. Let c_0 be the point of intersection of $\overline{N_s(a_0)}$ and the ray $\overrightarrow{a_0 b_0}$. By construction, we have $d(b_0, a_0) = d(b_0, c_0) + d(c_0, a_0)$ or that $d(b_0, c_0) = r - s$. Now we will show that $d(B, A + s) = d(b_0, c_0)$.

First we show that $d(b_0, A + s) = \min_{c \in A+s}\{d(b_0, c)\} = r - s$. Suppose there is a $\tilde{c} \in A + s$ so that $d(b_0, \tilde{c}) < d(b_0, c_0) = r - s$. Since $\tilde{c} \in A + s$, there is an element $\tilde{a} \in A$ so that $d(\tilde{c}, \tilde{a}) \leq s$. Then

$$d(b_0, \tilde{a}) \leq d(b_0, \tilde{c}) + d(\tilde{c}, \tilde{a}) < (r - s) + s = r.$$

However, we know that $r = d(b_0, a_0) \leq d(b_0, a)$ for any $a \in A$. This is a contradiction. Therefore, $d(b_0, A + s) = d(b_0, c_0)$. To complete the proof, we need to show that $d(b_0, A + s)$ is the maximum such distance.

Now we show that if $b \in B$, then $d(b, A + s) \leq d(b_0, A + s)$. Let $b \in B$. Recall that $r = d(b_0, A) \geq d(b, A)$. By lemma 2, there is an $a \in A$ so that $d(\{b\}, A) = d(b, A) = d(b, a) \leq r$. Let c be the point at which the ray $\overrightarrow{ab}$ intersects $N_s(a)$. Now $d(c, a) = s$, so $c \in A + s$. Since a, c, and b lie on a Euclidean line with c between a and b, we have

$$d(b, c) = d(b, a) - d(a, c) \leq r - s.$$

This shows that $d(b, A + s) = \min_{c \in A+s}\{d(b, c)\} \leq r - s = d(b_0, A + s)$. ■

The example in figure 4 shows that there is no corresponding lemma for $d(A + s, B)$.

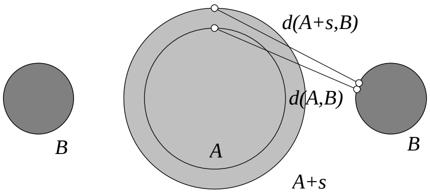

Figure 4: Left: $d(A+s,B)$ is not easily related to $d(A,B)$ and s.

Lemma 4. *Let $A \in \mathcal{H}(\mathbb{R}^n)$ and $s > 0$. If $x \in \partial(A+s)$, then $d(x,A) = s$.*

Proof: Let $x \in \partial(A+s)$. By theorem 2, we know that $A+s$ is closed. So $x \in A+s$. Therefore, there is an element $a \in A$ so that $d(x,a) \leq s$. So $d(x,A) \leq s$.

Suppose there is an $a \in A$ so that $d(x,a) = s' < s$. If $x' \in N_{s-s'}(x)$, then

$$d(x',a) \leq d(x',x) + d(x,a) < (s-s') + s' = s.$$

Thus, $x' \in A+s$. This shows that $N_{s-s'}(x) \subset A+s$, contradicting the fact that $x \in \partial(A+s)$. Therefore, $d(x,A) \geq s$. This gives us $d(x,A) = s$. ■

The extension of a set by a radius $r > 0$ also gives us an alternative way to view condition 1 of Theorem 1.

Lemma 5. *Let B be a set in $\mathcal{H}(\mathbb{R}^n)$. Then*

$$\bigcup_{b \in B} \overline{N_r(b)} = B + r$$

Proof: Let $B \in \mathcal{H}(\mathbb{R}^n), r > 0,$ and $x \in \bigcup_{b \in B} \overline{N_r(b)}$. Then there exists $b_0 \in B$ such that $x \in \overline{N_r(b_0)}$ and $d(x,b_0) \leq r$. Therefore, $x \in B+r$ and $\bigcup_{b \in B} \overline{N_r(b)} \subseteq B+r$.

Now let $x \in B+r$. Then $\exists\, b_0 \in B$ such that $d(x,b_0) \leq r$. But this means that $x \in \overline{N_r(b_0)}$ and so $x \in \bigcup_{b \in B} \overline{N_r(b)}$. Therefore $B+r \subseteq \bigcup_{b \in B} \overline{N_r(b)}$. ■

5 Lines

In this section we will present definitions, examples, and theorems about lines in $\mathcal{H}(\mathbb{R}^n)$. We will use the following notation for lines and rays in $\mathbb{R}^n$.

- $\overrightarrow{ab} = \{a + t(b-a) : t \geq 0\}$ is the Euclidean ray from the point a to the point b.
- $\overleftrightarrow{ab} = \{a + t(b-a) : t \in \mathbb{R}\}$ is the Euclidean line determined by the points a and b.
- $\overline{ab} = \{ta + (1-t)b : 0 \leq t \leq 1\}$ is the Euclidean segment between a and b.

A line connecting two ponts a and b in $\mathbb{R}^n$ can be thought of as the set of all points $c \in \mathbb{R}^n$ so that at least one of the equalities $d(a,b) = d(a,c) + d(c,b)$, $d(a,c) = d(a,b) + d(b,c)$, or $d(c,b) = d(c,a) + d(a,b)$ is satisfied, where d is the Euclidean distance on $\mathbb{R}^n$. We will use this notion of lines in $\mathbb{R}^n$ to define lines in $\mathcal{H}(\mathbb{R}^n)$.

Definition 4. *Let A, B be distinct elements of $\mathcal{H}(\mathbb{R}^n)$. The line determined by A and B is the set of elements $C \in \mathcal{H}(\mathbb{R}^n)$ so that at least one of*

1. $h(A,B) = h(A,C) + h(C,B)$,
2. $h(A,C) = h(A,B) + h(B,C)$,
3. $h(C,B) = h(C,A) + h(A,B)$

is satisfied.

We will call the lines in $\mathcal{H}(\mathbb{R}^n)$ *Hausdorff lines.*

As an example, we present a complete characterization of the points in $\mathcal{H}(\mathbb{R}^n)$ that lie on a Hausdorff line joining single point sets.

Theorem 3. *Let $a, b \in \mathbb{R}^n$. Let $A = \{a\}, B = \{b\}$ and $C \in \mathcal{H}(\mathbb{R}^n)$. Then C lies on the Hausdorff line defined by A and B if and only if there exist $r, s \in \mathbb{R}$ such that:*

1. *$C \subseteq (A+r) \cap (B+s)$*
2. *There exists $c_0 \in C$ such that $c_0 \in \partial(A+r) \cap \partial(B+s) \cap \overleftrightarrow{ab}$ and c_0 satisfies one of the following:*
    - *$d(a,b) = d(a,c_0) + d(c_0,b)$*
    - *$d(a,c_0) = d(a,b) + d(b,c_0)$*
    - *$d(c_0,b) = d(c_0,a) + d(a,b)$*

Proof: Let $a, b \in \mathbb{R}^n$. Let $A = \{a\}, B = \{b\}$ and $C \in \mathcal{H}(\mathbb{R}^n)$.

$\Rightarrow$: Suppose C is on the Hausdorff line defined by A and B. Let $r = h(A,C)$ and $s = h(C,B)$. Then $\forall c \in C$, we have $d(c,a) \leq r$ and $d(c,b) \leq s$. Furthermore, we have $C \in C_A(r)$ and $C \in C_B(r)$. Therefore, by condition 1 of theorem 1, we must have $C \subseteq \overline{N_r(a)}$ and $C \subseteq \overline{N_s(b)}$. By lemma 5, we know that $\overline{N_r(a)} = A + r$ and $\overline{N_s(b)} = B + s$ so that we have $C \subseteq (A+r) \cap (B+s)$. From condition 3 of Theorem 1, we know that either $C \cap \partial\left(\bigcup_{a \in A} N_r(a)\right) \neq \emptyset$ or there is an element $a_0 \in A$ such that C intersects only the boundary of $N_r(a_0)$. Since A = {a}, in either case, we will obtain $C \cap \partial N_r(a) \neq \emptyset$. Similarily, we must have $C \cap \partial N_s(b) \neq \emptyset$. Suppose no point in C is in $\partial N_r(a) \cap \partial N_s(b)$. Let $c_1 \in C \cap \partial N_r(a)$ and $c_2 \in C \cap \partial N_s(b)$. Then, $d(c_1,a) = r$ and $d(c_2,b) = s$. We know $c_1 \in B + s = \overline{N_s(b)} = \partial N_s(b) \cup N_s(b)$. Because c_1 is not in $\partial N_s(b)$, we must have $c_1 \in N_s(b)$. Therefore, $d(c_1,b) < s$. Similarily, $d(c_2,a) < r$. But this implies that

- $h(A,B) = d(a,b) \leq d(a,c_1) + d(c_1,b) < r + s = h(A,C) + h(B,C)$
- $h(A,C) = d(a,c_1) \leq d(a,b) + d(b,c_1) < d(a,b) + s = h(A,B) + h(B,C)$
- $h(C,B) = d(c_2,b) \leq d(c_2,a) + d(a,b) < r + d(a,b) = h(C,A) + h(A,B)$.

This contradicts the fact that C is on the Hausdorff line defined by A and B. Therefore there must be an element $c_0 \in C$ such that $c_0 \in \partial N_r(a) \cap \partial N_s(b)$. By lemma 5, this is equivalent to $c_0 \in \partial(A+r) \cap \partial(B+s)$. Thus, $h(A,C) =$

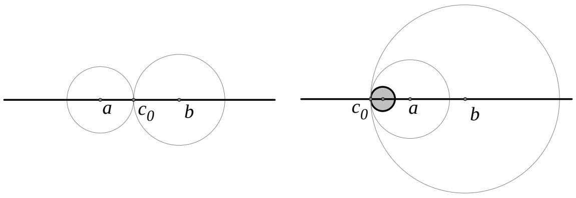

Figure 5: Let $A = a$ and $B = b$. On the right we see that the only sets that satisfy $h(A,B) = h(A,C) + h(C,B)$ are single element sets since $A + r$ and $B + s$ intersect in only one point in $\mathbb{R}^n$ for all $r, s > 0$. When we look for sets satisfying $h(A,C) = h(A,B) + h(B,C)$ or $h(C,B) = h(C,A) + h(A,B)$, we have a greater variety of sets C that satisfy the conditions of Theorem 3 as the example on the right shows.

$r = d(a, c_0)$, $h(B,C) = s = d(b, c_0)$, and $h(A,B) = d(a,b)$. Because C is on the Hausdorff line defined by A and B we have one of the following true:

- $d(a,b) = h(A,B) = h(A,C) + h(B,C) = d(a,c_0) + d(c_0,b)$
- $d(a,c_0) = h(A,C) = h(A,B) + h(B,C) = d(a,b) + d(b,c_0)$
- $d(c_0,b) = h(C,B) = h(C,A) + h(A,B) = d(c_0,a) + d(a,b)$

which implies that c_0 is on $\overleftrightarrow{ab}$. Therefore, we have shown that C satisfies conditions 1 and 2.

$\Leftarrow$: Assume C satisfies conditions 1 and 2. Since C satisfies condition 1, we know $\forall c \in C$, $d(c,a) \leq r$ and $d(c,b) \leq s$. Because C satisfies condition 2, $d(c_0,a) = r$ and $d(c_0,b) = s$. So we can conclude $d(c_0,a) = h(C,A)$ and $d(c_0,b) = h(C,B)$. Since C is on $\overleftrightarrow{ab}$, one of the following is true:

1. $d(a,b) = d(a,c_0) + d(c_0,b) \Leftrightarrow h(A,B) = h(A,C) + h(C,B)$
2. $d(a,c_0) = d(a,b) + d(b,c_0) \Leftrightarrow h(A,C) = h(A,B) + h(B,C)$
3. $d(c_0,b) = d(c_0,a) + d(a,b) \Leftrightarrow h(B,C) = h(C,A) + h(A,B)$

Therefore, C lies on the Hausdorff line defined by A and B. ■

Figure 5 gives examples of sets C lying on the Hausdorff line defined by A and B. Two interesting results of this classification are:

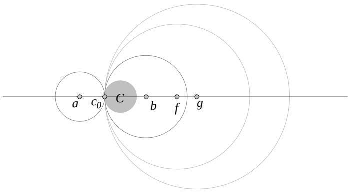

Figure 6: The closed disk C lies on the Hausdorff line defined by $F = \{f\}$ and $G = \{g\}$, but not on the Hausdorff line defined $A = \{a\}$ and $B = \{b\}$.

- Although we may have $a, b, f, g \in \mathbb{R}^n$ collinear in Euclidean space, the Hausdorff lines defined by $A = \{a\}$, $B = \{b\}$, $F = \{f\}$, and $G = \{g\}$ are the same if and only if $\{a, b\} = \{f, g\}$ (see Figure 6).
- If we restrict ourselves to the subspace of $\mathcal{H}(\mathbb{R}^n)$ consisting of single element sets, then we can see that the Hausdorff line through the points $A = \{a\}$ and $B = \{b\}$ contains exactly those sets $C = \{c\}$, where the point $c \in \mathbb{R}^n$ lies on the Euclidean line $\overleftrightarrow{ab}$. In this way, the standard Euclidean lines in $\mathbb{R}^n$ are embedded in the geometry of geometry of $\mathcal{H}(\mathbb{R}^n)$.

Now that we have a characterization of Hausdorff lines defined by single point sets, we turn our attention to the question of how such lines might intersect. In Euclidean space, we know that distinct lines are either parallel or intersect in exactly one point. For our purposes, we will consider two lines in $\mathcal{H}(\mathbb{R}^n)$ to be *parallel* if the lines have no points in common. The next theorems tell us how distinct lines defined by single point sets relate to each other.

Theorem 4. *Let $a, b, f, g \in \mathbb{R}^n$ such that $\overleftrightarrow{ab} \parallel \overleftrightarrow{fg}$ and $\overleftrightarrow{ab}, \overleftrightarrow{fg}$ are coplanar. Let p be the point where the perpendicular to $\overleftrightarrow{fg}$ at f intersects $\overleftrightarrow{ab}$. Then if we let $A = \{a\}, B = \{b\}, F = \{f\}, G = \{g\} \in \mathcal{H}(\mathbb{R}^n)$, we will have $\overleftrightarrow{AB} \parallel \overleftrightarrow{FG}$ if and only if for some relabeling of a, b, f, g we have $p \in \overline{ab}$.*

Proof: Let $a, b, f, g \in \mathbb{R}^n$ such that $\overleftrightarrow{ab} \parallel \overleftrightarrow{fg}$ are coplanar and $\overline{af} \cap \overline{bg} = \emptyset$. Let $p, q \in \mathbb{R}^n$ be the points where the perpendicular to $\overleftrightarrow{fg}$ at f and the perpendicular to $\overleftrightarrow{fg}$ at g intersect $\overleftrightarrow{ab}$. Let $m, n \in \mathbb{R}^n$ be the points where the perpendicular to $\overleftrightarrow{ab}$ at a and the perpendicular to $\overleftrightarrow{ab}$ at b intersect $\overleftrightarrow{fg}$. Also, let $A = \{a\}, B =$

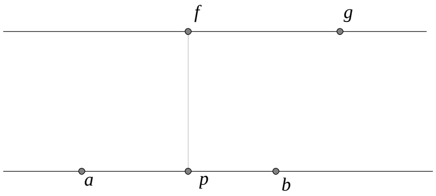

Figure 7: Condition for parallel lines

$\{b\}, F = \{f\}, G = \{g\}$.

$\Leftarrow$: Suppose that p is on $\overline{ab}$. That is, $d(a,b) = d(a,p) + d(p,b)$ (see Figure 7). Suppose we also have $C \in \overleftrightarrow{AB} \cap \overleftrightarrow{FG}$. Then from Theorem 3 we must have $r, s, t, u > 0$ so that $C \subseteq (A+r) \cap (B+s) \cap (F+t) \cap (G+u)$ and there exist $c_0, c_1 \in C$ with $c_0 \in \partial(A+r) \cap \partial(B+s) \cap \overleftrightarrow{ab}$ and $c_1 \in \partial(F+t) \cap \partial(G+u) \cap \overleftrightarrow{fg}$. Therefore we have $d(c_0, a) = r, d(c_0, b) = s, d(c_1, f) = t$, and $d(c_1, g) = u$.

Case 1: Suppose either $d(a,b) = d(a,c_0) + d(c_0,b)$ or $d(f,g) = d(f,c_1) + d(c_1,g)$. Without loss of generality, we will consider the case when $d(a,b) = d(a,c_0) + d(c_0,b)$. Since $C \subseteq (A+r) \cap (B+s)$ and $c_1 \in C$, we must have $d(a,c_1) \leq r = d(a,c_0)$ and $d(b,c_1) \leq s = d(b,c_0)$. So,

$$d(a,b) \leq d(a,c_1) + d(c_1,b) \leq d(a,c_0) + d(c_0,b) = d(a,b).$$

Obviously, $d(a,b) = d(a,b)$ so we must have

$$d(a,b) = d(a,c_1) + d(c_1,b).$$

But this would imply that we have $c_1 \in \overleftrightarrow{ab} \cap \overleftrightarrow{fg}$ which contradicts $\overleftrightarrow{ab} \parallel \overleftrightarrow{fg}$.

Case 2: Now suppose $d(a,c_0) = d(a,b) + d(b,c_0)$ and $d(f,c_1) = d(f,g) + d(g,c_1)$ or $d(c_0,b) = d(c_0,a) + d(a,b)$ and $d(c_1,g) = d(c_1,f) + d(f,g)$. Without loss of generality, we shall assume the former. Let $w \in \overleftrightarrow{fg}$ so that $\overline{c_0 w} \perp \overleftrightarrow{fg}$ and let $\gamma = d(w, c_0)$. Since $\overleftrightarrow{ab} \neq \overleftrightarrow{fg}$ we know $\gamma > 0$. We also know that since $C \subseteq B + s$ and $c_1 \in C$, we must have $d(b, c_1) \leq s = d(b, c_0)$. Suppose we have

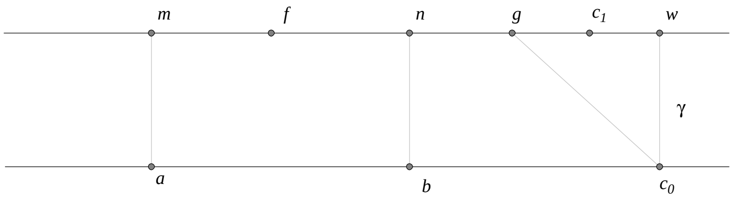

Figure 8: Case 2

$d(g, c_1) = d(g, w) + d(w, c_1)$. Then we would have

$$\begin{aligned} d(b, c_1)^2 &= d(b, n)^2 + (d(n, w) + d(w, c_1))^2 \\ &> d(b, n)^2 + d(n, w)^2 \\ &= \gamma^2 + d(b, c_0)^2 > d(b, c_0)^2. \end{aligned}$$

This would imply $d(b, c_1) > d(b, c_0)$, a contradiction. A similar argument shows $d(w, c_1) = d(w, g) + d(g, c_1)$ implies $d(b, c_1) > d(b, c_0)$ as well. Therefore we must have $d(g, w) = d(g, c_1) + d(c_1, w)$. (See figure 8) Consider the right triangle with vertices g, w and c_0 and hypotenuse $\overline{gc_0}$. From the Pythagorean Theorem, we have

$$\begin{aligned} d(g, c_0)^2 &= d(g, w)^2 + \gamma^2 \\ &= (d(g, c_1) + d(c_1, w)^2 + \gamma^2 \\ &= d(g, c_1)^2 + 2(d(c_1, w))(d(g, c_1)) + d(c_1, w)^2 + \gamma^2 > d(g, c_1)^2. \end{aligned}$$

This gives us $d(g, c_0) > d(g, c_1) = u$, which implies $c_0 \notin G + u$. Since $c_0 \in C$, this contradicts $C \subseteq G + u$.

Case 3: Suppose $d(a, c_0) = d(a, b) + d(b, c_0)$ and $d(c_1, g) = d(c_1, f) + d(f, g)$ or $d(c_0, b) = d(c_0, a) + d(a, b)$ and $d(f, c_1) = d(f, g) + d(g, c_1)$. Without loss of generality, we shall assume the former. Because $\overleftrightarrow{fp} \perp \overleftrightarrow{fg}$ and $\overleftrightarrow{fg} \parallel \overleftrightarrow{ab}$, we must have $\overleftrightarrow{fp} \perp \overleftrightarrow{ab}$. Therefore, f, p and c_0 form a right triangle with hypotenuse $\overline{fc_0}$. Similiarily, since $\overleftrightarrow{bn} \perp \overleftrightarrow{fg}$ we know b, n, and c_1 form a right triangle with hypotenuse $\overline{bc_1}$. Since we know that $d(b, c_1) \leq d(b, c_0)$ and the length of a leg

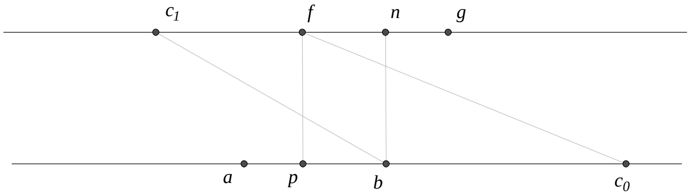

Figure 9: Case 3

of any right triangle is less than the length of the hypotenuse, we have

$$\begin{aligned} d(f,c_0) &> d(c_0,p) = d(c_0,b) + d(b,p) \\ &\geq d(c_0,b) \geq d(c_1,b) \\ &> d(n,c_1) = d(n,f) + d(f,c_1) \geq d(f,c_1). \end{aligned}$$

This implies that $t = d(f,c_1) < d(f,c_0)$ which contradicts $c_0 \in F + t$.

We can conclude that there can be no set C in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$ and therefore $\overleftrightarrow{AB} \parallel \overleftrightarrow{FG}$.

$\Leftarrow$: Suppose that we have $\overleftrightarrow{AB} \parallel \overleftrightarrow{FG}$ in $\mathcal{H}(\mathbb{R}^n)$ with $p \notin \overline{ab}$ for all relabelings of a, b, f, g. Without loss of generality, suppose that m, n satisfy $d(m,g) = d(m,f) + d(f,g)$ and $d(n,g) = d(n,f) + d(f,g)$ with $d(f,n) > 0$ while p, q satisfy $d(a,p) = d(a,b) + d(b,p)$ and $d(a,q) = d(a,b) + d(b,q)$ with $d(b,p) > 0$. Let $\gamma = d(b,n) = d(f,p)$ and $\alpha = d(n,f) = d(p,b)$ and consider the quantity $\frac{\gamma^2 - \alpha^2}{2\alpha}$. Since α is strictly positive, it follows that the denominator is nonzero and therefore this quantity is real. Therefore we can find some $\beta > 0$ such that $\frac{\gamma^2-\alpha^2}{2\alpha} \leq \beta$. Let c_1 be a point on $\overleftrightarrow{fg}$ such that $d(c_1,g) = d(c_1,f) + d(f,g)$ and $d(c_1,f) = \alpha + \beta$. Similarly, let c_0 be a point on $\overleftrightarrow{ab}$ such that $d(a,c_0) = d(a,b) + d(b,c_0)$ and $d(c_0,b) = \alpha + \beta$ (see Figure 10). Then

$$d(c_1,n) = d(c_1,f) - d(f,n) = \alpha + \beta - \alpha = \beta.$$

The same argument shows $d(c_0,p) = \beta$ as well. Therefore, by the Pythagorean theorem we have

$$d(b,c_1)^2 = d(b,n)^2 + d(n,c_1)^2 = \gamma^2 + \beta^2$$

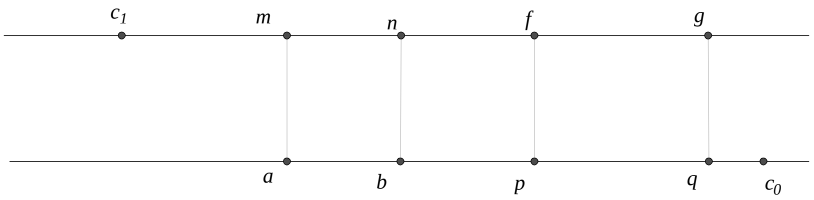

Figure 10: $d(n,f) = d(p,b) = \alpha$ and $d(p,f) = d(n,b) = \gamma$. We choose c_0, c_1 so that $d(c_1,f) = d(c_0,b) = \alpha + \beta$.

and

$$d(f,c_0)^2 = d(f,p)^2 + d(p,c_0)^2 = \gamma^2 + \beta^2.$$

Using these facts, we have

$$\begin{aligned} \frac{\gamma^2 - \alpha^2}{2\alpha} &\leq \beta \\ \gamma^2 - \alpha^2 &\leq 2\alpha\beta \\ \gamma^2 &\leq \alpha^2 + 2\alpha\beta \\ \gamma^2 + \beta^2 &\leq (\alpha+\beta)^2 \\ d(b,c_1)^2 &\leq d(b,c_0)^2. \end{aligned}$$

Because $d(b,c_1), d(b,c_0) \geq 0$, we can conclude that $d(b,c_1) \leq d(b,c_0)$. A similar argument will show that $d(f,c_0) \leq d(f,c_1)$. We may also repeat the above process to show that $d(a,c_1) \leq d(a,c_0)$ and $d(g,c_1) \leq d(g,c_0)$. Consider the set $C = \{c_0, c_1\}$. Let $r = d(a,c_0), s = d(b,c_0), t = d(f,c_1)$, and $u = d(g,c_1)$. Then since $d(a,c) \leq r \ \forall c \in C$ and $d(b,c) \leq s \ \forall c \in C$ we have $C \in (A+r) \cap (B+s)$. We also have $c_0 \in C$ such that $c_0 \in \partial(A+r) \cap \partial(B+s) \cap \overleftrightarrow{ab}$. Therefore, $C \in \overleftrightarrow{AB}$ by our previous theorem. Since $d(f,c) \leq t$ and $d(g,c) \leq u \ \forall c \in C$ with $c_1 \in C$ such that $c_1 \in \partial(F+t) \cap \partial(G+u) \cap \overleftrightarrow{fg}$, we also must have $C \in \overleftrightarrow{FG}$. Thus, $C \in \overleftrightarrow{AB} \cap \overleftrightarrow{FG}$ and $\overleftrightarrow{AB}$ is not parallel $\overleftrightarrow{FG}$, a contradiction (In fact, any subset of $(A+r) \cap (B+s) \cap (F+t) \cap (G+u)$ that contains c_0, c_1 will be in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$.) Therefore, we must have either the perpendicular to $\overleftrightarrow{ab}$ at a or b intersecting $\overline{fg}$ or the perpendicular to $\overleftrightarrow{fg}$ at f or g intersecting $\overline{ab}$. ■

Note that when $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{FG}$, the intersection of the extensions $A+r$, $B+s$, $F+u$, and $G+t$ will be a convex set. Consequently, we can find

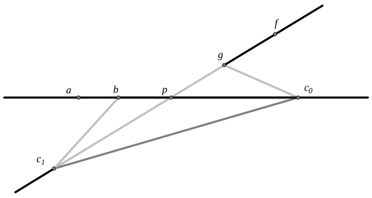

Figure 11: Here we have $d(a,b) \neq d(a,p)+d(p,b)$ and $d(f,g) \neq d(f,p)+d(p,g)$ (Case 1).

infinitely many sets in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$. Because of this, our definition of lines will not give us an incidence geometry. (For a discussion of incidence geometries and their properties see [5].)

We have now seen examples of distinct Hausdorff lines that parallel and distinct Hausdorff lines that intersect in infinitely many points. The next theorem shows us that these are not the only possibilities.

Theorem 5. *Let* $a, b, f, g, p \in \mathbb{R}^n$ *such that* $\overleftrightarrow{ab} \cap \overleftrightarrow{fg} = p$. *Let* $A = \{a\}, B = \{b\}, F = \{f\}$ *and* $G = \{g\}$ *in* $\mathcal{H}(\mathbb{R}^n)$. *Then* $\overleftrightarrow{AB} \cap \overleftrightarrow{FG} = \{p\}$ *if only if after some relabeling of* a, b, f, g *we have* $d(a,b) = d(a,p)+d(p,b)$ *and either* $d(f,g) = d(f,p)+d(p,g)$ *or* $d(g,p) \leq d(b,p)$ *where* $d(g,p) < d(f,p)$.

Proof: Let $a, b, f, g, p \in \mathbb{R}^n$ such that $\overleftrightarrow{ab} \cap \overleftrightarrow{fg} = p$. Let $A = \{a\}, B = \{b\}, F = \{f\}$ and $G = \{g\}$ in $\mathcal{H}(\mathbb{R}^n)$ and let $\theta = \angle gpb$. Without loss of generality, suppose $d(g,p) < d(f,p)$.

$\Rightarrow$: Suppose that be can find no labeling a, b, f, g such that $d(a,b) = d(a,p) + d(p,b)$ and either $d(f,g) = d(f,p)+d(p,g)$ or $d(g,p) \leq d(b,p)$.

Case 1: Suppose that $d(a,b) \neq d(a,p)+d(p,b)$ and $d(f,g) \neq d(f,p)+d(p,g)$. Without loss of generality, we shall assume $d(f,p) = d(f,g)+d(g,p)$ with $g \neq p$ and $d(a,p) = d(a,b)+d(b,p)$ with $b \neq p$. Let c_0 be a point on $\overleftrightarrow{ab}$ such that $d(b,c_0) = d(b,p)+d(p,c_0)$. Then let c_1 be the point on $\overleftrightarrow{fg}$ such that $d(p,c_1) = d(p,c_0)$ and $d(g,c_1) = d(g,p)+d(p,c_1)$. (See Figure F:sin2) Consider $\triangle c_0pc_1$. Since $d(p,c_1) = d(p,c_0)$, we must have $\mathbf{m}\angle pc_1c_0 = \mathbf{m}\angle pc_0c_1$. If we look at $\triangle gc_1c_0$, we have $\mathbf{m}\angle gc_0c_1 > \mathbf{m}\angle gc_1c_0$ and therefore $d(g,c_1) >$

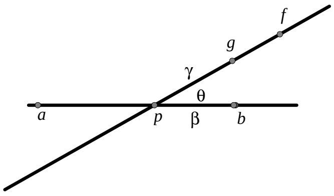

Figure 12: Here we have $d(a,b) = d(a,p)+d(p,b)$, $d(f,g) \neq d(f,p)+d(p,g)$, and $d(g,p)\cos(\theta) \leq d(b,p) < d(g,p)$ (Case 2).

$d(g,c_0)$. Similarily, if we consider triangles fc_1c_0, bc_1c_0, and ac_1c_0, we show that $d(f,c_1) > d(f,c_0)$, $d(a,c_0) > d(a,c_1)$, and $d(b,c_0) > d(b,c_1)$. Let $C = \{c_0,c_1\}$ and let $r = d(a,c_0)$, $s = d(b,c_0)$, $t = d(f,c_1)$, and $u = d(g,c_1)$. Then $C \subseteq (A+r)\cap(B+s)$ and we have $c_0 \in C$ with $c_0 \in \partial(A+R)\cap\partial(B+S)\cap\overleftrightarrow{ab}$ so that by Theorem $C \subseteq (F+t)\cap(G+u)$ and we have $c_1 \in C$ with $c_1 \in \partial(F+t)\cap\partial(G+u)\cap\overleftrightarrow{fg}$ so that again by Theorem we have $C \neq \{p\}$ such that $C \in \overleftrightarrow{AB}\cap\overleftrightarrow{FG}$. (Since we can repeat the above with any c_0, c_1 that satisfy $d(b,c_0) = d(b,p)+d(p,c_0)$, $d(p,c_1) = d(p,c_0)$ and $d(g,c_1) = d(g,p)+d(p,c_1)$, we actually have infinitely many $C \in \overleftrightarrow{AB}\cap\overleftrightarrow{FG}$).

Case 2: Suppose that we have $d(a,b) = d(a,p)+d(p,b)$, $d(f,g) \neq d(f,p)+d(p,g)$ and $d(g,p)\cos(\theta) \leq d(b,p) < d(g,p)$. Without loss of generality, assume that $d(f,p) = d(f,g)+d(g,p)$ with $d(g,p) \neq 0$ (see Figure 12 . Let $\beta = d(b,p)$, $\gamma = d(g,p)$, $\alpha = \frac{2\gamma\beta(\beta\cos(\theta)-\gamma)}{\beta^2-\gamma^2}$ and $\delta = \frac{2\gamma\beta(\gamma\cos(\theta)-\beta)}{\beta^2-\gamma^2}$. Notice that since $\gamma > \beta > \gamma\cos(\theta)$, both α and δ have strictly negative numerators and stricly negative denominators, and therefore, $\alpha,\delta \in \mathbb{R}^+$. We also know that

$$\begin{aligned} 2\gamma(\beta\cos(\theta)-\gamma) &< 2\gamma(\beta-\gamma) \\ &< (\gamma+\beta)(\beta-\gamma) \\ &= \beta^2-\gamma^2. \end{aligned}$$

Since $\beta^2-\gamma^2 < 0$, this implies $\frac{2\gamma(\beta\cos(\theta)-\gamma)}{\beta^2-\gamma^2} > 1$ and

$$\alpha = \frac{2\beta\gamma(\beta\cos(\theta)-\gamma)}{\beta^2-\gamma^2} > \beta.$$

Therefore, we can choose $c_0 \in \mathbb{R}^n$ on $\overleftrightarrow{ab}$ such that $\alpha = d(p, c_0) = d(p, b) + d(b, c_0)$. Let $c_1 \in \mathbb{R}^n$ be the point on $\overleftrightarrow{fg}$ such that $d(p, c_1) = \delta$ and $d(g, c_1) = d(g, p) + d(p, c_1)$. Let $r = d(a, c_0)$, $s = d(b, c_0)$, $t = d(f, c_1)$, $u = d(g, c_1)$ and let $C = \{c_0, c_1\}$. Algebra will show that with our given choice of α and δ, we have

$$(\alpha - \beta)^2 = \delta^2 + \beta^2 + 2\beta\delta\cos(\theta).$$

But then by the law of cosines, we have

$$\begin{aligned} d(b, c_0)^2 &= (\alpha - \beta)^2 \\ &= \delta^2 + \beta^2 + 2\beta\delta\cos(\theta) \\ &= d(b, c_1)^2. \end{aligned}$$

So $d(b, c_1) = d(b, c_0) = s$ and $C \subseteq B + s$. We also have

$$\begin{aligned} \mathbf{m}\angle ac_0c_1 &= \mathbf{m}\angle bc_0c_1 \\ &= \mathbf{m}\angle bc_1c_0 \\ &< \mathbf{m}\angle ac_1c_0. \end{aligned}$$

Therefore $d(a, c_1) < d(a, c_0)$ and $C \subseteq A + r$. Since $C \subseteq (A + r) \cap (B + s)$ and we have $c_0 \in C$ such that $c_0 \in \partial(A + r) \cap \partial(B + s) \cap \overleftrightarrow{ab}$, we must have $C \in \overleftrightarrow{AB}$. Again, we can show by algebra that

$$(\delta + \gamma)^2 = \alpha^2 + \gamma^2 - 2\alpha\gamma\cos(\theta).$$

It follows from the law of cosines that

$$d(g, c_1)^2 = (\alpha - \beta)^2 \quad = \delta^2 + \beta^2 + 2\beta\delta\cos(\theta) = d(g, c_0)^2.$$

Thus $d(g, c_0) = d(g, c_1) = u$ and $C \subseteq G + u$. We also have

$$\begin{aligned} \mathbf{m}\angle ac_1c_0 &= \mathbf{m}\angle gc_1c_0 \\ &= \mathbf{m}\angle bc_0c_1 \\ &< \mathbf{m}\angle fc_0c_1. \end{aligned}$$

This implies $d(f, c_1) < d(f, c_0) = t$ and hence, $C \subseteq F + t$. Therefore we have $C \subseteq (F + t) \cap (G + u)$ with $c_1 \in C$ such that $c_1 \in \partial(F + t) \cap \partial(G + u) \cap \overleftrightarrow{fg}$

and so $C \in \overleftrightarrow{FG}$. We can conclude that $\overleftrightarrow{AB} \cap \overleftrightarrow{FG} \neq \{p\}$. (Note that since $c_0, c_1 \in (A+r) \cap (B+s) \cap (F+t) \cap (G+u)$, an intersection of convex sets, we must have $\overline{c_0c_1} \subseteq (A+r) \cap (B+s) \cap (F+t) \cap (G+u)$. Thus if we choose $C' = \{c_0\} \cup \{c_1\} \cup Y$ where Y is a nonempty, compact subset of $\overline{c_0c_1}$ we will have $C' \in \overleftrightarrow{AB} \cap \overleftrightarrow{FG}$. Since we can choose infinitely such C', we again have an infinite number of points on $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$).

Case 2: Suppose that we have $d(a,b) = d(a,p) + d(p,b)$, $d(f,g) \neq d(f,p) + d(p,g)$ and $d(b,p) < d(g,p)\cos(\theta)$. Without loss of generality, assume that $d(f,p) = d(f,g) + d(g,p)$ with $d(g,p) \neq 0$. Let $\beta = d(b,p)$, $\gamma = d(g,p)$, and let c_0 be the point on $\overleftrightarrow{ab}$ such that $d(p,c_0) = 2\beta = d(p,b)+d(b,c_0)$. Let $C = \{p, c_0\}$ and take $r = d(a,c_0)$, $s = d(b,c_0)$, $t = d(u,p)$, and $u = d(g,p)$. Then $d(b,c_0) = \beta = d(b,p)$ so that $C \subseteq B+s$. Furthermore, $d(a,c_0) = d(a,p)+d(a,c_0) > d(a,p)$ so that $C \subseteq A+r$. Since $C \subseteq (A+r) \cap (B+s)$ and there exists $c_0 \in C$ such that $c_0 \in \partial(A+r) \cap \partial(B+s) \cap \overleftrightarrow{ab}$, we must have $C \in \overleftrightarrow{AB}$. We also have

$$\begin{aligned} d(g,c_0)^2 &= \gamma^2 + 4\beta^2 - 4\beta\gamma\cos(\theta) \\ &< \gamma^2 + 4\beta^2 - 4\beta^2 \\ &= \gamma^2 = d(g,p)^2. \end{aligned}$$

This implies $d(g,c_0) < d(g,p)$ and $C \subseteq (G+u)$. Furthermore, since $d(g,c_0) < d(g,p)$ gives us $\mathbf{m}\angle gcpc_0 < \mathbf{m}\angle gc_0p$, we can conclude that

$$\begin{aligned} \mathbf{m}\angle fpc_0 &= \mathbf{m}\angle gpc_0 \\ &< \mathbf{m}\angle gc_0p \\ &< \mathbf{m}\angle fc_0p \end{aligned}$$

which implies that $d(f,c_0) < d(f,p)$ and $C \subseteq (F+t)$. Since $C \subseteq (F+t) \cap (G+u)$ with $p \in C$ such that $p \in \partial(F+t) \cap \partial(G+u) \cap \overleftrightarrow{fg}$ we must have $C \in \overleftrightarrow{FG}$. Therefore we have found $C \neq \{p\}$ such that $C \in \overleftrightarrow{AB} \cap \overleftrightarrow{FG}$. (Note that again, since $\overline{pc_0}$ is also in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$, we have an infinite number of sets in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$.)

$\Leftarrow$: Suppose we have $d(a,b) = d(a,p) + d(p,b)$.

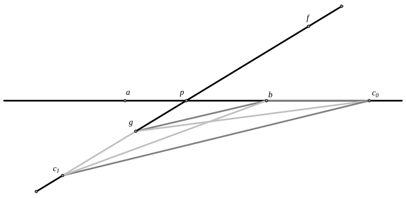

Figure 13: The case when $d(a, b) = d(a, p) + d(p, b)$ and $d(f, g) = d(f, p) + d(p, g)$ implying we have a single point interesection of distinct Hausdorff lines.

Case 1: Suppose that we also have $d(f, g) = d(f, p) + d(p, g)$ (see 13). Let C be a point in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$. Then we can find $c_0, c_1 \in C$ such that c_0 is on $\overleftrightarrow{ab}$, c_1 is on $\overleftrightarrow{fg}$ and for any $c \in C$, we must have $d(a, c) \leq d(a, c_0), d(b, c) \leq d(b, c_0), d(f, c) \leq d(f, c_1)$ and $d(g, c) \leq d(g, c_1)$.

Case 1a: Suppose that either $d(a, c_0) = d(a, b) + d(a, c_0)$ or $d(c_0, b) = d(c_0, a) + d(a, b)$ and either $d(f, c_1) = d(f, g) + d(g, c_1)$ or $d(c_1, g) = d(c_1, f) + d(f, g)$. Without loss of generality, we shall assume $d(a, c_0) = d(a, b) + d(b, c_0)$ and $d(f, c_1) = d(f, g) + d(g, c_1)$. Consider the quadrilateral formed by c_0, b, g and c_1 with diagonals $\overline{bc_1}$ and $\overline{gc_0}$. We know that $d(b, c_1) \leq d(b, c_0)$. Since longer sides are opposite larger angles in any Euclidean triangle, we have $\mathbf{m}\,\angle bc_0c_1 \leq \mathbf{m}\,\angle bc_1c_0$. This implies that

$$\begin{aligned} \mathbf{m}\,\angle gc_0c_1 &< \mathbf{m}\,\angle bc_0g + \mathbf{m}\,\angle gc_0c_1 \\ &= \mathbf{m}\,\angle bc_0c_1 \\ &\leq \mathbf{m}\,\angle bc_1c_0 \\ &< \mathbf{m}\,\angle gc_1b + \mathbf{m}\,\angle bc_1c_0 \\ &= \mathbf{m}\,\angle gc_1c_0. \end{aligned}$$

But $\angle gc_0c_1$ is opposite $\overline{gc_1}$ in triangle gc_0c_1 while $\angle gc_1c_0$ is opposite $\overline{gc_0}$. We can conclude that $d(g, c_1) < d(g, c_0)$ which contradicts the fact that $d(g, c_1) \geq d(g, c) < \forall c \in C$. Therefore, we cannot have $d(a, c_0) = d(a, b) + d(b, c_0)$ and $d(f, c_1) = d(f, g) + d(g, c_1)$.

Case 1b: Suppose that either $d(a, b) = d(a, c_0) + d(c_0, b)$ or $d(f, g) = d(f, c_1) + d(c_1, g)$. Without loss of generality, we shall assume the former. Let

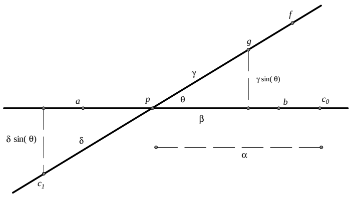

Figure 14: Here we have $d(a,b) = d(a,p) + d(p,b)$, $d(f,g) \neq d(f,p) + d(p,g)$ and $d(g,p) < d(b,p)$. Choosing $c_0, c_1 \in C$ satisfying $d(a,c_0) = d(a,b) + d(b,c_0)$ and $d(f,c_1) = d(f,g) + d(g,c_1)$ leads to a contradiction.

c' be any point in C. We know that $d(a,c') \leq d(a,c_0)$ and $d(b,c') \leq d(b,c_0)$. This implies that

$$d(a,b) \leq d(a,c') + d(c',b) \leq d(a,c_0) + d(c_0,b) = d(a,b).$$

So, $d(a,b) = d(a,c') + d(c',b)$ and c' is on $\overleftrightarrow{ab}$. But $c_1 \in C$ which implies that $c_1 \in \overleftrightarrow{ab} \cap \overleftrightarrow{fg}$. Therefore, $c_1 = p$ and we have $d(f,g) = d(f,c_1) + d(c_1,g)$. We can then use a similar argument to show that $c \in \overleftrightarrow{fg}$, $\forall c \in C$ so that we must have $C \in \overleftrightarrow{ab} \cap \overleftrightarrow{fg}$. It follows that $C = \{p\}$ and we are done.

Case 2:

Suppose that we have $d(a,b) = d(a,p) + d(p,b)$, $d(f,g) \neq d(f,p) + d(p,g)$ and $d(g,p) < d(b,p)$. Without loss of generality, we shall assume that p satisfies $d(f,p) = d(f,g) + d(g,p)$ with $d(g,p) \neq 0$. Let $C \in \mathcal{H}(\mathbb{R}^n)$ be in $\overleftrightarrow{AB} \cap \overleftrightarrow{FG}$. Then we can find $c_0, c_1 \in C$ such that c_0 is on $\overleftrightarrow{ab}$, c_1 is on $\overleftrightarrow{fg}$ and for any $c \in C$, we must have $d(a,c) \leq d(a,c_0), d(b,c) \leq d(b,c_0), d(f,c) \leq d(f,c_1)$ and $d(g,c) \leq d(g,c_1)$. Let $\beta = d(b,p)$, $\gamma = d(g,p)$, $\alpha = d(p,c_0)$, and $\delta = d(p,c_1)$.

Case 2a: Suppose that c_1 satisfies $d(f,g) = d(f,c_1) + d(c_1,g)$. Since $c_0 \in C$, we must have $d(f,c_0) \leq d(f,c_1)$ and $d(g,c_0) \leq d(g,c_1)$. It follows that

$$d(f,g) \leq d(f,c_0) + d(c_0,g) \leq d(f,c_1) + d(c_1,g) = d(f,g).$$

So, $d(f,g) = d(f,c_0) + d(c_0,g)$ and c_0 is on $\overleftrightarrow{fg}$. But this implies $c_0 \in \overleftrightarrow{ab} \cap \overleftrightarrow{fg}$. Therefore, $c_0 = p$ and p satisfies $d(f,g) = d(f,p) + d(p,g)$, a contradiction. Hence we cannot have $d(f,g) = d(f,c_1) + d(c_1,g)$.

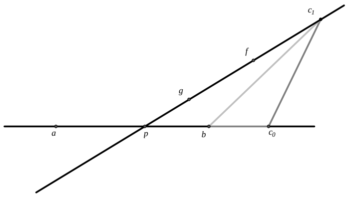

Figure 15: Again we have $d(a,b) = d(a,p)+d(p,b)$, $d(f,g) \neq d(f,p)+d(p,g)$ and $d(g,p) < d(b,p)$. Choosing $c_0, c_1/inC$ satisfying $d(a,c_0) = d(a,b) + d(b,c_0)$ and $d(c_1,g) = d(c_1,f) + d(f,g)$ leads to a contradiction as well.

Case 2b: Suppose we have $d(f,c_1) = d(f,g) + d(g,c_1)$ and $d(a,b) \neq d(a,c_0) + d(c_0,b)$. Without loss of generality, we shall assume $d(a,c_o) = d(a,b) + d(b,c_0)$ with $d(b,c_0) \neq 0$ (See Figure 14). Since $c_1 \in C$, we must have $d(b,c_1) \leq d(b,c_0)$. Therefore,

$$\mathbf{m}\angle pc_0c_1 = \mathbf{m}\angle bc_0c_1 \leq \mathbf{m}\angle bc_1c_0 < \mathbf{m}\angle pc_1c_0.$$

This implies that

$$\delta = d(p,c_1) < d(p,c_0) = \alpha.$$

We also have

$$\begin{aligned}(\alpha-\beta)^2 &= d(b,c_0)^2 \\ &\geq d(b,c_1)^2 \\ &= \beta^2+\delta^2+2\delta\beta\end{aligned} \tag{1}$$

and

$$\begin{aligned}(\gamma+\delta)^2 &= d(g,c_1)^2 \\ &\geq d(g,c_0)^2 \\ &= \alpha^2+\gamma^2-2\alpha\gamma.\end{aligned} \tag{2}$$

After simplifying 1 and 2 we have

$$2\gamma\delta + 2\alpha\gamma\cos(\theta) \geq \alpha^2 - \delta^2 \geq 2\alpha\beta + 2\delta\beta\cos(\theta)$$

and thus

$$\gamma + \alpha\gamma\cos(\theta) \geq \alpha\beta + \delta\beta\cos(\theta).$$

It follows from $\beta = d(b,p) \geq d(g,p) = \gamma$ that

$$\beta(\delta + \alpha\cos(\theta)) \geq \gamma(\delta + \alpha\cos(\theta)) \geq \beta(\alpha + \delta\cos(\theta)).$$

But this implies

$$\delta + \alpha\cos(\theta) \geq \alpha + \delta\cos(\theta)$$

$$\delta - \delta\cos(\theta) \geq \alpha - \alpha\cos(\theta)$$

$$\delta(1 - \cos(\theta)) \geq \alpha(1 - \cos(\theta))$$

$$\delta \geq \alpha$$

which contradicts the fact that $\alpha > \delta$. We can conclude that we cannot have c_0, c_1 that satisfy $d(f, c_1) = d(f, g) + d(g, c_1)$ and $d(a, c_o) = d(a, b) + d(b, c_0)$ with $d(b, c_0) \neq 0$.

Case 2c: Suppose we have $d(c_1, g) = d(c_1, f) + d(f, c_1)$ and $d(a, b) \neq d(a, c_0) + d(c_0, b)$. Without loss of generality, we shall assume $d(a, c_o) = d(a, b) + d(b, c_0)$ with $d(b, c_0) \neq 0$ (See Figure 15). Since $d(b, c_1) \leq d(b, c_0)$, we must have $\mathbf{m}\angle bc_0c_1 \leq \mathbf{m}\angle bc_1c_0$. But then we would have

$$\mathbf{m}\angle fc_1c_0 > \mathbf{m}\angle bc_1c_0 \geq \mathbf{m}\angle bc_0c_1 > \mathbf{m}\angle fc_0c_1$$

which implies that $d(f, c_0) > d(f, c_1)$, a contradiction. Therefore, we cannot have $d(c_1, g) = d(c_1, f) + d(f, c_1)$ and $d(a, b) \neq d(a, c_0) + d(c_0, b)$.

Case 2d: Suppose c_0 satisfies $d(a, b) = d(a, c_0) + d(c_0, b)$. Let c' be any point in C. We know that $d(a, c') \leq d(a, c_0)$ and $d(b, c') \leq d(b, c_0)$. This implies that

$$d(a, b) \leq d(a, c') + d(c', b) \leq d(a, c_0) + d(c_0, b) = d(a, b).$$

So, $d(a, b) = d(a, c') + d(c', b) = d(a, c_0) + d(c_0, b)$ and $c' = c_0$. But $c_1 \in C$ which implies that $c_1 = c_0$ and $c_1 \in \overleftrightarrow{ab} \cap \overleftrightarrow{fg}$. Therefore, $c = c_0 = c_1 = p$ for all $c \in C$ and we have $C = \{p\}$. ■

The previous two theorems illustrate the fact that there are three possibilities for the intersections of two distinct lines defined by single point sets in $\mathcal{H}(\mathbb{R}^n)$: no intersection, a single point intersection, or infinitely many points of intersection.

6 Conclusions

The geometry of $\mathcal{H}(\mathbb{R}^n)$ is a very rich one. There are many questions that remain to be answered. Among these are the following:

- As we saw, the standard Euclidean geometry is embedded in the geometry of $\mathcal{H}(\mathbb{R}^n)$ as the single element sets. However, the geometry of $\mathcal{H}(\mathbb{R}^n)$ is not itself an incidence geometry. Are there subspaces of $\mathcal{H}(\mathbb{R}^n)$ that form incidence geometries other than Euclidean geometry? For example, consider the subspace $\mathcal{B}(\mathbb{R}^n)$ of all closed n-balls in $\mathbb{R}^n$. We have been able to classify all elements in $\mathcal{B}(\mathbb{R}^n)$ that lie on lines defined in this subspace. However, a simple example shows that this subspace does not yield an incidence geometry. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Let A be the closed ball of radius 3 centered at $8e_1$, B the closed ball of radius 1 centered at $3e_1$, F the closed ball of radius 3 centered at $8e_2$, and G the closed ball of radius 1 centered at $3e_2$. Then any closed ball of radius less than 1 centered at the origin is on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{FG}$. Is it possible to find other subspaces that will give us an incidence geometry? If so, what kind of parallel lines will we obtain in such a geometry?

- In the previous section, we completely determined the intersection properties of lines defined by single element sets in $\mathcal{H}(\mathbb{R}^2)$. To obtain a complete classificiation of the intersections of lines defined by single element sets in $\mathcal{H}(\mathbb{R}^n)$, we must determine the outcomes of intersections of single point sets whose points lie on skew-parallel lines in $\mathbb{R}^n$.

- What other subspaces of $\mathcal{H}(\mathbb{R}^n)$ have interesting structures? For example, what, if anything, can we say about the geometry of $\mathcal{H}(S^n)$ or $\mathcal{H}(T^n)$, where S^n is the n-sphere and T^n is the n-torus?

- Is it possible to define induced geometries on interesting quotient spaces of $\mathcal{H}(\mathbb{R}^n)$. For example, if A and B are elements in $\mathcal{H}(\mathbb{R}^n)$, say that $A \sim B$ if there exist $r, s > 0$ so that $A + r = B + s$. It is not difficult to show that $\sim$ defines an equivalence relation on $\mathcal{H}(\mathbb{R}^n)$. Is there a well-defined quotient space geometry? If so, what type of geometry do we obtain? Are there other quotient spaces that yield interesting geometries?

- For certain subclasses of elements in $\mathcal{H}(\mathbb{R}^n)$, we are able to show that there are infinitely many points on the lines determined by these types of elements. A big question that remains is whether the same can be said for the line determined by any two arbitrary elements of $\mathcal{H}(\mathbb{R}^n)$.
- What analytic properties, if any, do lines in $\mathcal{H}(\mathbb{R}^n)$ posess? For example, does it make sense to talk about "continuous" lines in $\mathcal{H}(\mathbb{R}^n)$? One possible way to define "continuity" is, given distinct points A and B, C on the line $\overleftrightarrow{AB}$ and any $\epsilon > 0$, can we always find points $D \neq C$ so that D is on $\overleftrightarrow{AB}$ and $h(C, D) < \epsilon$? FOr our lines defined by single point sets, we can always do this. Let $A = \{a\}$ and $B = \{b\}$ in $\mathbb{R}^n$, with $a \neq b$, let $C \in \overleftrightarrow{AB}$ and let $\epsilon > 0$. From theorem 1 we know that there is a point c_0 on $\overleftrightarrow{ab}$ that is determined by C. If $c_0 \in \overline{ab}$, choose a point d_0 on $\overline{ab}$ that is within ϵ of c_0. Then $D = \{d_0\}$ is on $\overleftrightarrow{AB}$ and $h(C, D) < \epsilon$. If $c_0 \notin \overline{ab}$, then $D = C + \frac{\epsilon}{2}$ is on $\overleftrightarrow{AB}$ and $h(C, D) < \epsilon$. In either case, given any C on $\overleftrightarrow{AB}$, we can find elements on $\overleftrightarrow{AB}$ as close to C as we want. A question that remains is whether lines defined by any pair of elements in $\mathcal{H}(\mathbb{R}^n)$ are continuous in this sense. Is this version of continuity useful? Is there a similar notion of connectedness?

As you can see, there are many questions yet to be answered in this rich geometry.

References

[1] Micheal Barnsley, *Fractals Everywhere*, Academic Press, Inc., San Diego, 1988.

[2] Bouilott, Mikael and Normand Gregoire. Hausdorff distance between Convex Polygons. `http://cgm.cs.mcgill.ca`

[3] Gerald A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.

[4] I. Ginchev and A. Hoffman, The Hausdorff Nearest Circle to a Convex Compact Set in the Plane, *Journal for Analysis and its Applications*, Volume 17 (1998), No. 2, 479-499.

[5] Millman, Richard S. and Parker, George D., *Geometry: A Metric Approach with Models*, Springer-Verlag, New York, 1981

[6] Olson, Clark. Automatic Target Recognition by Matching Oriented Edge Pixels (1997). `http://citeseer.nj.nec.com/olson97automatic.html`

[7] Penkov, B.I and Sendov, Bl. Hausdorff Metric and its Applications, *Numeische Methoden der Approxstheorie*, Vol. 1 (1972), p. 127-146

[8] Sendov, Bl. On the Hausdorff Geometry of Polynomials `http://atlas-conferences.com/c/a/h/n/32.htm`