

Sierpinski N-Gons

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Introduction: The growing interest in chaos and fractal geometry has created a new field of mathematics that can be explored by faculty and undergraduates alike. Sierpinski triangles and Koch's curves have become common phrases in many mathematics departments across the country. In this paper we review some basic ideas from fractal geometry and generalize the construction of the Sierpinski triangle to form what we will call Sierpinski polygons.

The Sierpinski Triangle: In fractal geometry, the well known Sierpinski triangle can be constructed as a limit of a sequence of sets as follows: We begin with three points x_1 , x_2 , and x_3 that form the vertices of an equilateral triangle A_0 . For $i=1,2$, or 3 , let

$x_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$. Let \mathbb{R} represent the set of real numbers and let \mathbb{R}^2 be

the real plane. For $i=1,2$, or 3 , we define $\omega_i: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\omega_i \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} a_i \\ b_i \end{bmatrix}. \text{ Then } \omega_i, \text{ when applied to } A_0,$$

contracts A_0 by a factor of 2 and then translates the image of A_0 so that the i th vertices of A_0 and the image of A_0 coincide.

Define $A_{1,i}$ to be $\omega_i(A_0)$. Then $A_{1,i}$ is the set of all points half way between any point in A_0 and x_i , or $A_{1,i}$ is a triangle half the size of the original translated to the i th vertex of the original. Let

$$A_1 = \bigcup_{i=1}^3 A_{1,i}. \text{ } A_0 \text{ and } A_1 \text{ are shown in figures 1 and 2.}$$

respectively. We can continue this procedure replacing A_0 with

$$A_1. \text{ For } i=1,2, \text{ or } 3, \text{ let } A_{2,i} = \omega_i(A_1) \text{ and let } A_2 = \bigcup_{i=1}^3 A_{2,i}. \text{ } A_2$$

is pictured in figure 3. Again, we can continue this procedure, each time replacing A_i with A_{i+1} . A_4 and A_8 are shown in figures 4 and 5.

If we take the limit as i approaches infinity, the resulting figure is the Sierpinski triangle. This algorithm for building the Sierpinski triangle is called the deterministic algorithm.

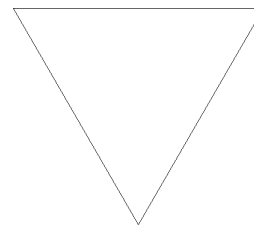


Fig. 1

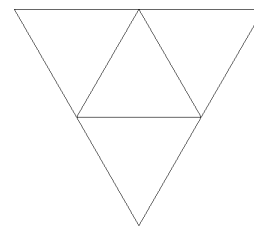


Fig. 2

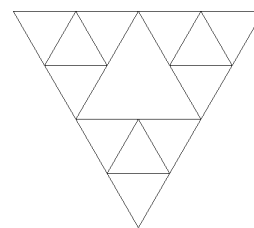


Fig. 3

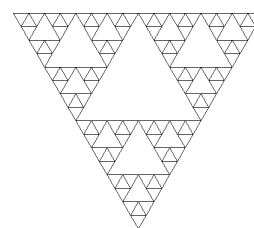


Fig. 4

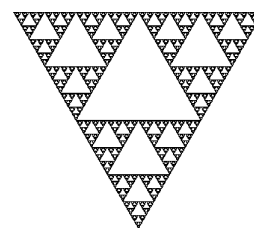


Fig. 5

Classifications: At this point it might be natural to ask what would happen if, in using the deterministic algorithm, instead of cutting distances in half, we cut the distances by a factor of 3, or 4, or 10? In other words, in the deterministic algorithm, what would happen if, for $r > 0$ we

defined $\omega_i \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{r} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{r-1}{r} \begin{bmatrix} a_i \\ b_i \end{bmatrix}$, let $A_{m,i}(r) = \omega_i(A_{m-1}(r))$ and let

$A_{m+1}(r) = \bigcup_{i=1}^3 A_{m,i}(r)$? As earlier, $A_{1,i}$ is a triangle translated to the i th vertex whose sides have

length $\frac{1}{r}$ of the lengths of the sides of the original triangle. In figures 6 and 7 we see $A_1(1.5)$ and $A_4(1.5)$ and in figures 8 and 9 we see $A_1(3)$ and $A_4(3)$.

It's easy to see that if $r > 2$, then the resulting A_m consist of a collection of 3^m disjoint triangles. In this case we say that the A_m are "totally disconnected". If $0 < r < 2$ and $r \neq 1$, A_m is a collection of 3^m intersecting triangles. In this case we say that the A_m are "overlapping".

However, if $r=2$ we have seen that A_m is a collection of 3^m triangles that intersect only at the edges. In this case we say that the A_m are "just touching". It seems that the most aesthetically pleasing situation is when $r=2$, where the triangles are "just touching".

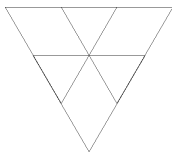


Fig. 6

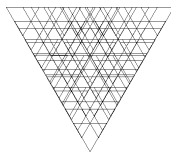


Fig. 7

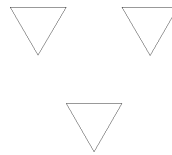


Fig. 8



Fig. 9

The Deterministic Algorithm Applied to Regular n -gons: In the above discussion there seems to be no reason why we should restrict ourselves to looking at only three points. Why not generalize to n points? Let v_1, v_2, \dots, v_n be n distinct points in the plane such that the line segments joining v_i to v_{i+1} , for $1 \leq i \leq n-1$, and v_n to v_1 are all of equal length and such that all angles determined by three consecutive sides have equal measure. The set of segments so formed is called a regular n -gon with vertices v_1, v_2, \dots, v_n . Let A_0 be a regular n -gon with vertices

v_1, v_2, \dots, v_n . For $i \in \{1, 2, 3, \dots, n\}$ let $v_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$, and for $r > 0$, let $\delta_i \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \frac{1}{r} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{r-1}{r} \begin{bmatrix} a_i \\ b_i \end{bmatrix}$,

$A_{m,i}(r) = \delta_i(A_{m-1}(r))$ and $A_{m+1}(r) = \bigcup_{i=1}^n A_{m,i}(r)$. Again, $A_{1,i}$ is an n -gon translated to the i th

vertex whose sides have length $\frac{1}{r}$ of the lengths of the sides of the original n -gon. Some examples of $A_1(r)$ can be seen in figures 10, 11, and 12.

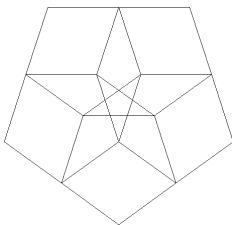


Fig. 10
n=5, r=2

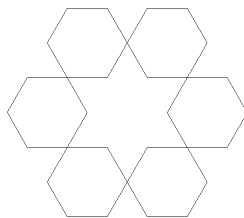


Fig. 11
n=6, r=3

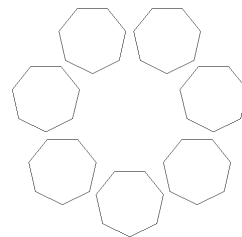


Fig. 12
n=7, r=3.5

Again we seem to have the situation that for some r 's, $A_m(r)$ consists of "overlapping" n -gons, for some r 's, $A_m(r)$ consists of "just touching" n -gons, and for other r 's, $A_m(r)$ consists of "totally disconnected" n -gons. This raises the question of which value of r makes the constituent n -gons "just touching".

Since (\mathbb{R}^2, d) is a complete metric space, where d is the Euclidean metric on \mathbb{R}^2 , the sequence $A_m(r)$ has a limit in \mathbb{R}^2 . (See [1] for details). For the "just touching" r we will call $SP_n = \lim_{m \rightarrow \infty} A_m(r)$ a regular Sierpinski n -gon.

In this paper we will determine, for each $n \geq 5$, the specific value of r that makes $A_m(r)$ just touching. For r 's bigger than the "just touching" r , the $A_m(r)$ will be disjoint and for r 's smaller than the "just touching" r , the $A_m(r)$ will be overlapping. In addition, we will determine the fractal dimension of each regular Sierpinski n -gon.

Note: The figures included in this paper were obtained beginning with the regular n -gon A_0 whose vertices are the points $v_i = (\cos(\alpha_0 + \frac{2\pi i}{n}), \sin(\alpha_0 + \frac{2\pi i}{n}))$ for $i = 1, 2, \dots, n$, where

$$\alpha_0 = \frac{\pi}{2} - \frac{\pi}{n}.$$

Regular Sierpinski Ngons: Given a regular n -gon, construct line segments from the center to each vertex. The radian measure of each central angle is clearly $\frac{2\pi}{n}$. Each of the triangles determined by the center and two adjacent vertices is isosceles, so the measure of the base angles of each of these triangles is $\frac{(n-2)\pi}{2n}$. Consequently, the measure of a vertex angle of a regular n -gon is $\frac{(n-2)\pi}{n}$.

Proposition 1: For $n \geq 5$, the value of r that determines a regular Sierpinski n -gon is

$$r_n = 2 \left[1 + \sum_{k=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2k\pi}{n}\right) \right],$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x .

Proof: Let n be an integer greater than or equal to 5. Without loss of generality, we can prove the theorem for a particular regular n -gon. We define v_1, v_2, \dots, v_n , A_0 , $A_{m,i}(r)$ and $A_m(r)$ as on page 3. We will call the $A_{m,i}(r)$ the constituent n -gons of $A_m(r)$. Since the length of any

side of one of the constituent n -gons of $A_m(r)$ is $\frac{1}{r}$ times the length of any side of one of the constituent n -gons of $A_{m-1}(r)$, for a given m all of the n -gons making up $A_m(r)$ are congruent, regular n -gons. Let d_0 be the length of one side of A_0 and let d_m be the length of one side of any constituent n -gons of $A_m(r)$. To find the value of r that makes the $A_{m,i}(r)$ "just touching", all we need do is find the ratio $\frac{d_0}{d_1}$.

In figure 13 we see a picture of a portion of $A_{1,1}$ and $A_{1,2}$.

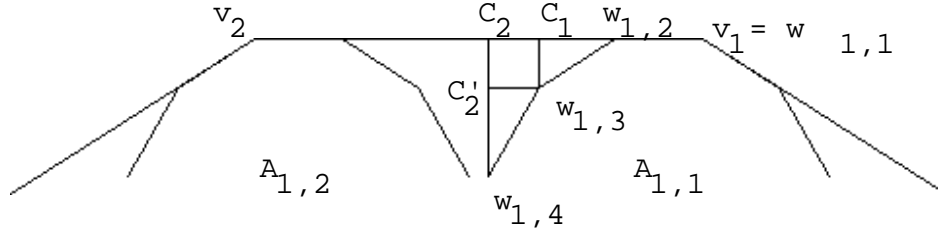


Fig. 13

Label the vertices of $A_{1,1}$ as $w_{1,1}, w_{1,2}, \dots, w_{1,n}$ starting with $w_{1,1} = v_1$ and proceeding counterclockwise. In figure 18 we see the sides v_1v_2 and v_nv_1 of A_0 and sides $w_{1,1}w_{1,2}$, $w_{1,2}w_{1,3}$, and $w_{1,3}w_{1,4}$ of $A_{1,1}$. Here, $|v_1v_2| = d_0$ and $|w_{1,i}w_{1,i+1}| = d_1$ for each i . Construct a line from $w_{1,3}$ perpendicular to v_1v_2 . Label the point of intersection C_1 . Now,

$m(\angle w_{1,3}w_{1,2}w_{1,1}) = \frac{(n-2)\pi}{n}$, so $m(\angle C_1w_{1,2}w_{1,3}) = \frac{2\pi}{n}$. So $|w_{1,2}C_1| = d_1 \cos\left(\frac{2\pi}{n}\right)$. Since the sum of the measures of the angles in a right triangle is π radians, it follows that

$m(\angle w_{1,2}w_{1,3}C_1) = \frac{(n-4)\pi}{2n}$. Now construct a line from $w_{1,4}$ perpendicular to v_1v_2 and call the point of intersection C_2 . The points C_1 , C_2 , and $w_{1,3}$ form three vertices of a rectangle.

Label the fourth vertex C'_2 . Construct rectangle $C_2C'_2C_1w_{1,3}$. Since

$$m(\angle w_{1,4}w_{1,3}C'_2) + m(\angle C'_2w_{1,3}C_1) + m(\angle w_{1,2}w_{1,3}C_1) + m(\angle w_{1,4}w_{1,3}w_{1,2}) = 2\pi,$$

$m(\angle w_{1,4}w_{1,3}C'_2) = \frac{4\pi}{n}$. So $m(\angle w_{1,3}w_{1,4}C'_2) = \frac{(n-8)\pi}{2n}$. Then $|C_2C_1| = |w_{1,3}C'_2| = d_1 \cos\left(\frac{4\pi}{n}\right)$.

We can continue this process inductively, at the t th stage obtaining an angle $\angle w_{1,t+1}w_{1,t+2}C'_t$ with measure $m(\angle w_{1,t+1}w_{1,t+2}C'_t) = \frac{(n-4t)\pi}{2n}$, as long as $n \geq 4t$.

By the division algorithm we can find an integer k so that $n=4k+r$, where $0 \leq r < 4$. So we can continue our construction up to the k th step. If $r=0$, then $n=4k$. In this situation, the k th side of $A_{1,1}$ in this progression coincides with a corresponding side of $A_{2,1}$. If $r>0$, then $n>4k$. In this situation $A_{1,1}$ and $A_{2,1}$ intersect at a vertex. Now we can see that if the $A_{m,i}(r)$ are "just

touching", then, by symmetry, the sum of the lengths $|w_{1,1}w_{1,2}| + |w_{1,2}C_1| + \sum_{i=1}^{k-1} |C_i C_{i+1}|$ will be

half of d_0 . So $\frac{d_0}{2} = |w_{1,1}w_{1,2}| + |w_{1,2}C_1| + \sum_{i=2}^k |C_{i-1}C_i| = d_1 + \sum_{i=1}^k d_1 \cos\left(\frac{2i\pi}{n}\right)$.

Therefore, the contractivity factor necessary to obtain "just touching" $A_{m,i}(r)$ is

$$\frac{d_0}{d_1} = 2 \left[1 + \sum_{i=1}^k \cos\left(\frac{2i\pi}{n}\right) \right].$$

Since $k = \lfloor n/4 \rfloor$ our proof is complete.

The Sequence $r_n = 2 \left[1 + \sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2i\pi}{n}\right) \right]$

What can be said about the sequence of distances $r_n = 2 \left[1 + \sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2i\pi}{n}\right) \right]$? Intuitively, as n increases, the polygons are approaching circles, so one would expect that the numbers $\frac{1}{r_n}$ would converge to 0. It is easy to see that this is the case.

Proposition 2: *The sequence r_n , as defined above, diverges to infinity.*

Proof: It suffices to show that $\sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2\pi i}{n}\right)$ diverges to infinity. Let n be greater than or equal

to 5 and let $f(k) = \cos\left(\frac{2\pi k}{n}\right) - \frac{k}{n}$ for $k \in \left[0, \frac{n}{6}\right]$. Then $f'(k) = -\left(\frac{1}{n}\right) \left[2\pi \sin\left(\frac{2\pi k}{n}\right) + 1 \right]$. From

this we can see that f is a decreasing function of k on $\left[0, \frac{n}{6}\right]$. Now, $f\left(\frac{n}{6}\right) > 0$ so $f(k) > 0$ on

$\left[0, \frac{n}{6}\right]$. It follows then that $\cos\left(\frac{2\pi k}{n}\right) > \frac{k}{n}$ for $k \in \left[0, \frac{n}{6}\right]$. So

$$\sum_{i=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2i\pi}{n}\right) > \sum_{i=1}^{\lfloor n/6 \rfloor} \cos\left(\frac{2i\pi}{n}\right) > \sum_{i=1}^{\lfloor n/6 \rfloor} \left(\frac{i}{n}\right) = \frac{1}{2n} \left[\frac{n}{6} \right] \left(\left[\frac{n}{6} \right] + 1 \right) > \frac{n-6}{72}$$

which diverges to infinity.

Fractal Dimension: The Sierpinski polygons we have been discussing are all examples of a wider class of objects known as fractals. Every fractal has a number associated to it, the fractal dimension, that determines, in some sense, how much of the overlying space it occupies. In this section we see that the Sierpinski n -gons are really attractors of iterated function systems and we will determine the fractal dimensions of each of the Sierpinski n -gons. All definitions in this

section can be found in Michael Barnsley's book Fractals Everywhere. We begin with a discussion of iterated function systems. As earlier, let SP_n be the Sierpinski n -gon.

For a given n , in constructing SP_n we used n contraction mappings of the form

$$\delta_i(z) = \frac{1}{r_n} I_2 z + u_i, \text{ for } z, u_i \in \mathbb{R}^2, \text{ where } I_2 \text{ is the } 2 \times 2 \text{ identity matrix. This set of mappings}$$

forms what is called an Iterated Function System (IFS) on \mathbb{R}^2 and is denoted

$\{\mathbb{R}^2; \delta_1, \delta_2, \dots, \delta_n\}$. We next need to view SP_n as the attractor of this iterated function system.

The attractor of an IFS $\{\mathbb{R}^2; \omega_1, \omega_2, \dots, \omega_N\}$ is found as follows: Let $B \subset \mathbb{R}^2$ be a

compact set. Let $W(B) = \bigcup_{i=1}^N \omega_i(B)$. It turns out that W is a contraction mapping on the metric

space of all non-empty compact subsets of \mathbb{R}^2 with the Hausdorff metric. As such, W has a unique fixed point A in \mathbb{R}^2 . In other words, there is a non-empty compact subset A of \mathbb{R}^2 so that

$$W(A) = \bigcup_{i=1}^N \omega_i(A) = A. \text{ Another way to think of } A \text{ is that } A = \lim_{i \rightarrow \infty} W^{\alpha_i}(B) \text{ for any non-empty}$$

compact subset $B \subset \mathbb{R}^2$. The set A is called the attractor of the IFS. In our situation, we chose $B=A_0$ to be a regular n -gon. We then constructed sets $A_1(r), A_2(r), \dots, A_m(r), \dots$. In following this construction of attractors, for each i the set $A_i(r)$ is equal to $W^{\alpha_i}(A_0)$. The attractor of the constructed IFS is then the set we are calling SP_n .

Next we discuss the definition of fractal dimension in \mathbb{R}^2 .

Definition 4: Let A be a non-empty compact subset of \mathbb{R}^2 . For each $\varepsilon > 0$ let $N(A, \varepsilon)$ denote the smallest number of closed balls of radius ε needed to cover A . If

$$D(A) = \lim_{\varepsilon \rightarrow 0} \frac{\ln(N(A, \varepsilon))}{\ln(\frac{1}{\varepsilon})}$$

exists, then $D=D(A)$ is the fractal dimension of A .

In Fractals Everywhere there is a wonderful theorem [Theorem 3, p. 184] which allows us to easily determine the fractal dimensions of the Sierpinski n -gons. We state this theorem for \mathbb{R}^2 but it holds in all dimensions. A complete proof of this theorem can be found in [2], [4], or [5].

Theorem 1: Let $\{\mathbb{R}^2; \omega_1, \omega_2, \dots, \omega_N\}$ be a just touching hyperbolic iterated function system and let A be its attractor. Suppose ω_k is a similitude of scaling factor s_k for each $k \in \{1, \dots, N\}$. Then $D(A)$, the fractal dimension of A , is the unique solution to

$$\sum_{k=1}^N |s_k|^{D(A)} = 1, D(A) \in [0, 2].$$

Proposition 3: *The fractal dimension of a Sierpinski n -gon is $\frac{\ln(n)}{\ln(r_n)}$, where*

$$r_n = 2 \left[1 + \sum_{k=1}^{\lfloor n/4 \rfloor} \cos\left(\frac{2k\pi}{n}\right) \right].$$

Proof: Earlier we showed that SP_n is the attractor of a just touching iterated function system in which the contraction mappings $\delta_1, \delta_2, \dots, \delta_n$ all had the same contractivity factor $\frac{1}{r_n}$. Then, by

Theorem 1, $1 = n \left| \frac{1}{r_n} \right|^{D(SP_n)}$. As a result, $D(SP_n) = \frac{\ln(n)}{\ln(r_n)}$.

At this point it seems natural to ask what happens to the sequence $\{D(SP_n)\}$ as $n \rightarrow \infty$. As mentioned earlier, as n increases, the polygons we start with are approaching circles. Intuitively, then, one would expect that, as $n \rightarrow \infty$, $D(SP_n)$ should approach the fractal dimension of a circle, which is 1. This is, in fact, exactly what happens.

Corollary: $\lim_{n \rightarrow \infty} D(SP_n) = 1$.

We omit the proof.

References:

- [1] Michael Barnsley, *Fractals Everywhere*, Academic Press Inc., San Diego, 1988.
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- [3] F. Croom, *Principles of Topology*, Saunders College Publishing, Philadelphia, 1989.
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- [5] J. Hutchinson, "Fractals and self-similarity", *Indiana University Journal of Mathematics*, 1981, v.30, p. 713-747.

A few words about the authors:

This paper is the result of a senior project completed by Kevin Dennis and supervised by Steve Schlicker while both were at Luther College in Iowa. Kevin is now at Michigan State University where he is a Graduate Assistant working on his doctorate in the field of analysis. Steve has since moved to Allendale, Michigan where he is an Associate Professor of Mathematics at Grand Valley State University.