Final Progress Report

Lisa Morales & Daniel Schultheis

September 8, 2006

1 Introduction

The Hausdorff metric is a rigorous way of measuring the "distance" between the nonempty closed and bounded subsets of a metric space (X, d). By applying this metric to even the most familiar of metric spaces $(\mathbb{R}^n, d_E)$, we discover a broad array of geometric properties about the resulting metric space $\mathcal{H}(\mathbb{R}^n)$. We will study some of the interesting properties of metric segments in $\mathcal{H}(\mathbb{R}^n)$, as well as formulate a notion of "angle" in the Hausdorff metric geometry. To that end, we begin by providing several topological terms we will need to define the Hausdorff metric. We take these topological terms from [2].

2 The Hausdorff Metric

Definition 1. *A metric on a set X is a function*

$$d : X \times X \to \mathbb{R}$$

having the following properties:

- $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
- $d(x, y) \geq 0$ *for all* $x, y \in X$*; equality holds if and only if* $x = y$.
- $d(x, y) + d(y, z) \geq d(x, z)$ *for all* $x, y, z \in X$.

A set X with a metric d defined on it is called a metric space, and is denoted (X, d).

Definition 2. *Let (X, d) be a metric space. A subset A of X is said to be bounded if there is some number $M > 0$ such that $d(a_1, a_2) \leq M$ for every pair a_1, a_2 of points of A.*

As a simple example, let us consider the unit disc $\bar{D}_1$ in $\mathbb{R}^2$, with distance defined by the Euclidean metric d_E. Then given points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ such that $p_1, p_2 \in \bar{D}_1$, we may use Figure 1 as a visual guide and find $d_E(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \leq 2$. Hence, the unit disk is bounded in $\mathbb{R}^2$ with the Euclidean metric.

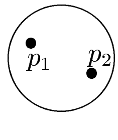


Figure 1: The unit disc $\bar{D}_1$, a bounded subset of $\mathbb{R}^2$.

Definition 3. *Consider a metric space (X, d). The r-ball about the point p is $b(p, r) = \{q \in X : d(p, q) < r\}$.*

Definition 4. *Let (X, d) be a metrix space. A set $S \subset (X, d)$ is open if, for every point $p \in S$, there exists $r > 0$ such that $b(p, r) \subset S$.*

A classic example of an open set is the aptly named open interval $I = (0, 1)$ in $\mathbb{R}$ together with the Euclidean metric. Using Figure 2 as a visual guide, let us pick a point $a \in I$. Then let $r = \min\{d_E(a, 0), d_E(a, 1)\}$. With r thus defined, it is clear that $b(a, r) \subset I$. Since a was arbitrary, I is open.

While we have not shown it here, it should be mentioned that r-balls are always open sets.

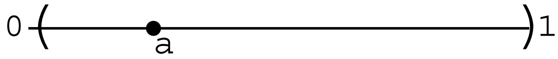


Figure 2: The open unit interval in $\mathbb{R}$ as an example of an open set.

Definition 5. *Let A be a subset of $\mathbb{R}^n$ and let x be a point in $\mathbb{R}^n$. We say that x is a limit point of A if every open set containing x intersects A at some point other than x itself.*

Definition 6. *Let A' be the set of all limit points of A. If $A' \subseteq A$, then A is closed.*

In keeping with our example of an open set above, we shall show that the closed interval $\bar{I} = [0, 1] \subset \mathbb{R}$ is indeed a closed set. If a is a point such that $a < 0$, then the r-ball $b(a, r)$ with $r < |a|$ is an open set about a which does not intersect $\bar{I}$. Similarly, if $a' > 1$, then the r-ball $b(a', r)$ with $r < a' - 1$ is an open set about a which does not intersect $\bar{I}$. Thus, no point in $\mathbb{R} - \bar{I}$ can be a limit point of $\bar{I}$, so $\bar{I}$ is closed. Figure 3 gives a visual to accompany our example.

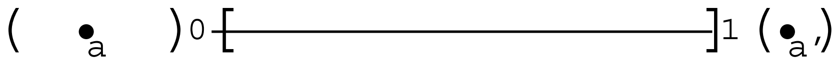


Figure 3: The closed unit interval in $\mathbb{R}$ as an example of a closed set.

Definition 7. *A subset Y of a metric space (X, d) is compact if every open covering $\mathcal{A}$ of Y contains a finite subcollection that also covers Y. Here an open covering $\mathcal{A}$ is a collection of open subsets of X such that the union of the elements of $\mathcal{A}$ contains Y.*

The compact subsets of $\mathbb{R}^n$ can be described particulary easily with the Heine-Borel Theorem. For a complete statement of the theorem see [6].

Theorem 1 (Heine-Borel Theorem). *In Euclidean space, compact subsets of $\mathbb{R}^n$ are precisely those sets which are closed and bounded.*

Before defining the Hausdorff metric, we need the Extreme Value Theorem, which we take from [2].

Theorem 2 (Extreme Value Theorem). *Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$. That is, f attains its maximal and minimal values when defined on a compact domain.*

We may now proceed with the definition of the Hausdorff metric. Let $\mathcal{H}(\mathbb{R}^n)$ be the set of nonempty closed and bounded subsets of $\mathbb{R}^n$. We will define the Hausdorff metric h on this set $\mathcal{H}(\mathbb{R}^n)$, and so create the metric space $(\mathcal{H}(\mathbb{R}^n), h)$. To prevent confusion, we will refer *elements* in $\mathcal{H}(\mathbb{R}^n)$ and denote them with upper case letters $A, B, \ldots$. We reserve the word *points* to be singletons in $\mathbb{R}^n$, and we shall denote them with lower case letters $a, b, \ldots$.

Definition 8. *Let A and B be elements in $\mathcal{H}(\mathbb{R}^n)$ and let $a \in A$ and $b \in B$.*

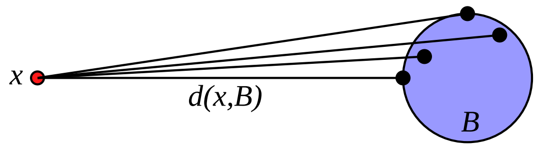


Figure 4: Distance from a point to an element.

- *If $x \in \mathbb{R}^n$, the "distance" from x to B is*

$$d(x, B) = \min_{b \in B}\{d_E(x, b)\}.$$

As Figure 4 shows, we may think of finding the distance from a point x to a compact set B as finding the distance from x to each point $b \in B$, and then selecting the minimum distance. Note that since B is compact, this minimum is guaranteed to exist by Theorem 2.

- *The "distance" from A to B is*

$$d(A, B) = \max_{x \in A}\{d(x, B)\}.$$

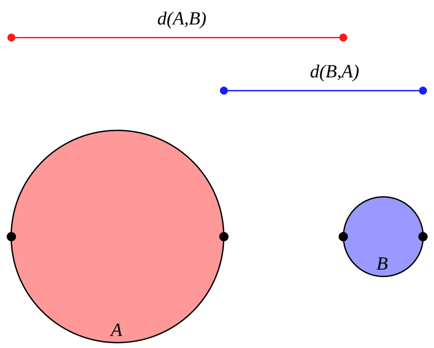


Figure 5: Distance from an element to an element.

The second step in our definition calls for us to take the maximum of $d(x, B)$ over all $x \in A$, as shown in Figure 5. Again, this maximum exists by virtue of Theorem 2. As Figure 5 shows, $d(A, B)$ need not be symmetric with respect to A and B. This leads to the third and final step in our definition.

- *The Hausdorff distance, $h(A, B)$, between A and B is*

$$h(A, B) = \max\{d(A, B), d(B, A)\}.$$

Now we prove that the Hausdorff distance satisfies the three conditions given in the definition of a metric.

Theorem 3. *The Hausdorff distance $h : \mathcal{H}(\mathbb{R}^n) \times \mathcal{H}(\mathbb{R}^n) \to \mathbb{R}$ is a metric.*

Proof. We address each of the three criteria for a metric space.

1. $h(A, B) = h(B, A)$

   By definition, $h(A, B) = \max\{d(A, B), d(B, A)\} = \max\{d(B, A), d(A, B)\} = h(B, A)$.

2. $h(A, B) \geq 0$ with equality if and only if $A = B$.

First let us show $h(A,B) \geq 0$. Since $h(A,B) = \max\{d(B,A), d(A,B)\}$, it suffices to show $d(A,B) \geq 0$. Now

$$\begin{aligned} d(A,B) &= \max_{x\in A}\{d(x,B)\} \\ &= \max_{x\in A}\{\min_{b\in B}\{d_E(x,b)\}\}. \end{aligned}$$

But we know that $d_E(x,b) \geq 0$ for any $x, b \in \mathbb{R}$, and so $d(A,B) \geq 0$. Hence, $h(A,B) \geq 0$ as well.

Next we show $h(A,B) = 0$ if and only if $A = B$. Suppose $A = B$, then for any $x \in A$ we have $x \in B$ as well. Then $d(x,B) = \min_{b\in B}\{d_E(x,b)\} = d_E(x,x) = 0$ for all $x \in A$. Similarly $d(y,A) = 0$ for all $y \in B$. Then $d(A,B) = \max_{x\in A}\{d(x,B)\} = 0$. Similarly, $d(B,A) = 0$. Thus, $h(A,B) = \max\{d(A,B), d(B,A)\} = \max\{0,0\} = 0$.

Now we must show that if $h(A,B) = 0$, then $A = B$. We shall prove the contrapositive. Suppose $A \neq B$. Then either there exists $a_0 \in A$ such that $a_0 \notin B$ or there exists $b_0 \in B$ such that $b_0 \notin A$. Without loss of generality, assume that there exists $a_0 \in A$ such that $a_0 \notin B$. Then $d(a_0, B) = \min_{b\in B} d_E(a_0, b)$. Let us choose b_1 such that $d_E(a_0, b_1) = d(a_0, B)$. We know that such a b_1 exists by Theorem 2. Note that, by the definition of a metric, $d_E(a_0, b_1) \geq 0$. Furthermore, since $a_0 \neq b_1$ then $d_E(a_0, b_1) \neq 0$. Hence $d(a_0, B) > 0$. Thus, $d(A,B) = \max_{x\in A}\{d(x,B)\} \geq d(a_0,B) > 0$ and so $h(A,B) = \max\{d(A,B), d(B,A)\} > 0$, as desired.

3. $h(A,C) \leq h(A,B) + h(B,C)$

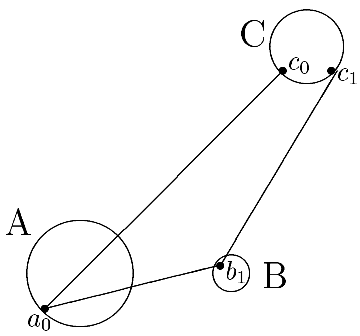


Figure 6: Proving Triangle Inequality in Hausdorff Metric.

We know that either $h(A,C) = d(A,C)$ or $h(A,C) = d(C,A)$. Let us suppose first that $h(A,C) = d(A,C) = r$. Choose $a_0 \in A$ so that $d(a_0, C) = r$. Then since $d_E(a_0, c) : C \to \mathbb{R}$ is a continuous function defined on a compact space, Theorem 2 tells us there exists $c_0 \in C$ such that $d_E(a_0, c_0) = \min_{c\in C}\{d_E(a_0, c)\}$. Hence, there exists a $c_0 \in C$ such that $d_E(a_0, c_0) = r$.

Next, note that there exists $b_1 \in B$ such that $h(A,B) \geq d_E(a_0, b_1)$. To see this, suppose that there existed no such b_1. Then $d_E(a_0, b) > h(A,B)$ for all $b \in B$. This would imply $d(a_0, B) > h(A,B) \geq d(A,B)$, which would violate the inequality $d(A,B) \geq d(a_0, B)$. Similarly, there exists $c_1 \in C$ such that $h(B,C) \geq d_E(b_1, c_1)$, for if not, then $d(b_1, C) > d(B,C)$, which would be a contradiction. See Figure 6 for a visual. Finally, let us note that $d_E(a_0, c_0) \leq d_E(a_0, c_1)$. Then by combining the inequalities and applying the triangle inequality for the Euclidean metric, we find

$$h(A,C) = d_E(a_0, c_0) \leq d_E(a_0, c_1) \leq d_E(a_0, b_1) + d_E(b_1, c_1) \leq h(A,B) + h(B,C).$$

If, on the other hand, $h(A,C) = d(C,A)$, the proof is completely analogous. Hence, $h(A,C) \leq h(A,B) + h(B,C)$, and so $h(A,B)$ is a metric. $\square$

3 Configurations, Betweenness, and Adjacencies in the Hausdorff Metric Geometry

In order to study the structure of such geometric constructions as lines in the Hausdorff metric geometry, we would do well to first revisit several concepts from Euclidean space for which our understanding is more intuitive than rigorous.

In Euclidean geometry we have a familiar notion of betweenness. If $a, b, c \in \mathbb{R}^n$ and $d_E(a, b) = r$, $d_E(a, c) = s$, and $d_E(b, c) = t$, then c is between a and b if $r = t + s$, as shown in Figure 7. We adapt this idea to define betweenness in $\mathcal{H}(\mathbb{R}^n)$.

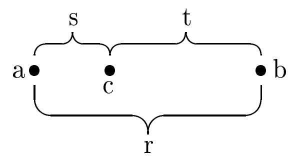


Figure 7: A point c between points a and b in Euclidean space.

Definition 9. *Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$. The set $C \in \mathcal{H}(\mathbb{R}^n)$ lies between A and B if*

$$h(A, B) = h(A, C) + h(C, B).$$

If we so desire, we can expand upon the preceding definition and say that C lies to the right of A if $h(B, C) = h(B, A) + h(A, C)$ and C lies to the left of B if $h(C, A) = h(C, B) + h(B, A)$.

Given this definition, we use our intuition from Euclidean space to define the Hausdorff segment with endpoints A and B, denoted as $S(A, B)$, as the collection of all compact sets between A and B. In $\mathbb{R}^n$, we take it for granted that if $d_E(a, b) = r$, then there is exactly one point c such that c lies between a and b at a distance s from a and $r - s$ from b. In the Hausdorff metric geometry, however, for certain elements A and B such that $h(A, B) = r$, we will find that there are multiple elements $C_1, \ldots, C_\mu$ that lie between A and B and such that $h(A, C_i) = s$ and $h(C_i, B) = r - s$. In such cases we say that $C_1, \ldots, C_\mu$ all lie at the same location between A and B. In some cases, there may even be infinitely many element at the same location between A and B. To more easily give examples of multiple elements at the same location, we introduce the notion of an extension.

Definition 10. *The extension $(A)_s$ of an element A by a positive length s is $\{x \in \mathbb{R}^n \mid d_E(x, a) \leq s$ for some $a \in A\}$.*

Let us note that if $a \in A$, then $a \in (A)_s$ as well, since $d_E(a, a) \leq s$ for any $s > 0$. Now we apply this idea to identify an element on $S(A, B)$ at a distance s from A and $r - s$ from b where $r = h(A, B)$.

Theorem 4. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$, $h(A, B) = r$ and let $M(s) = (A)_s \cap (B)_{r-s}$ for each $s \in (0, r)$. Then $h(A, M(s)) = s$ and $h(M(s), B) = r - s$. Thus, $M(s)$ lies on the metric segment $S(A, B)$.*

Bogdewicz proves Theorem 4 in [7]. If, however, we consider some element C which contains a point c_0 which is not in $M(s)$, then this point must either lie outside of $(A)_s$ or $(B)_{r-s}$. Without loss of generality, let us assume $c_0 \notin (A)_s$. Then $d(c_0, A) > s$, and so C cannot lie between A and B at a distance s from A. Using the similar result for B, this tells us that any element C at a particular distance s from A on the segment $S(A, B)$ must satisfy $C \subseteq (A)_s \cap (B)_{r-s}$. We shall sometimes refer to $(A)_s \cap (B)_{r-s}$ as the largest element at its location between A and B because any other element at the same location is a subset of $(A)_s \cap (B)_{r-s}$.

For an example of $(A)_s \cap (B)_{r-s}$, let us consider the example shown in Figure 8. If A is the solid disk on the left, and B is the circle on the right, then for some choice of s, the set C marks the largest element at its location between A and B. Any other element between A and B at a distance s from A must be a subset of C shown in the picture.

It turns out that for the specific example in Figure 8, there exist infinitely many elements between A and B at the same location as C. We see this through the following theorem, which is proved in [8].

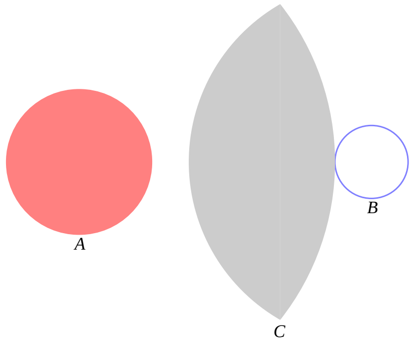


Figure 8: The set $C = (A)_s \cap (B)_{r-s}$ marks the largest element at its location between A and B.

Theorem 5. *Let $A \neq B \in \mathcal{H}(\mathbb{R}^n)$ and let $r = h(A, B)$. Let $s \in \mathbb{R}$ with $0 < s < r$,. If C' is a compact subset of $C = (A)_s \cap (B)_{r-s}$ containing $\partial((A)_s \cap (B)_{r-s})$, then C' lies between A and B with $h(A, C) = s$ and $h(B, C) = r - s$.*

Going back to the example in Figure 8 once more, we can quickly note that there are infinitely many compact subsets of C which include $\partial(C)$. For example, the sets $\{\{c\} \cup \partial(C)\}$, where c is any point in the interior of C, all lie between A and B at the same location as C itself.

Since our explorations will pay particular attention to those sets A and B, for which $S(A, B)$ has finitely many elements at each location, it is useful to define several necessary, though not sufficient, conditions for this outcome.

Theorem 6. *Given compact sets A and B, in order for $S(A, B)$ to have finitely many elements at each location, it is necessary that $h(A, B) = d(b, A) = d(a, B) = r$ for all $a \in A$ and $b \in B$.*

We refer to the above conditions as the PFAEL or Possibly Finite at Each Location conditions. For a proof of Theorem 6, see [3].

Definition 11. *A finite configuration is a set $X = A \cup B$ where A and B are finite sets and A and B satisfy the PFAEL conditions.*

It is entirely possible to have infinite configurations (so that A and B are infinite point sets) which satisfy the PFAEL conditions. We can even construct infinite configurations for which the number of elements at every location between A and B is finite. However, for reasons which will later become clear, we will focus our attention on finite configurations for the time being.

Before we move on to address exactly how we count the number of elements at each location between A and B when the number is indeed finite, we provide a visual of a finite configuration in Figure 9, in which $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$. Clearly, $h(A, B) = d(b_i, A) = d(a_i, B)$ for $i = 1, 2, 3$.

4 $\#(X)$ for a Configuration

In the case of a finite configuration satisfying the PFAEL conditions, let $a \in A$ and $b \in B$, with $d_E(a, b) = h(A, B)$. The extensions $\{a\}_s$ and $\{b\}_{r-s}$ can only intersect at a single point, just as in the Euclidean case, since extensions of single point elements in the Haudorff metric geometry are equivalent to extensions of points in the Euclidean metric. With A and B being finite point sets, this implies that the intersection of $(A)_s$ and $(B)_{r-s}$ is a finite point set. Let us say that $C = (A)_s \cap (B)_{r-s}$. If $|C| = t$, then the number of subsets of C is 2^t, since we may choose to include or exclude every point in C. Thus, the number of elements at a particular locations between A and B is less than or equal to 2^t, so finite configurations do indeed have a finite number of elements at each location between A and B.

While we have now shown that the number of elements at each location between A and B is finite when X is a finite configuration satisfying the PFAEL conditions, there remains the possibility that

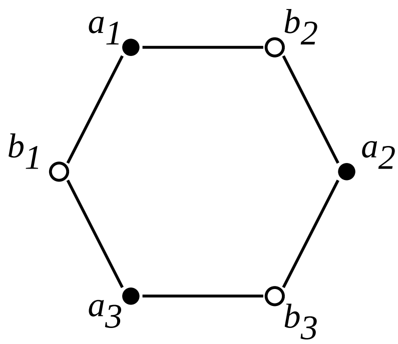


Figure 9: An example of a finite configuration, in which A and B are both 3 point sets.

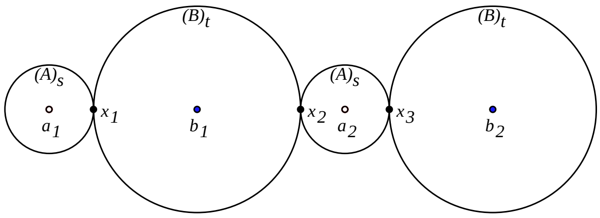


Figure 10: For finite configurations, extensions $(A)_s$ and $(B)_{r-s}$ intersect at only finitely many points.

we could have different numbers of elements at different locations between A and B. This problem is addressed in [3], where the authors prove that, except at the endelements A and B themselves, the number of elements at each location between A and B is a constant. Thus, this number is a property of the configuration X alone, and so we shall denote it by $\#(X)$.

For an example of calculating $\#(X)$, let us consider the configuration shown in Figure 10. Here, $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $C = \{x_1, x_2, x_3\}$. With only 8 subsets of C to consider, we can check each subset in turn and verify that only the sets C and $\{x_1, x_3\}$ satisfy the betweenness criteria. Hence, for the configuration in Figure 10, $\#(X) = 2$. We will see several more efficient methods for determining $\#(X)$ in the next section.

5 Counting Techniques

Given a finite configuration X, we can in theory calculate $\#(X)$ by the brute force approach outlined at the end of the previous section. To make more clear which subsets of C which will lie between A and B, we give the following two definitions.

Definition 12. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and let $a \in A$ and $b \in B$. The point a is said to be adjacent to b, denoted $a \doteqdot b$, if and only if $d_E(a,b) = h(A,B)$. The adjacency set $[a]_C$, is defined as the set $\{c \in C : c \doteqdot a.\}$ We will let $[A]_C$ denote the set $\cup_{a \in A}[a]_C$.*

Definition 13. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions. Define the functions $q_A : C \to A$ and $q_B : C \to B$ by $q_A(c) = a$ when $c \in [a]_C$ and $q_B(c) = b$ when $c \in [b]_C$. Then $q_A(c)$ is the element of $a \in A$ that is adjacent to c.*

The following theorem shows q_A and q_B are well-defined and surjective. For a proof of the theorem, see [3].

Theorem 7. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ satisfy the PFAEL conditions and let $C = (A)_s \cap (B)_{r-s}$, where $r = h(A,B)$ and $0 < s < r$.*

1. *For all $c \in C$ we have $d(c, A) = s$ and $d(c, B) = r - s$.*

2. *If $C' \subseteq C$, then $h(A, C) \leq h(A, C')$.*

3. *Let $C' \subseteq C$ lie between A and B. If $a_0 \in A$ then $[a_0]_{C'} \neq \emptyset$. If $b_0 \in B$, then $[b_0]_{C'} \neq \emptyset$.*

4. *Given $c \in C$, there exists exactly one $a \in A$ such that $c \in [a]_C$ and exactly one $b \in B$ such that $c \in [b]_C$.*

Proposition 1. *Let U be a subset of C, and $C' = C - U$. If for all $a \in q_A(U)$ there exists $c \in [a]_{C'}$ and for all $b \in q_B(U)$ there exists $c \in [b]_{C'}$, then C' lies between A and B at the same location as C. Furthermore, only those C' which satisfy the restrictions above lie between A and B.*

Proof. We shall begin with the forward direction of the proof. First, let $a \notin q_A(U)$, if such an a exists. Then a is a point for which no adjacent points $c \in U$ have have been removed in creating C'. Let $h(A, B) = r$ and let $h(A, C) = s$, where $0 \leq s \leq r$. Then there must exist a $c_a \in C'$ such that $d_E(a, c_a) = s$. First we consider the case in which $U = \emptyset$, so $C' = C$. The certainly for all $a \in q_A(U)$ there exists $c \in [a]_{C'}$ and for all $b \in q_B(U)$ there exists $c \in [b]_{C'}$, since C' here is the largest element between A and B. If $U \neq \emptyset$, the consider a point $a' \in q_A(U)$. By assumption, there exists a point $c_{a'} \in [a]_{C'}$, so that $d_E(a', c_{a'}) = s$. At this point, we may see that $d(A, C') = s$. Since every point $c \in C'$ lies between some $a \in A$ and $b \in B$, it remains true that $d(C', A) = s$ as well. Hence, $h(A, C') = s$. Through an entirely similar argument, we may show that when C' has the assumed form, $h(B, C') = r - s$. Then C' lies between A and B, as desired.

To prove the second half of our if and only if statement, we proceed by contradiction. Suppose that there exists a $C' \subseteq C$ which lies between A and B, and for which either there exists $a \in q_A(U)$ such that $[a]_{C'} = \emptyset$ or there exists $b \in q_B(U)$ for which $[b]_{C'} = \emptyset$. Without loss of generality, we address the former case. Since $[a]_{C'} = \emptyset$, then a is adjacent to no points in C'. There cannot exist any points in $c \in C'$ for which $d_E(a, c) < s$ since $C' \subseteq C$, and so we must conclude that $d(a, C) > s$. Hence, $h(A, C) > s$ as well. Let us now apply part 2 of Theorem 7 to say that $h(B, C') \geq h(B, C)$. Then

$$h(A, C') + h(B, C') > h(A, C) + h(B, C) = s + (r - s) = r = h(A, B),$$

so C' cannot possibly lie between A and B. This completes the proof. □

We may understand this proof as stating that in order for $C' \subseteq C$ to lie between A and B, then the removal of points to create C' cannot isolate any points in A or B. Let us consider the implications in the context of the next example.

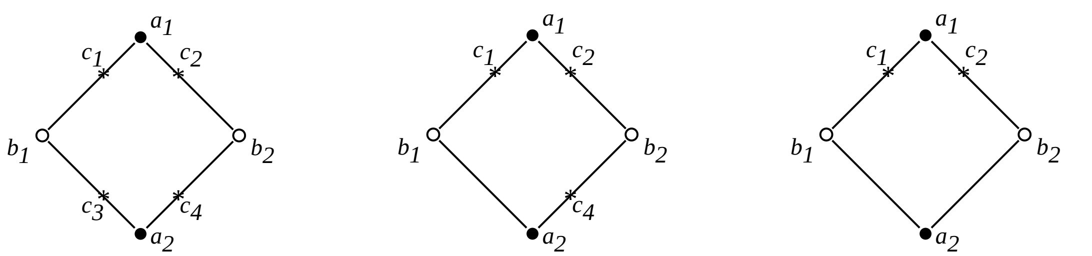


Figure 11: Subsets of $C = \{c_1, c_2, c_3, c_4\}$ satisfy betweenness when they do not "isolate a point" $a \in A$ or $b \in B$.

Let X be the finite configuration given in the left picture of Figure 11, so that $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$ and the largest element between A and B is $C = \{c_1, c_2, c_3, c_4\}$. Then if we remove the point c_3 to create $C' = \{c_1, c_2, c_4\}$, as in the middle figure, we see that for each point $a \in A$ and $b \in B$, there remains a $c \in C'$ such that $d_E(a, c) = s$ and $d_E(b, c) = r - s$. Hence, C' does indeed lie between A and B.

If, on the other hand, we consider the subset $C'' = \{c_1, c_2\}$ shown in the right picture of Figure 11, we see immediately that the point a_2 has been isolated. Since there exists no $c \in C''$ such that $d_E(a_2, c) = s$, then $d(a_2, C'') > s$. In fact, $d(a_2, C'') = h(A, C'') > h(A, B)$, and so C'' cannot lie between A and B.

5.1 String Configurations

We now possess the tools to calculate $\#(X)$ for some simple classes of finite configurations. We therefore define two commonly occuring classes of finite configurations, namely string and polygonal configurations, which will occur frequently through the remainder of this paper. The definition for string configuration comes from [3].

Definition 14. *Let l be a positive integer. The l-string configuration S_l is the set of points $\{1,2,\ldots,l\}$ on the real line, with A and B in $\mathcal{H}(\mathbb{R}^n)$ defined as follows: A is the set of odd points in S_l and B is the set of even points in S_l.*

With this construction, the l-string is indeed a finite configuration, since $d(a,B) = d(b,A) = h(A,B)$ for all $a \in A$ and $b \in B$. This can been seen quite intuitively in Figure 12.

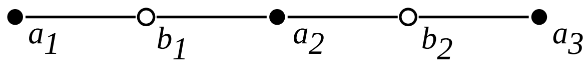


Figure 12: A sample string configuration, S_5.

It turns out that we can calculate $\#(S_l)$ quite easily for an arbitrary $l \in \mathbb{N}$. We will derive a closed form for $\#(S_l)$ using an argument based on the dicussion of string configurations in [5]. First let us consider a few bases cases. Let $A = \{a\}$ and $B = \{b\}$ form the string configuration S_2. There is only one point $c \in C$, where C is the largest element at its location between A and B as usual. Then we can clearly note that only C itself lies at its location between A and B, so $\#(S_2) = 1$.

Next consider the case in which $A = \{a_1, a_2\}$ and $B = \{b\}$, as shown in Figure 13.

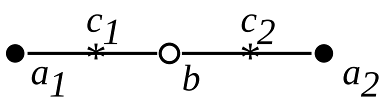


Figure 13: The S_3 string configuration.

If $C = \{c_1, c_2\}$, then we may quickly observe that removing either point from the set will isolate some point, preventing betweenness. Specifically, removing c_1 will isolate a_1 and removing c_2 will isolate a_2. Hence, the only subset of C which satisfies betweenness is C itself, so $\#(S_3) = 1$.

We are now set to consider the general string configuration. Let S_l be defined by the sets $A = \{a_1, \ldots, a_i\}$ and $B = \{b_1, \ldots, b_j\}$, where $i + j = l$. Then C is the $l-1$ point set $\{c_1, \ldots, c_{l-1}\}$. Now let C' be a subset of C. We see immediately that in order for C' to lie between A and B, C' must contain the points c_1 and c_{l-1}. We now consider two cases: either $c_2 \in C'$ or $c_2 \notin C'$.

If $c_2 \in C'$, then the number of such C' satisfying betweenness is the same as the number satisfying betweenness for the string S_{l-1}, since we have effectively just shortened the string by 1. If, on the other hand, $c_2 \notin C'$, then c_3 must be in C' in order to satisfy betweenness. The number of such C' is equal to $\#(S_{l-2})$. Thus, we find that $\#(S_l) = \#(S_{l-1}) + \#(S_{l-2})$. This is a Fibonacci-type relationship. Given the initial data for S_2 and S_3 we find that

$$\#(S_l) = F_{l-1},$$

where F_l is the l-th Fibonacci number.

5.2 Polygonal Configurations

For the second of our common finite configurations, we take the definition of a polygonal configuration from [5].

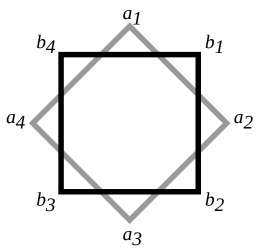


Figure 14: Example of the polygonal configuration P_4

Definition 15. *Let A and B consist of the vertices of regular m-gons with $m \in \mathbb{N}$, in which the m-gons share the same center point and initially are stacked such that the vertices correspond. Then B is rotated $\frac{\pi}{m}$ radians with respect to A about the center point. Let P_m be the configuration $A \cup B$.*

At this point, we could show that $\#(P_m) = L_{2m}$, where L_{2m} is the $2m$-th Lucas number. Readers eager to see such a proof should may read [5]. The proof can be done much more elegantly once we have developed the Looping Algorithm, however, so we defer the proof for now.

5.3 Adjoining a point to an existing configuration

For a particular finite configuration X, we now know that the elements C' which lie between A and B are those which do not isolate any points $a \in A$ or $b \in B$. For larger configurations, it requires quite a bit of work to find $\#(X)$. We would like to present a recursive algorithm for finding $\#(X)$ in order to simplify the calculations for certain configurations.

Theorem 8. *Let X be a finite configuration defined by sets A and B. Define a new configuration X' by adjoining a point y to X at the point a. Furthermore, assume a is adjacent to k points $b_1, b_2, \ldots, b_k$ in X, none of which are endpoints. Then $\#(X') = \#(X) + \#(X - \{a\})$.*

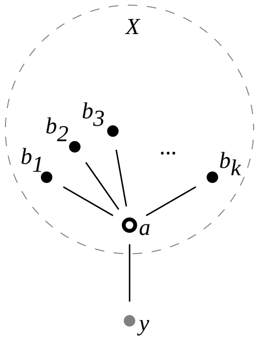


Figure 15: Adjoining a point to a pre-existing configuration X.

We model our proof on the one given in [5].

Proof. First, let us clarify that when we say b_j is not an endpoint, we mean that its adjacency to the point a is not its only adjacency. In general, we will say that a point in a finite configuration is an endpoint if it has only one adjacency.

Now let X be a finite configuration defined by the sets A and B, and create a new configuration X' by adjoining the point y to the point a, as shown in Figure 15. Furthermore, let a in turn be adjacent to the points $b_1, b_2, \ldots, b_k$, none of which are endpoints in X. Let $h(A, B) = r$, and let $h(A, C) = s$, where $0 < s < r$ and $C = (A)_s \cap (B)_{r-s}$. Next, define $A' = A$ and $B' = B \cup \{y\}$, and let $C' = (A')_s \cap (B')_{r-s}$, c_0 be the point in C' between a and y. We will denote a subset of C as C^* and a subset of C' as C'^*.

We can note immediately that in order for $C^{'*}$ to lie between A' and B', it is necessary that $c_0 \in C'^*$. Now if C^* lies between A and B, then $C'^* = C^* \cup \{c_0\}$ lies between A' and B' as well because each for each point $a \in A'$ and $b \in B'$, the $[a]_C \neq \emptyset$ and $[b]_C \neq \emptyset$. The number of such C'^* is equal to $\#(X)$.

Since $[a]_C$ now contains the point c_0, however, this list is not exhaustive. We are free to consider C'^* in which we remove all of the points $c_1, c_2, \ldots, c_k$ which lie between a and the points $b_1, b_2, \ldots, b_k$. The number of such C'^* which lie between A' and B' will be equal to $\#(X - \{a\})$. Hence, we find

$$\#(X') = \#(X) + \#(X - \{a\}).$$

□

While it is not a direct result of adjoining points to pre-existing configurations, many configurations can best be described in terms of adjoining smaller configurations. We will invoke a labeling scheme for finite configurations in which a large configuration will be broken into string and polygon configurations, using a $\oplus$ symbol to show that the smaller configurations have been adjoined. For example, the configuration shown in Figure 16 would be given the label $P_2 \oplus S_1[b_2]$. The $[b_2]$ specifies the point of the base configuration P_2 to which the S_1 adjoins.

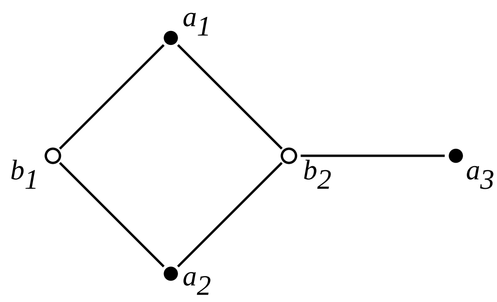


Figure 16: This configuration can be written $P_2 \oplus S_1[b_2]$.

5.4 Rules and Observations for Finding $\#(X)$

The following rules and observations are quite useful for finding $\#(X)$. We do not prove all of our assertions. For the remainder of the proofs, see [3]. We first provide a definition which we shall need for Rule 3.

Definition 16. *Let $X = A \cup B$ and $Y = C \cup D$ be finite configurations such that $h(A, B) = h(C, D) = r$. Furthermore, let us assume that $h(X, Y) > r$. Then we may creat the new configuration $X \cup Y = (A \cup C) \cup (B \cup D)$. We will call such a configuration a* disconnected configuration *for reasons which are made clear in Figure 17.*

1. Adjoining a point y to a configuration $X = A \cup B$ at a point x to create a new configuration X', then $\#(X') \geq \#(X)$.

   This is more than just an application of the formula given in Section 5.3, as it applies even when we adjoin y next to another endpoint. As a proof, let $C = (A)_s \cap (B)_{r-s}$ as usual, and let c_0 be the point between y and x which lies distance s from one and $r - s$ from the other. Then if $C' \subset C$ lies between A and B, we see also that $C' \cup c_0$ lies between A' and B'.

2. If we create a configuration X' by adjoining a point y to a configuration X at the point x, and x is already adjacent to an endpoint, then $\#(X') = \#(X)$.

3. Let $X \cup Y$ be a disconnected configuration. Then $\#(X \cup Y) = \#(X) \cdot \#(Y)$.

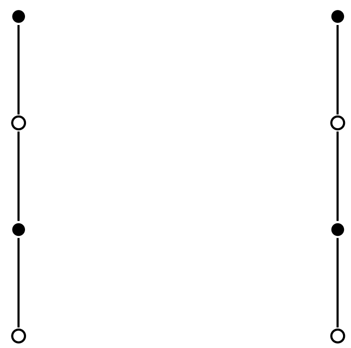


Figure 17: A sample disconnected configuration.

6 Looping Algorithm

By combining our formula for adjoining a point to a configuration shown in Theorem 8 with the formula for $\#(P_m)$ for polygonal configurations, we can calculate $\#(X)$ for any configuration involving no more than one polygonal subconfiguration. If we want to consider a configuration with more than one polygonal subconfiguration, however, we are brought back to brute force counting techniques. We therefore decided to explore the possibility of adding adjacencies to pre-existing configurations. If we choose to be formal about the derivation, then there are two cases to consider; we can add an adjacency between the two halves of a disconnected configuration, or we can add an adjacency between two points on an already connected configuration. We shall derive the former case and simply state the result for the latter, since it is so similar.

Let us suppose that we have a disconnected configuration $X \cup Y$, so that both $X = A \cup B$ and $Y = F \cup G$ are finite configurations themselves, with $h(A, B) = h(F, G)$. Figure 18 provides a visual for a specific instance in which both X and Y are P_2 configurations. If we adjoin the configuration X at some point i to the configuration Y at some point j, we shall denote this $X[j] \oplus Y[i]$. Continuing with our example from 18, the adjoined configuration $P_2[j] \oplus P_2[i]$ is shown if Figure 19. We will prove the following theorem about $\#(X[j] \oplus Y[i])$.

Theorem 9. *Let $X \cup Y$ be a disconnected configuration, and let us consider the configuration $X[j] \oplus Y[i]$ obtained by adjoining X and Y at the points i and j. Then we find*

$$\#(X[j] \oplus Y[i]) = \#(X) \cdot \#(Y) + \#(X \oplus S_1[i]) \cdot \#(Y \oplus S_1[j]).$$

If we instead add an adjacency between two points i and j of a configuration X which is already connected, then for the resulting configuration $X \oplus [i, j]$, we find

$$\#(X \oplus [i, j]) = \#(X) + \#(X \oplus \{S_1[i], S_1[j]\}).$$

We shall explicitly prove the first part of Theorem 9. The proof of the second part is very similar.

Proof. Let C be the largest element that lies between A and B at a distance s from A, and let H be the largest element that lies between F and G at a distance s from F. Then in the configuration $X[j] \oplus Y[i]$, the largest element between $A \cup F$ and $B \cup G$ will be $C \cup H \cup \{c_0\}$, where c_0 is the point which lies between the newly adjacent points i and j.

We must now consider which subsets D of $C \cup H \cup \{c_0\}$ satisfy betweenness. There are two cases to consider. First, consider the case in which $c_0 \notin D$. In this case, for every point $a \in A$, $b \in B$, $f \in F$, and $g \in G$, we need that $[a]_D \neq 0$, $[b]_D \neq 0$, $[f]_D \neq 0$, and $[g]_D \neq 0$. Thus, D must contain some $C' \subseteq C$ which lies between A and B and some $H' \subseteq H$ which lies between F and G. By Rule 3 from Section 5.4, the number of such D is $\#(X) \cdot \#(Y)$.

Next, consider the case in which $c_0 \in D$. Then we still require that $[a]_D \neq 0$, $[b]_D \neq 0$, $[f]_D \neq 0$, and $[g]_D \neq 0$ for all a, b, f, g, but we begin with $c_0 \in [i]_D$ and $c_0 \in [j]_D$. By assumption, c_0 is in D, so the

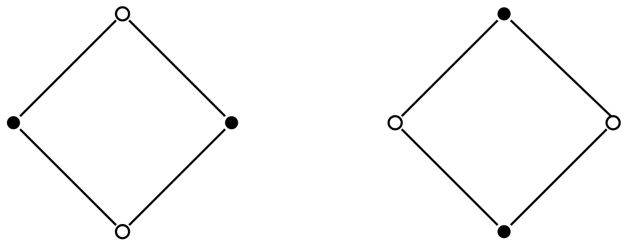


Figure 18: Two disconnected P_2 configurations, soon to be adjoined.

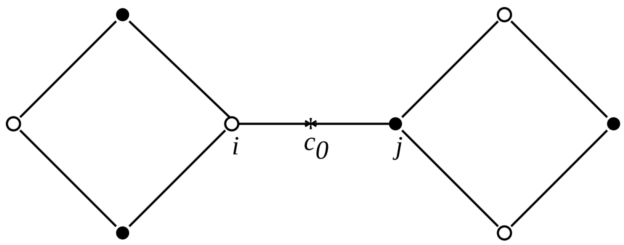


Figure 19: The newly adjoined configuration $P_2[j] \oplus P_2[i]$.

number of such D fitting these requirements will be equal to the number of D which satisfy $[a]_D \neq 0$, $[b]_D \neq 0$, $[f]_D \neq 0$, and $[g]_D \neq 0$ when we already have $c_0 \in [i]_D$ and $c_0 \in [j]_D$. This is equivalent to having a disconnected configuration $(X \oplus S_1[i]) \cup (Y \oplus S_1[j])$. The number of D which contain c_0 will be equal to $\#(X \oplus S_1[i]) \cdot \#(Y \oplus S_1[j])$. By combining the results of the two cases here, we arrive at the expected formula $\#(X[j] \oplus Y[i]) = \#(X) \cdot \#(Y) + \#(X \oplus S_1[i]) \cdot \#(Y \oplus S_1[j])$. □

As a specific example, let us consider the adjoined configuration $P_2[j] \oplus P_2[i]$ shown in Figure 19. By Theorem 9, which we shall sometimes refer to as the looping algorithm, $\#(P_2[j] \oplus P_2[i])$ is equal to the sum of the nums of the disconnected configurations in Figures 18 and 20. Thus,

$$\begin{aligned}\#(P_2[j] \oplus P_2[i]) &= \#(P_2) \cdot \#(P_2) + \#(P_2 \oplus S_1[i]) \cdot \#(P_2 \oplus S_1[j]) \\ &= 7 \cdot 7 + 8 \cdot 8 \\ &= 113.\end{aligned}$$

As we mentioned at the beginning of this section, it is possible to add an adjacency between two points of an already connected finite configuration X. When we add such an adjacency between the points i and j in X, we shall denote the new configuration $X \oplus [i, j]$. Since the derivation is quite similar to the case in which we add an adjacency between the parts of a disconnected configuration, we simply state the result:

$$\#(X \oplus [i, j]) = \#(X) + \#(X \oplus \{S_1[i], S_1[j]\}).$$

As an example of this, we can form the $2m$-gon configuration P_m by adding an adjacency between the end points of the string configuration S_{2m}. Figure 21 shows the case when $m = 3$, with the polygonal configuration P_3 on the left, the string configuration S_6 in the center, and the string configuration $S_6 \oplus \{S_1[i], S_1[j]\} = S_8$ on the right.

Then by the looping algorithm, $\#(P_3) = \#(S_6) + \#(S_8) = F_5 + F_7 = 5 + 13 = 18$, where F_m is the m-th Fibonacci number. With a bit of careful observation, the reader should become convinced that $\#(P_m) = F_{2m-1} + F_{2m+1}$. We know from [5] that $F_{2m-1} + F_{2m+1} = L_{2m}$, where L_{2m} is the $2m$-th Lucas number. Thus, we have finally justified the assertion that $\#(P_m) = L_{2m}$.

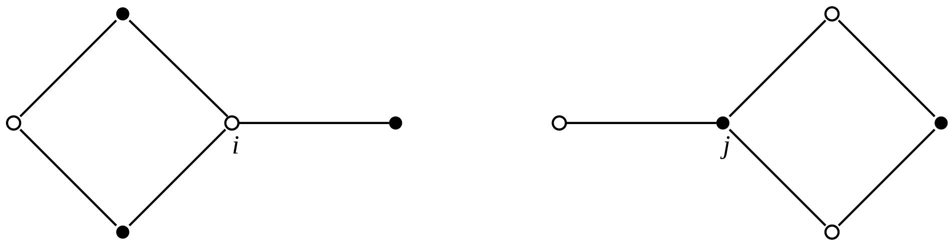


Figure 20: Two disconnected configurations $P_2 \oplus S_1[j]$ and $P_2 \oplus S_1[i]$.

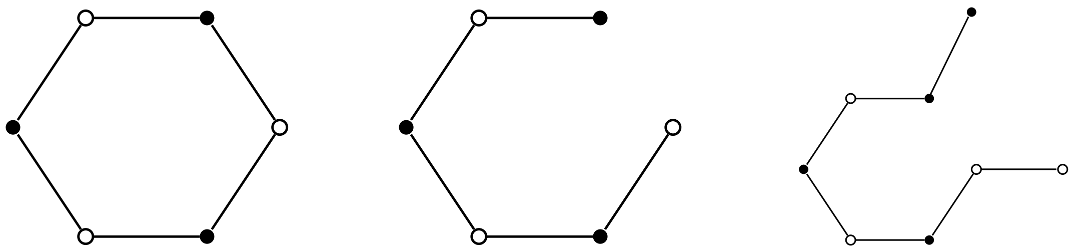


Figure 21: Subsets of $C = \{c_1, c_2, c_3, c_4\}$ satisfy betweenness when they do not "isolate a point" $a \in A$ or $b \in B$.

6.1 Polygon Chains

From Section 5.3, we know that the Fibonacci numbers appear as $\#(S_l)$ for string configurations and the Lucas numbers appear as $\#(P_m)$ for polygonal configurations. The appearance of these well known sequences in string and polygonal configurations motivated us to look for other well known sequences in configurations combining both string and polygonal configurations. For our purposes, we will consider those configurations which consist of a series of identical polygon configurations adjoined in between by string configurations of a particular length. We will refer to these configurations as polygon chains. We considered the most arbitrary case in which we vary 3 parameters: the size of the polygonal configuration, the total number of polygonal configurations, and the length of the string configuration between each polygonal configuration. We represent a polygon chain with the notation $P_m^k(S_l)$, where $2m$ is the number of points in each set of the polygonal configuration, k is the number of polygonal configurations, and l is the length of the string configuration between each polygonal configuration. For example, Figure 22 shows a $P_2^2(S_1)$. It is formed by linking two P_2 configurations with an S_1 string.

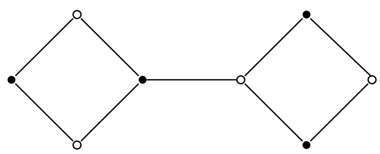


Figure 22: The polygon chain $P_2^2(S_1)$.

Now that we possess Theorems 8 and 9, we have the tools necessary to calculate $\#(P_m^k(S_l))$ using recursive formulas. The derivation of the formulas for finding $\#(P_2^k(S_1))$ is as follows:

Recall the example corresponding to Figure 19 that we used to illustrate the Looping Algorithm. We explained that the Looping Algorithm allows us to find $\#(X)$ for Figure 19 by adding the product of $\#(X)$ of each of the two configurations shown in Figures 18 and 20. Now that we have established notation, this example really took us through finding $\#(P_2^2(S_1))$.

Figure 23 is a visual representation of the Looping Algorithm being applied to the $P_2^2(S_1)$ configuration.

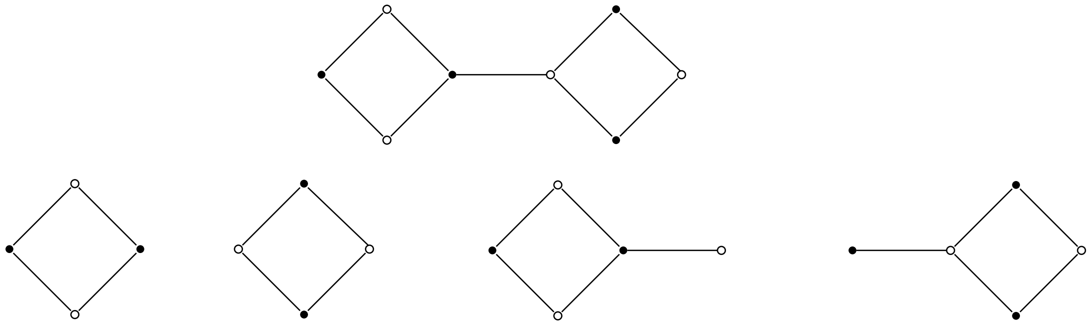


Figure 23: Applying the Looping Algorithm to the polygon chain $P_2^2(S_1)$.

Again, as we saw from the Looping Algorithm example in section 6, it turned out that $\#(P_2^2(S_1)) = 113$.

Now let us consider $P_2^3(S_1)$ shown at the top of Figure 24. The looping algorithm states that

$$\#(P_2^3(S_1)) = \#(P_2) \cdot \#(P_2^2(S_1)) + \#(P_2 \oplus S_1) \cdot \#(P_2^2(S_1) \oplus S_1). \quad (1)$$

(Notice that we have suppressed the [i] and [j] notation that we saw in Section 6 telling us where to adjoin a new point for $\#(P_2 \oplus S_1)$ and $\#(P_2^2(S_1) \oplus S_1)$. For the remainder of this section, we are adjoining new points only to the locations where the original adjacency has been removed. This implies that new points being adjoined will occur at either of the farthest endpoints of each new configuration and therefore we will continue to suppress this notation.) In Figure 24, we can pictorially see the Looping Algorithm applied to $P_2^3(S_1)$. Notice that we have removed the single adjacency following the first P_2 configuration to end up with a single P_2 and a $P_2^2(S_1)$ as shown in the bottom left of Figure 24. Again, we will "add a point" to each of these separate configurations where they were disconnected from their original adjacency and draw the new corresponding adjacencies. Therefore, the polygon chain $P_2^3(S_1)$ can be broken down into the addition of two simplified multiplicative cases as shown at the bottom of Figure 24.

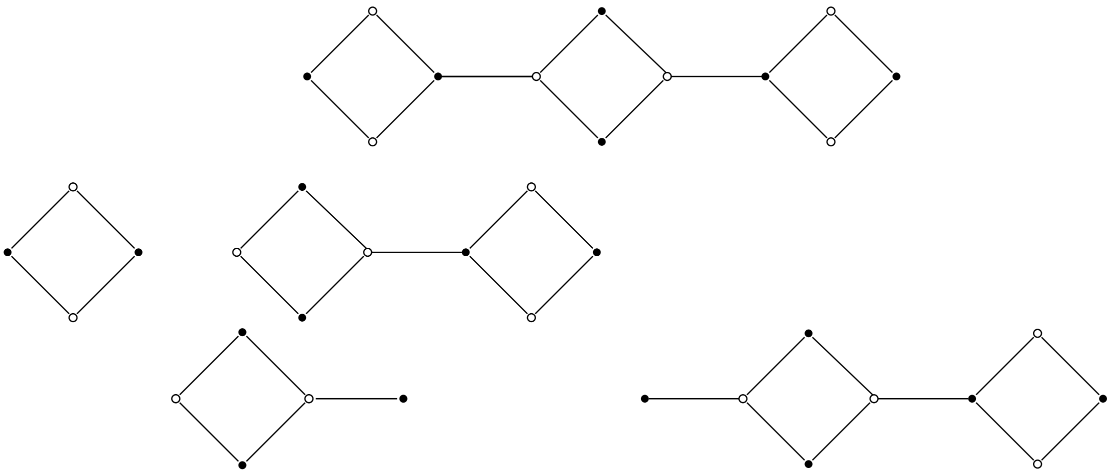


Figure 24: Applying the Looping Algorithm to the polygon chain $P_2^3(S_1)$.

Notice however that Equation (1) contains the term $\#(P_2^2(S_1) \oplus S_1)$, which we have not yet calculated. Using Theorem 8, we see that

$$\#(P_2^2(S_1) \oplus S_1) = \#(P_2^2(S_1)) + \#(P_2 \oplus S_2) \quad (2)$$

as shown at the bottom of Figure 25. (We note that $\#(X)$ of the configuration on the bottom right of Figure 25 is equivalent to $\#(P_2 \oplus S_2)$.)

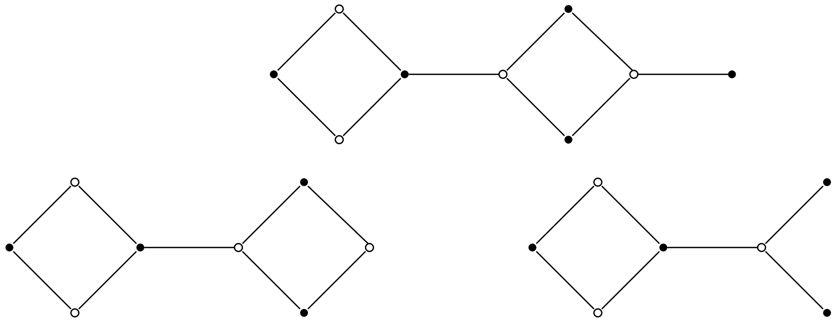


Figure 25: Applying the Looping Algorithm to the polygon chain $P_2^2(S_1) \oplus S_1$.

Again, by using Theorem 8, we know that

$$\#(P_2 \oplus S_2) = \#(P_2 \oplus S_1) + \#(P_2)$$

as shown in the final step of Figure 26.

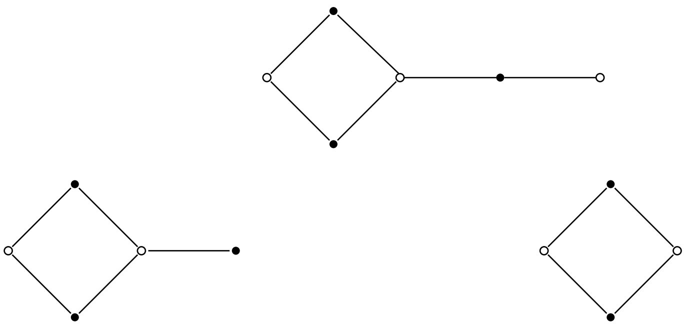


Figure 26: The polygon chain $P_2^2(S_1) \oplus S_1$ divided into two separate cases.

Therefore, substituting this information into (2) we see that

$$\begin{aligned}\#(P_2^2(S_1) \oplus S_1) =& P_2^2(S_1) + \#(P_2 \oplus S_1) + \#(P_2) \\ =& 113 + (8 + 7) \\ =& 128\end{aligned}$$

Now we can substitute that answer into (1) to obtain

$$\begin{aligned}\#(P_2^3(S_1)) =& \#(P_2) \cdot \#(P_2^2(S_1)) + \#(P_2 \oplus S_1) \cdot \#(P_2^2(S_1) \oplus S_1) \\ =& 7 \cdot 113 + 8 \cdot 128 \\ =& 1815\end{aligned}$$

Notice that a pattern for calculating $\#(X)$ is emerging for the case $P_2^k(S_1)$. Assuming we know $\#(P_2)$ and $\#(P_2 \oplus S_1)$, the recursive formulas for $\#(P_2^k(S_1))$ if $k \geq 2$ are as follows (Note: When $k = 1$, the linking term (S_l) can be disregarded as there are not 2 or more polygons to link):

$$\#(P_2^k(S_1)) = \quad \#(P_2) \cdot \#(P_2^{k-1}(S_1)) + \#(P_2 \oplus S_1) \cdot \#(P_2^{k-1}(S_1) \oplus S_1) \tag{3}$$

where

$$\#(P_2^k(S_1) \oplus S_1) = \quad \#(P_2^k(S_1)) + \#(P_2^{k-1}(S_1) \oplus S_1) + \#(P_2^{k-1}(S_1)) \tag{4}$$

So now we have formulas to calculate $\#(P_m^k(S_l))$ when $m = 2$ and $l = 1$.

We now want to consider polygon chains with $l \geq 2$. Intuitively, we can see that this will create configurations adjoined to various string lengths as shown in Figure 27. This brings us to the following theorem.

Theorem 10. *Let $P_m^k(S_l) \oplus S_l$ represent a finite configuration. Then*

$$\#(P_m^k(S_l) \oplus S_l) = F_{l-1} \cdot \#(P_m^k(S_l)) + F_l \cdot \#(P_m^k(S_l) \oplus S_1)$$

where $F_0 = 0$, $F_1 = 1$, $F_l = F_{l-2} + F_{l-1}$ are the Fibonacci numbers for $l \leq 2$. (Note: When $k = 1$, the linking term (S_l) can be disregarded as there are not 2 or more polygons to link.)

Proof. Consider the configuration represented by $P_m \oplus S_l$ where $l = 2$. Observe that

$$\begin{aligned}\#(P_m \oplus S_2) =& F_1 \cdot \#(P_m \oplus S_1) + F_2 \cdot \#(P_m)\\ =& 1 \cdot \#(P_m \oplus S_1) + 1 \cdot \#(P_m).\end{aligned}$$

We know this is true because we obtain the same answer by applying the algorithm for adjoining a point,

$$\#(P_m \oplus S_2) = 1 \cdot \#(P_m \oplus S_1) + 1 \cdot \#(P_m)$$

.

Now let us assume that for a positive integer t

$$\#(P_m^k(S_l) \oplus S_t) = F_{t-1} \cdot \#(P_m^k(S_l)) + F_t \cdot \#(P_m^k(S_l) \oplus S_1)$$

.

We now show that for $l = t + 1$,

$$\#(P_m^k(S_l) \oplus (S_{t+1})) = F_t \cdot \#(P_m^k(S_l)) + F_{t+1} \cdot \#(P_m^k(S_l) \oplus S_1)$$

.

Again by the applying the algorithm for adjoining a point we have

$$\#(P_m^k(S_l) \oplus S_{t+1}) = \#(P_m^k(S_l) \oplus S_t) + \#(P_m^k(S_l) \oplus S_{t-1}). \tag{5}$$

Solving for $\#(P_m^k(S_l) \oplus S_t)$ we obtain

$$\#(P_m^k(S_l) \oplus S_t) = \#(P_m^k(S_l) \oplus S_{t-1}) + \#(P_m^k(S_l) \oplus S_{t-2})$$

.

Substituting this back in to Equation (5) we can see that

$$\begin{aligned}\#(P_m^k(S_l) \oplus S_{t+1}) =& \#(P_m^k(S_l) \oplus S_{t-1}) + \#(P_m^k(S_l) \oplus S_{t-2}) + \#(P_m^k(S_l) \oplus S_{t-1})\\ =& 1 \cdot \#(P_m^k(S_l) \oplus S_{t-2}) + 2 \cdot \#(P_m^k(S_l) \oplus S_{t-1})\\ =& F_2 \cdot \#(P_m^k(S_l) \oplus S_{t-2}) + F_3 \cdot \#(P_m^k(S_l) \oplus S_{t-1}).\end{aligned} \tag{6}$$

Now let us apply the adjoining a point algorithm to $\#(P_m^k(S_l) \oplus S_{t-1})$. This yields

$$\#(P_m^k(S_l) \oplus S_{t-1}) = \#(P_m^k(S_l) \oplus S_{t-2}) + \#(P_m^k(S_l) \oplus S_{t-3})$$

.

Substituting this into Equation (6) we have

$$\begin{aligned}\#(P_m^k(S_l)\oplus S_{t+1}) =&1\cdot\#(P_m^k(S_1)\oplus S_{t-2})+2\cdot(\#(P_m^k(S_l)\oplus S_{t-2})+\#(P_m^k(S_l)\oplus S_{t-3}))\\ =&2\cdot\#(P_m^k(S_l)\oplus S_{t-3})+3\cdot\#(P_m^k(S_1)\oplus S_{t-2})\\ =&F_3\cdot\#(P_m^k(S_l)\oplus S_{t-3})+F_4\cdot\#(P_m^k(S_1)\oplus S_{t-2}). \qquad (7)\end{aligned}$$

Notice that in both Equations (6) and (7), we obtain the Fibonacci terms F_t and F_{t+1}. As we carry this through until we get the degenerate cases $\#(P_m^k(S_l))$ and $\#(P_m^k(S_l)\oplus S_1)$, we continue to obtain the Fibonacci terms F_t and F_{t+1}.

Therefore,

$$\#(P_m^k(S_l)\oplus S_{t+1})=F_t\cdot\#(P_m^k(S_l))+F_{t+1}\cdot\#(P_m^k(S_l)\oplus S_1)$$

thereby verifying S_{t+1}.

Hence, by induction,

$$\#(P_m^k(S_l)\oplus S_l)=F_{l-1}\cdot\#(P_m^k(S_l))+F_l\cdot\#(P_m^k(S_l)\oplus S_1)$$

□

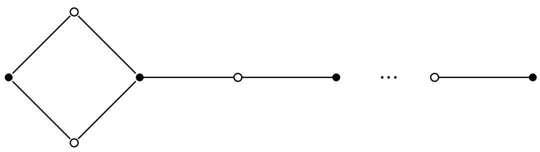


Figure 27: The configuration given by $P_2\oplus S_l$.

With Theorem 10, we can now calculate $\#(P_m^k(S_l))$ with $l\geq 2$.

Consider the polygon chain $P_2^2(S_2)$ shown in Figure 28.

By applying the looping algorithm, shown pictorially at the bottom of Figure 28, we arrive at

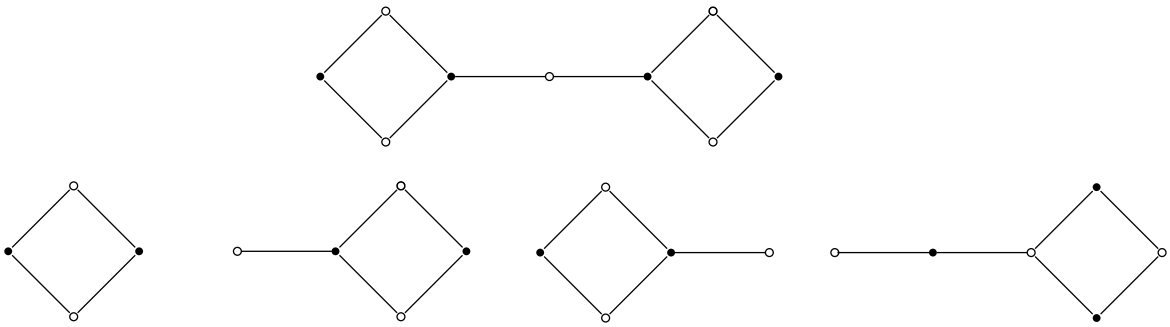


Figure 28: The polygon chain $P_2^2(S_2)$ divided into two separate cases.

$$\#(P_2^2(S_2))=\#(P_2)\cdot\#(P_2\oplus S_1)+\#(P_2\oplus S_1)\cdot\#(P_2\oplus S_2) \qquad (8)$$

where by Theorem 10 we know that

$$\#(P_2\oplus S_2)=F_1\cdot\#(P_2)+F_2\cdot\#(P_2\oplus S_1)$$

Substituting this into equation (8) arrive at

$$\#(P_2^2(S_2)) = \#(P_2)\cdot\#(P_2\oplus S_1) + \#(P_2\oplus S_1)\cdot(F_1\cdot\#(P_2) + F_2\cdot\#(P_2\oplus S_1))$$

Again, assuming we know $\#(P_2)$ and $\#(P_2\oplus S_1)$, we can find $\#(P_2^k(S_l))$ for values where $l\geq 2$ and $k\geq 2$ using the following recursive formulas. Again let $F_0=0, F_1=1, F_2=1, F_l=F_{l-2}+F_{l-1}$ be the Fibonacci numbers for $l\geq 2$, then

$$\begin{aligned}\#(P_2^k(S_l)) &= \#(P_2)\cdot(F_{l-2}\cdot\#(P_2^{k-1}(S_l)) + F_{l-1}\cdot\#(P_2^{k-1}(S_l)\oplus S_1)) \\ &+ \quad \#(P_2\oplus S_1)\cdot(F_l\cdot\#(P_2^{k-1}(S_l)\oplus S_1) + F_{l-1}\cdot P_2^{k-1}(S_l))\end{aligned} \tag{9}$$

where

$$\begin{aligned}\#(P_2^k(S_l)\oplus S_1) =&\#(P_2^k(S_l)) + \#(P_2^{k-1}(S_l)\oplus S_{l+1}) \\ =&\#(P_2^k(S_l)) + F_{l+1}\cdot\#(P_2^{k-1}(S_l)\oplus S_1) + F_l\cdot\#(P_2^{k-1}(S_l))\end{aligned} \tag{10}$$

Having investigated $m=2$, we consider letting m take on values greater than or equal to 3. Stemming from the previous recursive formulas, we derived two more formulas to find $\#(P_m^k(S_1))$ when $m\geq 3$ and $k\geq 2$. Again, to use these formulas, one must know $\#(P_m)$ and $\#(P_m\oplus S_1)$. Let $L_1=1, L_2=3, L_m=L_{m-2}+L_{m-1}$ be the Lucas numbers for $m\geq 2$. We find

$$\#(P_m^k(S_1)) = L_{2m}\cdot\#(P_m^{k-1}(S_1)) + (L_{2m}+F_{2m-2})\cdot\#(P_m^{k-1}(S_1)\oplus S_1) \tag{11}$$

where

$$\begin{aligned}\#(P_m^k(S_1)\oplus S_1) =&\#(P_m^k(S_1)\oplus S_1) + [(F_{m-1})^2 + 2(F_{m-1})(F_{m-2})]\cdot\#(P_m^{k-1}(S_1)) + \\ &[(F_{m-1})^2 + 2(F_{m-1})(F_{m-2}) + (F_{m-2})^2]\cdot\#(P_m^{k-1}(S_1)\oplus S_1)\end{aligned} \tag{12}$$

Finally, to calculate $\#(P_m^k(S_l))$ with $m\geq 3, l\geq 2$ and $k\geq 2$ we can use the following recursive formulas assuming we know $\#(P_m)$ and $\#(P_m\oplus S_1)$:

$$\begin{aligned}\#(P_m^k(S_l)) = \#(P_m)\cdot F_{l-2}\cdot(\#(P_m^{k-1}(S_l)) + F_{l-1}\cdot(\#(P_m^{k-1}(S_l)\oplus S_1)) + \\ \#(P_m\oplus S_1)\cdot F_l\cdot(\#(P_m^{k-1}(S_l)\oplus S_1) + F_{l-1}\cdot\#(P_m^{k-1}(S_l)))\end{aligned} \tag{13}$$

where

$$\begin{aligned}\#(P_m^k(S_l)\oplus S_1) =&\#(P_m^k(S_l)) + [F_{l+1}(F_{m-1})^2 + 2(F_{m-1})(F_{m-2}) + F_l(F_{m-1})^2]\cdot\#(P_m^{k-1}(S_l)\oplus S_1) + \\ &[F_l((F_{m-1})^2 + 2(F_{m-1})(F_{m-2})) + F_{l-1}(F_{m-2})^2]\cdot\#(P_m^{k-1}(S_l))\end{aligned} \tag{14}$$

Varying one index at time, either k, l or m using one of four pairs of recursive formulas, (18)(19), (20)(10), (11)(12) or (13)(14), we calculated $\#(P_m^k(S_l))$ for numerous configurations to try and find well known patterns. A few of the sequences we found are listed below:

When $m=2$, $l=1$ and $k=2,3,4,5$ $\#(P_m^k(S_l)) = 113, 1815, 29153, 468263$
When $m=3$, $l=1$ and $k=2,3,4$ $\#(P_m^k(S_l)) = 765, 32733, 1400634$
When $m=2$, $l=2$ and $k=2,3,4$ $\#(P_m^k(S_l)) = 176, 4393, 109649$

After checking these sequences and many others in [1] we found no matches and we are currently looking into submitting these new sequences into the database.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

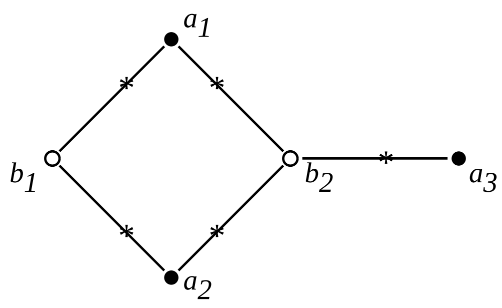


Figure 29: The $P_2 \oplus S_1$ configuration with biadjacency matrix.

7 Biadjacency Matrices

While the combination of Theorems 8 and 9 theoretically enables us to determine $\#(X)$ for any finite configuration, the recursive formulas which we derived were exceedingly complicated. Also, the highly pictoral nature of the computations would prevent computerized calculation of $\#(X)$ by current means. With these observations in mind, we decided to examine the structure of finite configurations by using adjacency matrices.

Definition 17. *Let $A, B \in \mathcal{H}(\mathbb{R}^n)$ be finite point sets, with $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, and let $X = A \cup B$ satisfy the PFAEL conditions. Now let*

$$x_i = \begin{cases} a_i & \text{if } 1 \leq i \leq m \\ b_i & \text{if } m+1 \leq i \leq m+n \end{cases} .$$

Then an adjacency matrix is an $(m+n) \times (m+n)$ binary matrix $M = [m_{i,j}]$ in which $m_{i,j} = 1$ if x_i is adjacent to x_j and $m_{i,j} = 0$ if x_i is not adjacent to x_j.

Since adjacency is a symmetric property, we can note immediately that $m_{i,j} = m_{j,i}$. That is, the adjacency matrix for a finite configuration is a symmetric matrix. Also, the main diagonal will always be all 0, since points cannot be adjacent to themselves. Furthermore, $m_{i,j} = 0$ if points i and j lie in the same set A or B.

With these observations, we note that we can actually contain all of the information in the adjacency matrix within a smaller $m \times n$ matrix.

Definition 18. *Let $X = A \cup B$ be a finite configuration where A and B satisfy the PFAEL conditions. Consider an $m \times n$ binary matrix $M = [m_{i,j}]$ in which the ith row will correspond to a point a_i in $A = \{a_1, \ldots, a_m\}$, and the jth column will correspond to a point b_j in $B = \{b_1, \ldots, b_n\}$. The entries $m_{i,j}$ will be either 0 or 1, as determined by the presence of an adjacency between a_i and b_j. We will call this the biadjacency matrix of the configuration X.*

Figure 29 gives an example of a finite configuration along with the corresponding biadjacency matrix.

Now let us consider some configuration $X = A \cup B$ in which A and B are finite point sets that satisfy the PFAEL conditions. We reiterate that any element which lies between A and B at a distance s from A and a distance $r - s$ from B is a subset of $C = (A)_s \cap (B)_{r-s}$, where $0 < s < r$ and $r = h(A, B)$. If $\mid C \mid = u$, then there are 2^u potentional elements at each location between A and B, corresponding to the inclusion or exclusion of each point in C.

We recall, however, that the only elements which truly lie between A and B are those $C' \subseteq C$ for which $h(A, C') = s$ and $h(B, C') = r - s$. This is equivalent to saying that for each $a \in A$, there exists some point $c_a \in C$ such that $d_E(a, c_a) = s$ and for each $b \in B$, there exists some point $c_b \in C$ such that $d_E(b, c_b) = r - s$. At the risk of sounding repetive, we remind the reader that this means that by removing points c from C to create C', we have not "isolated" a point in A or B. Thus, to count the number of valid elements at some location between A and B, and therefore calculate $\#(X)$, it is equivalent to count those subsets $C' \subseteq C$ which do not isolate a point in A or B.

Let us note that since the biadjacency matrix for the finite compact sets A and B contains all of the adjacency information for the points in those sets, the matrix also contains the information for the locations of the points $c \in C$. That is, if there is an adjacency between some point $a \in A$ and $b \in B$, then the element C will have a point lying between a and b. Then each $C' \subseteq C$ will correspond to a substate of the biadjacency matrix, where a substate is defined below.

Definition 19. *Let $M = [m_{i,j}]$ be the $m \times n$ biadjacency matrix corresponding to some finite configuration X. Then a substate $N = [n_{i,j}]$ of M is another binary $m \times n$ matrix with the property that if $m_{i,j} = 0$, then $n_{i,j} = 0$. However, if $m_{i,j} = 1$, then $n_{i,j}$ may be either 0 or 1.*

The creation of a substate is analogous to removing some points from C to create C'. For example, if the entry $m_{i,j}$ in the biadjacency matrix M is a 1, then we could create a substate of M in which $m_{i,j}$ is set to 0, corresponding to the $C' \subset C$ in which the point c between a_i and b_j has been removed. Thus, a point $a \in A$ will be "isolated" if and only if the corresponding row of the biadjacency matrix has had all of the entries set to 0. Similarly, a point $b \in B$ will be "isolated" if and only if the corresponding column of the biadjacency matrix has had all of the entries set to 0. In this light, we have recast the problem of calculating $\#(X)$ as a counting problem for the number of substates of the biadjacency matrix in which no row or column has all 0 entries.

7.1 The Inclusion-Exclusion Principle

We now present a counting technique from set theory which we will soon employ to help us calculate $\#(X)$.

Theorem 11. *Let $A_1, A_2, \ldots, A_n$ be finite sets. The the cardinality of the union of all of the A_i is given by*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{I \subseteq \{1,\ldots,n\}, I \neq \emptyset} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|.$$

See [4] for further explanation of the Inclusion-Exclusion Principle. We examine the case with three sets below to further clarify the Inclusion-Exclusion Principle.

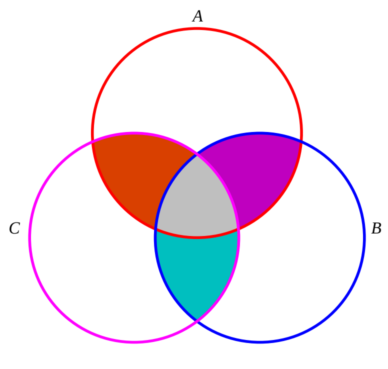


Figure 30: The Inclusion Exclusion Principle applied to three sets.

Let us consider the case in which we have 3 finite sets A, B, C, as shown in Figure 30. If we wish to find the cardinality of $\mid A \cup B \cup C \mid$, then we begin by adding the cardinalities of A, B, and C themselves. However, this would lead us to double count all of the intersections between two of the sets. We therefore proceed to subtract off the cardinalities of the intersections of each pair of sets. Finally, we must make one more correction, for in subtracting the two set intersections, we remove the intersection of all 3 sets one too many times. We therefore add back in $\mid A \cap B \cap C \mid$, so that the entire calculation is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

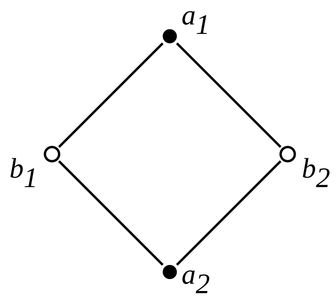


Figure 31: The P_2 configuration with biadjacency matrix.

$$\begin{aligned} \mid A \cup B \cup C \mid = & \mid A \mid + \mid B \mid + \mid C \mid \\ & - \mid A \cap B \mid - \mid A \cap C \mid - \mid B \cap C \mid \\ & + \mid A \cap B \cap C \mid . \end{aligned}$$

This agrees with the formula given in Theorem 11 when $n = 3$. The reader may imagine similar use of the Inclusion-Exclusion Principle for larger n.

7.2 Applying Inclusion-Exclusion

We have already explained how we can frame the calculation of $\#(X)$ as identifying the substates of the biadjacency matrix which do not have either rows or columns with all zero entries. We now bring the Inclusion-Exclusion Principle to bear on this counting problem.

Let $X = A \cup B$ be a finite configuration. Let N be the set of all potential elements $C' \subseteq C$, so that $|N| = 2^u$, where u is the number of 1's in the biadjacency matrix of X. Now let R_1 denote the set of all substates in which the first row has all 0 entries. Then $|R_1| = 2^{u-r_1}$, where r_1 is the number of 1's in the first row of the biadjacency matrix. Define similarly $R_2 \ldots R_{m+n}$, so that each R_i is the set of substates of the biadjacency matrix in which the ith row or column has all 0 entries. Then $\#(X) = |N| - |R_1 \cup R_2 \cup \ldots \cup R_{m+n}|$. By applying the Inclusion-Exclusion Principle, we can actually calculate $\#(X)$ using this algorithm. To clarify this new algorithm, let us consider its application to the example below.

Consider the configuration P_2, in which $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Both a picture of this configuration and its biadjacency matrix are given in Figure 31. We will calculate $\#(P_2)$ using the I.E. (Inclusion-Exclusion) Algorithm.

To use the I.E. Algorithm here, we note that we are looking for the union of four finite sets R_1, R_2, C_1, C_2. Here R_1 is the set of all substates of the biadjacency matrix in which the first row has all 0 entries, R_2 is the set of all substates of the biadjacency matrix in which the second row has all 0 entries, and similarly for the first and second columns. Then noting that the biadjacency matrix has four 1's in it, we will be looking for $\#(P_2) = 2^4 - |R_1 \cup R_2 \cup C_1 \cup C_1|$. In the calculations below, each row of matrices corresponds to the forcing to 0 of some number of rows or columns. The first row therefore gives the unique substate in which none of the rows or columns are set to zero. The next row of matrices outlines the four ways in which one row or column can be set to zero. For each matrix, we raise 2 to the number of 1's remaining in the substate matrix to consider how many ways this substate may arise.

					$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ $2^4 = 16$
		$\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$ $2^2 = 4$	$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$ $2^2 = 4$	$\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$ $2^2 = 4$	$\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ $2^2 = 4$
$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$	$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ $2^1 = 2$	$\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ $2^1 = 2$	$\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ $2^1 = 2$	$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ $2^1 = 2$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$
		$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$	$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$
					$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ $2^0 = 1$

We now add up all of the powers of 2 which we calculated in the table, assigning signs to the terms as dictated by the Inclusion-Exclusion Principle. Then

$$\begin{aligned}\#(P_2) &= 16 - (4+4+4+4) + (1+2+2+2+2+1) - (1+1+1+1) + 1\\ &= 16 - 16 + 10 - 4 + 1\\ &= 7\end{aligned}$$

When we discussed polygonal configurations, we stated that $\#(P_m) = L_{2m}$. Thus, we would expect that $\#(P_2) = L_4 = 7$, which does indeed coincide the result using the I.E. Algorithm.

The reader may notice in the preceding example that the number of computations which go into finding $\#(X)$ makes the process quite cumbersome. While this makes the I.E. Algorithm difficult to use by hand except in the most symmetric of cases, we have found that the process actually lends itself quite well to computer use.

7.3 Using Binary Numbers for Bookkeeping

When we ran through the example of calculating $\#(P_2)$ in the last section, we needed to consider all combinations of setting the various rows and columns of the biadjacency matrix to zero. While the method which we employed in that example (considering increasing numbers of rows and columns sent to zero) is one way to keep track of the calculations, it is hardly the only way.

The reader may notice, for example, that if I want to set the the second row of the biadjacency matrix for P_2 to zero, the number of 1's left in the resulting matrix can be obtained by calculating

$$\begin{bmatrix}1 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} \cdot \begin{bmatrix}1\\1\end{bmatrix} = 2.$$

Taking 2 to the power which we get as a result of the matrix product, we find that there are $2^2 = 4$ substate matrices satisfying the condition that the second row has all zero entries.

More generally, if we have an $m \times n$ biadjacency matrix M, and we wish to see the number of substate matrices which satisfy setting some combination of rows and columns to zero, then it is equivalent to multiply M by a $1 \times m$ row vector on the left and an $n \times 1$ column vector on the right, and then raise 2 to the power which we obtain from the matrix product. A zero in the i-th position of the row vector will set the i-th row of M to zero, and a zero in the j-th position of the column vector will set the j-th column of M to zero. All other entries in the row and column vectors should be ones.

The problem setting all combinations of rows and columns to zero therefore becomes a matter of multiply M by all row and column vectors of the correct sizes. The numbers which we arrive at after each matrix multiplication are analogous to the number of 1's remaining in the substate matrix after setting some rows and columns to all zero, so we take 2 to the product of the row, biadjacency, and column matrices, and sum up over all row and column matrices. The sign of each summand is determined by the number of 0's in both the row and the column matrices combined. Thus, in the case of P_2, we could write

$$\#(P_2) = \sum_{a=0}^{1}\sum_{b=0}^{1}\sum_{c=0}^{1}\sum_{d=0}^{1}(-1)^{4-a-b-c-d}\begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix}$$

Using methods based on this form for determining $\#(X)$, we were able to develop a Java program which can calculcate $\#(X)$ given only the biadjacency matrix for X. This program has in turn allowed us a quick and easy was to verify $\#(X)$ for specific configurations as we look for patterns in various classes of configurations, like the polygon chains. We will see another application of our program in the next section.

8 SPACK numbers

Previous research groups studying the Hausdorff metric at the Grand Valley State University REU in 2004 and 2005 theorized that we might find interesting patterns by classifying configurations based on the smallest dimension in which those configurations can exist.

Definition 20. *A positive integer k is a SPACK-n number if there exists a configuration of two sets A and B in $\mathbb{R}^n$ having exactly k elements at each location between A and B, and no such configuration exists in $\mathbb{R}^{n-1}$. If no such configuration exists in any dimension, then k is called a SPACK-0 number. If a SPACK-n number p is prime, then we call p a SPACK-n prime.*

The SPACK-1 numbers admit a fairly simple classification. Note that the only connected configurations which can exist in $\mathbb{R}$ are string configurations. Disconnected configurations will just consist of several string configurations. Therefore, the only SPACK-1 numbers are the Fibonacci numbers and products of Fibonacci numbers.

We do not yet have a complete classification of SPACK-2 numbers, but we know from considering polygonal configurations that those Lucas numbers L_{2m} which are not already SPACK-1 will be SPACK-2.

This summer we were able to prove that 57 is the smallest SPACK-3 number. The proof is not particularly enlightening, as it is an exhaustive proof by cases. We simply identified all configurations in $\mathbb{R}$ and $\mathbb{R}^2$ such that $\#(X) \leq 58$, and showed that there were none for which $\#(X) = 57$. We were able to identify a finite configuration which could exist in $\mathbb{R}^3$, however. Due to the difficulty of drawing 3-dimensional configurations in LaTeX, we will simply state that it has the biadjacency matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a proof that 57 is SPACK-3, refer to Appendix B.

The fact that SPACK-0 numbers exist at all came as something of a surprise; the definition of SPACK-0 was actually added on to the SPACK classification when previous REU's found that there existed no configuration X such that $\#(X) = 19$. Before the summer of 2006, the prime 19 was the only confirmed SPACK-0 number. In the process of proving that 57 is SPACK-3, we were able to show that

37 and 41 are not SPACK-1 or SPACK-2 numbers. At this point, we suspect that both 37 and 41 are SPACK-0 but we still need to formalize a proof for dimensions greater than 3.

We were very intrigued by the existence of SPACK-0 numbers this summer, and we decided to try to find a few more such numbers, if possible, in an attempt to find a pattern to the SPACK-0 numbers if such a pattern exists. While we have found no such pattern to date, we developed one particularly interesting helpful tool in locating SPACK-0 numbers. We reasoned that if we used our $\#(X)$ calculating program on all biadjacency matrices and created a table of all the numbers which were the output for some configuration, then any gaps in the table would be strong candidates for SPACK-0 numbers. Due to the similarity between this method of looking for SPACK-0 numbers and the Sieve of Eratosthones in looking for prime numbers, we have taken to calling the program the SPACK-0 Sieve. We say "strong candidates" because an actual proof by this method would be computationally impractical. Clearly showing even that 19 is SPACK-0 by means of this program would require considering all biadjacency matrices for configurations $X = A \cup B$ in which $|A| + |B| \leq 21$. At present, we have only run the SPACK-0 Sieve on matrices for which $|A| \cdot |B| \leq 25$. For a complete listing of the results of the Sieve so far, refer to Appendix A.

8.1 Does the SPACK Sieve Actually Find SPACK-0 Numbers?

Up until now, we have said that small numbers which do not appear in the output of the SPACK Sieve are "strong candidates" for SPACK-0 numbers. We would like to make this more rigorous by placing some bounds on which cases we must consider in the Sieve before we can definitely say that a particular number is SPACK-0. Throughout this paper, $X = A \cup B$ will be a finite configuration. Furthermore, we reserve the letter m for the cardinality of A and n for the cardinality of B. We also define the magnitude of X to be $|X| = m + n$.

Lemma 1. *If a prime p is not SPACK-0, then there exists a connected finite configuration X' such that $\#(X') = p$.*

Proof. Since p is not SPACK-0, there exists a finite configuration X such that $\#(X) = p$. If X is connected, then the proof is complete. If X is not connected, the X consists of some number of disconnected configurations $X_1, \ldots, X_k$ such that $X = \cup_{1 \leq i \leq k} X_i$ and $\#(X) = \prod_{1 \leq i \leq k} \#(X_i)$. Since p is prime, however, $\#(X_i) = 1$ for all but one of the i, and for the unique index j for which $\#(X_j) \neq 1$, we find $X_j = p$. Then $X' = X_j$ is the connected finite configuration which we sought. □

Thus, in looking for SPACK-0 primes, it suffices to look at connected finite configurations.

Lemma 2. *If $X = A \cup B$ is a connected finite configuration such that $\#(X) = q$, then there exists a finite configuration $X' = A' \cup B'$ for which $\#(X') = q$ and with the property that, for all $a \in A'$ and $b \in B'$, a and b are adjacent to no more than one endpoint. Furthermore, $|X'| \leq |X|$.*

Proof. Given our configuration $X = A \cup B$, if $a \in A$ is adjacent to more than one endpoint, simply remove one of them. Proceed similarly for all points $b \in B$. Having removed the excess endpoints, define A' and B' to be the sets of points remaining from A and B. Then clearly, $|X'| \leq |X|$. Also, by Rule 2 in Section 5.4, we know that $\#(X') = \#(X)$. □

In light of the two preceding Lemmas, we will give the following definition.

Definition 21. *Let $X = A \cup B$ be a connected finite configuration with the property that for all $a \in A$ and all $b \in B$, a and b are adjacent to no more than one endpoint. Then we shall call X a minimal configuration.*

We can immediately see that if a prime p is not SPACK-0, then there exists a minimal configuration X such that $\#(X) = p$. We will use this fact to help determine when a number is SPACK-0.

Lemma 3. *Let $X = A \cup B$ be a finite configuration. If we create a new configuration X' from X either by adding a new adjacency or by adjoining a new point y to some point $x \in X$, where x is not already adjacent to an endpoint, then $\#(X') > \#(X)$.*

Proof. Let $X = A \cup B$ be a finite configuration. If we add an adjacency between two points i and j in X to create a new configuration X', then by the looping algorithm,

$$\#(X') = \#(X) + \#(X \oplus \{S_1[i], S_1[j]\}).$$

Thus, $\#(X') \geq 2\#(X)$, so certainly $\#(X') > \#(X)$.

Next, say instead that we create the configuration X' by adjoining to X the point y to the point $x \in X$, where x is not already adjacent to any endpoints. Then the algorithm for adjoining a point to a configuration applies, yielding

$$\#(X') = \#(X) + \#(X - \{x\}).$$

We know that $\#(X - \{x\}) \geq 1$, so $\#(X') > \#(X)$, as desired. □

We know that the largest minimal configuration X such that $\#(X) = 1$ is S_3. We also know that when seeking to show that a prime p is SPACK-0, it suffices to examine minimal configurations. Note that $|S_3| = 3$. Thus, if we build upon S_3 by adjoining points and adding adjacencies so that the configuration X is minimal after each addition, Lemma 3 guarantees that $\#(X)$ increases by at least one with each new point adjoined. Thus, in order for $\#(X)$ to be equal to p, $|X| \leq p + 2$ for a configuration built in this manner.

This still leaves us with a problem, however. Given some minimal configuration X with $|X| > 3$, can we create X from S_3 only by adding adjacencies and adjoining new points to points which are not adjacent to endpoints? Or to put it another way, given some minimal configuration X with $|X| > 3$, can we create X from S_3 through a series of intermediate configurations X' such that X' is always minimal? If we can prove this last statement, then we will have shown that in looking for configurations such that $\#(X) = p$, we need only examine those configurations for which $|X| \leq p + 2$.

We will proceed by induction on the magnitude of X. As our base case, we take $|X| = 4$. A little thought should convince the reader that the only minimal configurations with 4 points are S_4 and P_2. Clearly S_4 can be created from S_3 by adjoining a single point to one of the endpoints of S_3. This configuration is minimal at every step of the process. Furthermore, we can create P_2 from S_3 by first creating S_4 and then adding an adjacency to complete the loop. Thus, the intermediate configurations when forming P_2 from S_3 are all minimal. This proves the base case.

Now suppose that for all $4 \leq k' < k$, it remains true that every minimal configuration of magnitude k' can be formed from S_3 by adding individual points and adjacencies such that all of the intermediate configurations are minimal. We will show that every minimal configuration of magnitude k can be so formed as well.

Consider a minimal configuration X of magnitude k. If X contains an S_1 string adjoined at some point, where x is the point at the end of this string, then let X' be the configuration $X' = X - \{x\}$. By assumption, X' can be formed from S_3 by adding individual adjacencies and points such that all of the intermediate configurations are minimal. Then by forming X from X' by adjoining x, we observe that X can also be formed from S_3 in the desired way.

If X does not contain an S_1 string, but it does contain some endpoint y, then proceed as follows. Let X' be the configuration $X' = X - \{y\}$. Removing the point y still leaves X' a finite connected configuration, and there cannot be more than one endpoint adjacenct to any given point by the assumption that there were no strings of length exactly 1. Hence, X' is a minimal configuration such that $|X'| = k - 1$. Hence, by forming X from X' through the adjoining of y, we observe that X can be formed from S_3 with the adjoining of individual points and adding of adjacencies such that each intermediate configuration is minimal.

Finally, if X did not have any endpoints at all, then X must consist entirely of polygonal configurations. Then select an adjacency which, if removed, will not cause X to become disconnected. Create X' by removing this adjacency. If X' does not have any endpoints, then locate another adjacency which, if removed, will not cause the configuration to become disconnected, and remove this adjacency as well. Eventually, this process will create a configuration X' with some strings. Note that this process cannot introduce more than one endpoint adjacent to any point, as removing one adjacency from a polygon can only create two strings, which cannot both be S_1's. Then X' fits into one of the two cases previously considered, and so X can be created from S_3 such that every intermediate configuration is minimal.

This exhausts the possibilities, so for any $k \geq 4$, a minimal configuration X of magnitude k can be created from S_3 through a series of adding individual points and adjacencies, such that every intermediate configuration is also minimal. As we already explained, this guarantees that the num of the intermediate configurations increases by at least 1 with each new point adjoined. Hence, $\#(X) \geq k-2$.

Corollary 1. *Let p be prime. If p does not appear in the output of the SPACK Sieve after having considered all configurations for which $|A|+|B| \leq p+2$, then p is SPACK-0.*

9 Bipartite Graphs

Each finite configuration can also be viewed as a linked labeled bipartite graph. So our algorithm can be used to compute the number of linked labeled subgraphs of a linked bipartite graph with the same vertex set. In some cases, when the biadjacency matrix is symmetric for example, we can use the Inclusion-Exclusion Principle to find a closed form sum for $\#(X)$ and therefore for certain types of bipartite graphs. The most obvious case of this is when our biadjacency matrix contains all 1's: this is when we have all possible adjacencies in our configuration or when the corresponding graph is a complete linked bipartite graph. Let's examine this case when our biadjacency matrix $M = [m_{i,j}]$ is an $m \times n$ matrix of all 1's, corresponding to a configuration $X = A \cup B$ where $|A| = m$ and $|B| = n$ or a complete linked bipartite graph G with vertex sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$.

In this case, the maximal set C between A and B is the set containing mn points (one point between each pair of adjacent points in X) or 2^{mn} possible linked bipartite subgraphs of G with the same vertex set. The elements between A and B (or the linked bipartite subgraphs G' of G with the same vertex set) will correspond to matrices $M' = [m'_{i,j}]$ so that $m'_{i,j} = 0$ if $m_{i,j} = 0$ and $m'_{i,j}$ can be either 1 or 0 when $m_{i,j} = 1$, and which also satisfy the property that every row and column is non-zero. This ensures that no point in X (and no vertex of G') is isolated. To count the number of matrices M', we sum the number of matrices with zero rows or columns (including no zero rows or columns), adding or subtracting these numbers according to the Inclusion-Exclusion Principle.

To do this, we count the number of ways we can have zero rows. We illustrate the general case first by considering the number of ways we can have a single zero row. There are $\binom{m}{1}$ ways to obtain 1 row of zeros. We can combine this zero row with any number of zero columns. There are $\binom{n}{j}$ ways to obtain j columns of zeros. Each column of zeros removes m 1s from our matrix M, so j columns of zeros remove jm 1s. The single zero row removes $n-j$ additional 1s from M. This leaves $mn - (mj + (n-j)) = (mn-n) + j(1-m)$ 1s in our matrix. This gives us

$$\binom{m}{1} \sum_{j=0}^{n} \binom{n}{j} 2^{(mn-n)+j(1-m)}$$

matrices M' with exactly 1 zero row. If we make 1 zero row and j zero columns, we have made a total of $j+1$ zero rows and columns. So the Inclusion-Exclusion Principle tells us we need to attach the sign $(-1)^{j+1}$ to the j^{th} summand. Therefore, we will add the terms

$$\binom{m}{1} \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} 2^{(mn-n)+j(1-m)} = \binom{m}{1} 2^{mn-n} \sum_{j=0}^{n} (-1)^{j-n}(-1)^{n+1} \binom{n}{j} 2^{j(1-m)}$$

or (since $(-1)^{j-n} = (-1)^{n-j}$)

$$(-1)^{n+1} \binom{m}{1} 2^{mn-n} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (2^{(1-m)})^{j} \tag{15}$$

to the total.

The Binomial Theorem tells us

$$(x-1)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} x^j, \tag{16}$$

so

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} (2^{(1-m)})^j = \left(2^{1-m}-1\right)^n$$

and (15) becomes

$$(-1)^{n+1} \binom{m}{1} 2^{mn-n} \left(2^{1-m}-1\right)^n.$$

Now we argue the general case. There are $\binom{m}{i}$ ways to obtain i rows of zeros. We can combine these zero rows with any number of zero columns. There are $\binom{n}{j}$ ways to obtain j columns of zeros. Each column of zeros removes m 1s from our matrix M, so j columns of zeros remove jm 1s. Each zero row removes $n-j$ additional 1s from M, so i zero rows remove $i(n-j)$ 1s from M. This leaves $mn-(mj+i(n-j)) = (mn-in)+j(i-m)$ 1s in our matrix. This gives us

$$\binom{m}{i} \sum_{j=0}^{n} \binom{n}{j} 2^{(mn-in)+j(i-m)}$$

matrices M' with exactly i zero rows. If we make i zero rows and j zero columns, we have made a total of $j+i$ zero rows and columns. So the Inclusion-Exclusion Principle tells us we need to attach the sign $(-1)^{j+i}$ to the j^{th} summand. Therefore, we will add the terms

$$\binom{m}{i} \sum_{j=0}^{n} (-1)^{j+i} \binom{n}{j} 2^{(mn-in)+j(i-m)} = \binom{m}{i} 2^{mn-in} \sum_{j=0}^{n} (-1)^{n-j} (-1)^{n+i} \binom{n}{j} (2^{(i-m)})^j.$$

to the total. Using (16) again gives us

$$\binom{m}{i} 2^{mn-in} (-1)^{n+i} \left(2^{i-m}-1\right)^n.$$

Now we sum as i goes from 0 to m (note that this takes into account all ways to set columns to 0 as well) to obtain our formula for the number of linked bipartite subgraphs of G with the same vertex set as G:

$$\begin{aligned}
\sum_{i=0}^{m} \binom{m}{i} (-1)^{n+i} 2^{mn-in} \left(2^{i-m}-1\right)^n &= \sum_{i=0}^{m} \binom{m}{i} (-1)^{n+i} \left(2^{m-i}\right)^n \left(2^{i-m}-1\right)^n \\
&= \sum_{i=0}^{m} \binom{m}{i} (-1)^{n+i} \left(1-2^{m-i}\right)^n \\
&= \sum_{i=0}^{m} \binom{m}{i} (-1)^{i} \left(2^{m-i}-1\right)^n. \qquad (17)
\end{aligned}$$

Formula 17 was previously submitted to the Online Encyclopedia of Integer Sequences by Vladeta Jovovic as the generating function for several integer sequences. The sequence A058481 begins with $1, 7, 25, 79, 241, \ldots$ and gives the number of $2 \times n$ matrices with no zero rows or columns. The sequence A058482 begins with $1, 25, 265, 2161, \ldots$ and gives the number of $3 \times n$ matrices with no zero rows or columns. Finally, the sequence A048291 begins with $1, 25, 265, 41503, \ldots$ and gives the number of $n \times n$ matrices with no zero rows or columns.

Our derivation of equation (17) therefore gives a method to confirm these three integer sequences, and also provides another interpretation of their meaning: they represent the number of linked subgraphs of complete labeled bipartite graphs. Furthermore, although we do not provide more closed forms for sequences like those mentioned from [1], we should point out that equation (17) allows us to find the number of linked subgraphs of *any* labeled complete bipartite graph, not just the $2 \times n$, $3 \times n$, and $n \times n$ cases which we explicitly reference.

10 Exploring Angles in $\mathcal{H}(\mathbb{R}^n)$

In a general metric space, the only piece of information which we know we have at our disposal is a notion of distance. By analyzing the distance function in $\mathcal{H}(\mathbb{R}^n)$, researchers at the GVSU REU have been able to define and study line segments, lines, and circles. Defining many other familiar "geometric" figures from Euclidean geometry, however, also requires some notion of an angle. We know that Euclidean geometry is embedded in the Hausdorff metric geometry, so it should not seem too unexpected to hope for reasonable "angles" in $\mathcal{H}(\mathbb{R}^n)$. Below we present the most logical generalization that we could find for an angle in $\mathcal{H}(\mathbb{R}^n)$.

Intuitively, we think of an angle in Euclidean space as some measure of the intersection of two lines or line segments. If we define the special cases of tangent and orthogonal line segments, then we can proceed still further to let an angle be a measure of how closely the intersection of two line segments approaches tangency or orthogonality. In keeping with this intuitive approach to angles, we shall first present plausible generalizations for tangency and orthogonality of metric segments in $\mathcal{H}(\mathbb{R}^n)$.

Let us start with tangency, and build up from Euclidean geometry. Let $a, b, c \in \mathbb{R}^n$ be three points in Euclidean space. Then we can define the line segments $\overline{ab}$ and $\overline{bc}$. We shall say that $\overline{ab}$ and $\overline{bc}$ are tangent in if a, b, and c are collinear, as shown in Figure 32. Within the confines of Euclidean geometry, we can go still further to say that $\angle abc \equiv \pi$ if b lies between a and c, and $\angle abc \equiv 0$ if a lies between b and c or c lies between a and b. We can immediately generalize the first part of this definition to $\mathcal{H}(\mathbb{R}^n)$.

Figure 32: Tangent line segments $\overline{ab}$ and $\overline{bc}$ in $\mathbb{R}^n$.

Definition 22. *Let $A, B, C \in \mathcal{H}(\mathbb{R}^n)$ and let $\overline{AB}$ and $\overline{BC}$ be the metric segments defined by A, B and B, C respectively. We say that $\overline{AB}$ and $\overline{BC}$ are tangent to each other if A, B, and C are collinear.*

As we suggested above, the next most important special case for angles is that of a right angle. In Euclidean geometry, we say that $\angle abc = \pi/2$ if $\overline{ab}$ and $\overline{bc}$ are orthogonal. We shall therefore generalize orthogonality to $\mathcal{H}(\mathbb{R}^n)$.

First, we will try to frame orthogonality in terms of purely metric concepts in $\mathbb{R}^n$. Let $a, b, c \in \mathbb{R}^n$ and let $\overline{ab}$ be the line segment between a and b and let $\overline{bc}$ be the line segment between b and c. We can extend the line segment $\overline{bc}$ to create the line $\overleftrightarrow{bc}$. Now locate the point d on $\overleftrightarrow{bc}$ which lies closest to a, as shown in Figure . Then $\overline{ab}$ and $\overline{bc}$ are orthogonal if an only if $b = d$. See Figure 33 for a visual. Note that it was necessary to extend $\overline{bc}$, because, in the case of an obtuse angle, d does not actually lie on $\overline{bc}$, but rather to the left of b. Now we shall generalize orthogonality to $\mathcal{H}(\mathbb{R}^n)$, as promised.

Definition 23. *Let A, B, and C be elements in $\mathcal{H}(\mathbb{R}^n)$ such that each pair of elements satisfies the PFAEL conditions. Furthermore, let $\overline{AB}$ be the metric segment with endelements A and B, let $\overline{BC}$ be the metric segment with endelements B and C, and if $h(A, B) = s$, then let $\overline{BC}^+$ be the extension of $\overline{BC}$ which also includes all elements E collinear with B and C such that $h(B, E) \leq 2s$. We shall say that $\overline{AB}$ and $\overline{BC}$ are orthogonal if*

$$\min_{E \in \overline{BC}^+} \{h(A, E)\} = h(A, B).$$

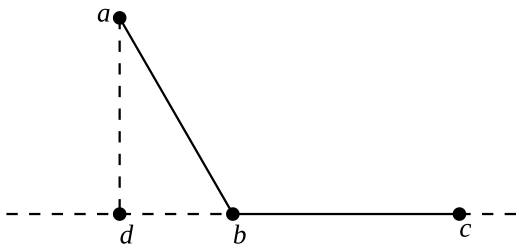


Figure 33: Line segments $\overline{ab}$ and $\overline{bc}$ in $\mathbb{R}^n$ are orthogonal if $b = d$.

This definition immediately begs clarification. Let us suppose for a moment that the minimum of the distance from A to $\overline{BC}^+$ is even defined. First we shall show that $\overline{BC}^+$ is indeed the correct extension of $\overline{BC}$ to consider over which to find a minimum distance.

Just as in the Euclidean example, it is entirely possible that the element (or elements) for which the minimum distance from A to the line $\overleftrightarrow{BC}$ is attained might not lie on the line segment $\overline{BC}$. We must therefore consider *some* extension of $\overline{BC}$. However, as explained in [8], we cannot always extend line segments in $\mathcal{H}(\mathbb{R}^n)$. The authors give necessary and sufficient conditions for the extension of a line segment, and we can check that requiring that A, B, and C to pairwise satisfy the PFAEL conditions is indeed sufficient to permit the extension of $\overline{BC}$.

Now we must ask how far we should extend $\overline{BC}$. Recall that we will be searching for an element on the line $\overleftrightarrow{BC}$ for which the distance to the element A is a minimum. Note also that we know $B \in \overleftrightarrow{BC}$, so the minimum distance from A to the line $\overleftrightarrow{BC}$ is no more than $h(A, B)$. Let $h(A, B) = s$ and consider an element E' on the line $\overleftrightarrow{BC}$ for which $h(B, E') > 2s$. By the triangle inequality in the Hausdorff metric, $h(A, E') > s$. Thus, when searching for an element $E \in \overleftrightarrow{BC}$ for which $h(A, E)$ is a minimum, it suffices to consider just those elements on $E \in \overleftrightarrow{BC}$ for which $h(B, E) \leq 2s$. We define $\overline{BC}^+$ to be precisely this extension of $\overline{BC}$.

Having now justified our choice for the extension of $\overline{BC}$, we turn our attention to the minimizing of the distance from A to $\overline{BC}^+$. We shall now prove that a minimum exists. Recall that by Theorem 2, if a continuous function is defined on a compact set, then the function attains its maximal and minimal values on that compact set. Thus, if we can demonstrate that $\overline{BC}^+$ is compact, then the minimum of the distance from A to $\overline{BC}^+$ will be well-defined.

Proposition 2. *The metric segment extension $\overline{BC}^+$ is compact in $\mathcal{H}(\mathbb{R}^n)$.*

The proof of Proposition 2 will take some time and a number of intermediate steps. We begin by showing that there exists a compact subset M of $\mathcal{H}(\mathbb{R}^n)$ which contains both A and $\overline{BC}^+$.

We shall simply assert that we can create a large enough closed ball $M \subset \mathbb{R}^n$ around B so that all of A and $\overline{BC}^+$ lies within this box. M is clearly bounded, and by construction, M is closed, so we know that M is compact in $\mathbb{R}^n$. We shall now demonstrate that the collection of compact subsets of M is compact in $\mathcal{H}(\mathbb{R}^n)$. That is, we shall show that $\mathcal{H}(M)$ is compact. In particular, we will need the following lemma, proved in [2].

Lemma 4. *A metric space (X, d) is compact if and only if it is complete and totally bounded.*

To clear up the lemma above, we give the following definitions, taken from [9].

Definition 24. *A sequence (x_n) in a metric space (X, d) is called a Cauchy sequence if for every $\epsilon > 0$ there exists n_0 such that $d(x_m, x_n) < \epsilon$ for every $n \geq n_0$ and $m \geq n_0$. We call X complete if every Cauchy sequence in X is convergent.*

Definition 25. *A subset E of a metric space (X, d) is said to be totally bounded if for every $\epsilon > 0$, there exists a finite subset $\{x_1, \ldots, x_n\}$ of X such that $E = \cup_{k=1}^{n} B(x_k, \epsilon)$.*

The author of [9] quickly goes on to show that any bounded set in $\mathbb{R}^n$ is totally bounded, giving us the total boundedness of M. Furthermore, in the same book the author shows that $\mathbb{R}^n$ is a complete metric space and that any closed subset of $\mathbb{R}^n$ is also complete. Thus, M is both complete and totally bounded in $\mathbb{R}^n$.

We take our next tool from [10], where the author proves that if (X, d) is a complete and totally bounded metric space, then $(\mathcal{H}(X), h)$ is complete and totally bounded as well. Applying this to our specific case tells us that $\mathcal{H}(M)$, the collection of compact subsets of M, is indeed complete and totally bounded. Hence, by Lemma 4, $(\mathcal{H}(M), h)$ is compact.

We now know that $(\mathcal{H}(M), h)$ is compact and that $\overline{BC}^+$ is a subset of $(\mathcal{H}(M), h)$. We shall use the following lemma, proved in [2], to show that $\overline{BC}^+$ is compact as well.

Lemma 5. *Every closed subspace of a compact space is compact.*

Thus, regarding $\overline{BC}^+$ as a subspace of $(\mathcal{H}(M), h)$,we have now reduced the problem of showing that $\overline{BC}^+$ is compact to showing that $\overline{BC}^+$ is merely closed.

We will prove that $\overline{BC}^+$ is closed by showing that it contains all of its limit points. Let T be a limit point of $\overline{BC}^+$. Then apply the following lemma, proved in [2].

Lemma 6. *Let X be a topological space and let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$; the converse holds if X is metrizable.*

We have not introduced the concept of a topological space, but suffice it to say that metric spaces are a specific class of topological spaces. Therefore, since T is a limit point of $\overline{BC}^+ \subset M$, there exists a sequence of points in $\overline{BC}^+$ which converges to T. By this we mean that there exists a collection of elements $T_n \in \overline{BC}^+$ with the property that, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $n > N$, we have $h(T, T_n) < \epsilon$.

We shall need to consider several cases at this point, and it is advantageous to introduce the following notation. Since $\overline{BC}^+$ is an extension of $\overline{BC}$, it can potentially have points in up to five regions.

- $\overline{BC}$: the collection of elements in $\overline{BC}^+$ between B and C
- $\overline{BC}^l$: the collection of elements in $\overline{BC}^+$ to the left of B
- $\overline{BC}^r$: the collection of elements in $\overline{BC}^+$ to the right of C
- B: the element B itself
- C: the element C itself.

Since $\mathcal{H}(M)$ is a metric space, [2] tells us that T_n converges to only one element. First, we know that if $T_n = B$ (or C) for all n greater some fixed $N \in \mathbb{N}$, then T_n converges to B (or C) as n tends to infinity.

Next, suppose that there exists some $N \in \mathbb{N}$ such that, for all $n < N$, $T_n \in \overline{BC}$. Then for all $n > N$, define the sequences U_n and V_n by

$$U_n = h(B, T_n) \qquad V_n = h(C, T_n).$$

Since T_n lie between B and C for all $n > N$, we know that $U_n + V_n = h(B, C)$, a constant. Note that h is a continuous function, so we know that

$$\lim_{n\to\infty} h(B, T_n) = h(B, \lim_{n\to\infty} T_n) = h(B, T).$$

Similarly, $\lim_{n\to\infty} h(C, T_n) = h(C, \lim_{n\to\infty} T_n) = h(C, T)$. Then since U_n and Vn are convergent, [9] demonstrates that

$$\lim_{n\to\infty} U_n + \lim_{n\to\infty} V_n = \lim_{n\to\infty} (U_n + V_n) = \lim_{n\to\infty} h(B, C) = h(B, C).$$

Thus,

$$h(B,C) = h(B,T) + h(C,T).$$

This is precisely the definition of betweenness, so T lies between B and C. Thus the limit point T is an element of $\overline{BC}$, and so is an element of $\overline{BC}^+$. In quite a similar manner, we could suppose that for all $n > N \in \mathbb{N}$, $T_n \in \overline{BC}^l$ (or $T_n \in \overline{BC}^r$). Then through nearly identical logic, we would find that T lies in $\overline{BC}^+$.

Next we must consider what happens if there exists no $N \in \mathbb{N}$ such that for all $n > N$, T_n lie entirely in one of $\overline{BC}$ or $\overline{BC}^l$ or $\overline{BC}^r$. Certainly, there must exist an $M \in \mathbb{N}$ such that for all $n > M$, $T_n \notin \overline{BC}^l$ or $T_n \notin \overline{BC}^r$. To see this point, let A_n be the subset of T_n for which $T_n \in \overline{BC}^l$ and let B_n be the subset of T_n for which $T_n \in \overline{BC}^r$. Then by the triangle inequality in $\mathcal{H}(\mathbb{R}^n)$, either $h(T, A_n) \geq h(B,C)/2$ or $h(T, B_n) \geq h(B,C)/2$. Either of these would directly contradict the definition of convergence.

Then we are left with two remaining cases. In the first case, $T_n \in \overline{BC}^l \cup \{B\} \cup \overline{BC}$ for all $n > N$, but there exists no $M \in \mathbb{N}$ such that $T_n \in \overline{BC}^l$ or $T_n \in \overline{BC}$ for all $n > M$. In the second case, $T_n \in \overline{BC} \cup \{C\} \cup \overline{BC}^r$ for all $n > N$, but there exists no $M \in \mathbb{N}$ such that $T_n \in \overline{BC}$ or $T_n \in \overline{BC}^r$ for all $n > M$. These two cases can be handled with nearly identical logic, so we shall focus on the former. We shall show then that $T_n \to B$.

We shall proceed by contradiction. Let us suppose that the sequence T_n does not converge to B. Then there exists $\epsilon > 0$ such that there exists no $L > 0$ such that, for all $n > L$, $h(B, T_n) < \epsilon$. Intuitively, this means that for some $\epsilon > 0$ and any $L > 0$, there exists an $n_0 > L$ such that $h(B, T_{n_0}) \geq \epsilon$. Let us find and fix an $\epsilon_0 > 0$ for which this is true. Since the sequence T_n is comprised of points in $\overline{BC}^l$, $\{B\}$, and $\overline{BC}$, this tells us that there is an infinite subsequence W_m of T_n which lies at lies at least ϵ_0 to the left of B or to the right (between B and C). Let us assume, without loss of generality, that W_m lies to the left of B (lies in $\overline{BC}^l$). Note, however, that since the sequence T_n has infinitely many elements in $\overline{BC}$, then T_n must also have an infinite subsequence which lies entirely in $\overline{BC}$. Since T_n converges, these two infinite subsequences must converge as well, and to the same element as T_n, yet they converge to elements of $\mathcal{H}(\mathbb{R}^n)$ which are separated by a distance of at least ϵ_0. This is a contradiction of the assumption that T_n does not converge to B. Then in this final case, note that the limit point once again lies in $\overline{BC}^+$.

This exhausts all possible cases, and all cases, limit points of $\overline{BC}^+$ are elements of $\overline{BC}^+$. Thus, $\overline{BC}^+$ is closed, and by virtue Lemma 5, $\overline{BC}^+$ is compact as well, thereby proving Proposition 2. Hence, the minimum distance from an element A to $\overline{BC}^+$ does exist by the Extreme Value Theorem. In keeping with our earlier definition, we shall say that $\overline{AB}$ and $\overline{BC}$ are orthogonal if

$$\min_{E \in \overline{BC}^+} \{h(A,E)\} = h(A,B).$$

This summer, extending the notions of tangency and orthogonality in $\mathcal{H}(\mathbb{R}^n)$ was as far as we went while exploring angles in $\mathcal{H}(\mathbb{R}^n)$. If time permits, we would very much recommend further research into this area by future REU's, as angles have proved to have some surprising interpretations in $\mathcal{H}(\mathbb{R}^n)$.

11 Recent Findings

11.1 Additional Patterns in $\#(X)$

Many patterns exist in $\#(X)$ for classes of configurations beyond Fibonacci numbers and the Lucas numbers. Here we will show that we can always find configurations such that $\#(X)$ has the form $3^n - 2$, $3^n - 1$, and 3^n for all $n \geq 1$. Furthermore, such configurations can exist in $\mathbb{R}^3$ or lower dimension.

Let $X^{2,n}$ denote the finite configuration which has as its biadjacency matrix a $2 \times n$ matrix with entries of all 1's. Then $A = \{a_1, a_2\}$ and $B = \{b_1, \ldots, b_n\}$, and a_i is adjacent to b_j for $i = 1, 2$ and $1 \leq j \leq n$.

Let us examine first the configuration $X^{2,1}$, which is actually the string configuration S_3 which we have seen earlier. Then we know immediately that $\#(X^{2,1}) = 1$. Also, we can note that $\#(X^{2,1} \oplus S_1[a_1]) = 2$ and $\#(X^{2,1} \oplus \{S_1[a_1], S_1[a_2]\}) = 3$.

We will now show that $\#(X^{2,n}) = 3^n - 2$, as well as $\#(X^{2,n} \oplus S_1[a_1]) = 3^n - 1$ and $\#(X^{2,n} \oplus \{S_1[a_1], S_1[a_2]\}) = 3^n$. The proof will be by induction on n, and we take our observations for $n = 1$ to be the base case.

Now let us assume that for all n' satisfying $1 \leq n' \leq n$ it holds true that

$$\#(X^{2,n'}) = 3^{n'} - 2 \tag{18}$$

$$\#(X^{2,n'} \oplus S_1[a_1]) = 3^{n'} - 1 \tag{19}$$

$$\#(X^{2,n'} \oplus \{S_1[a_1], S_1[a_2]\}) = 3^{n'} \tag{20}$$

We shall prove that (18), (19), and (20) hold for n as well.

By the Looping Algorithm, we observe

$$\begin{aligned}\#(X^{2,n}) &= \#(X^{2,n-1} \oplus S_1[a_1]) + \#(X^{2,n-1} \oplus \{S_2[a_1], S_1[a_2]\}) \\ &= \#(X^{2,n-1} \oplus S_1[a_1]) + \#(X^{2,n-1} \oplus \{S_1[a_1], S_1[a_2]\}) + \#(X^{2,n-1} \oplus S_1[a_2])\end{aligned}$$

But by symmetry, $\#(X^{2,n-1} \oplus S_1[a_1]) = \#(X^{2,n-1} \oplus S_1[a_2])$, so

$$\begin{aligned}\#(X^{2,n}) &= (3^{n-1} - 1) + 3^{n-1} + (3^{n-1} - 1) \\ &= 3^n - 2.\end{aligned}$$

Next consider the configuration $X^{2,n} \oplus S_1[a_1]$. Using the formula for adjoining a point to a configuration, we get

$$\#(X^{2,n} \oplus S_1[a_1]) = \#(X^{2,n}) + \#(X^{2,n} - \{a_1\}).$$

The configuration $X^{2,n} - \{a_1\}$ will leave a single point a_2 in A, with all points $b_1, \ldots, b_n$ radiating out like spokes on a tire. This means that every point b_j is an endpoint, so $\#(X^{2,n} - \{a_1\}) = 1$. Hence,

$$\#(X^{2,n} \oplus S_1[a_1]) = 3^n - 1.$$

Finally, consider the configuration $X^{2,n-1} \oplus \{S_1[a_1], S_1[a_2]\}$. Through logic similar to the previous case, we find

$$\begin{aligned}\#(X^{2,n-1} \oplus \{S_1[a_1], S_1[a_2]\}) &= \#(X^{2,n} \oplus S_1[a_1]) + \#(X^{2,n} \oplus S_1[a_1] - \{a_2\}) \\ &= 3^n - 1 + 1 \\ &= 3^n.\end{aligned}$$

Thus, by induction on n, we have found

$$\begin{aligned}\#(X^{2,n}) &= 3^n - 2 \\ \#(X^{2,n} \oplus S_1[a_1]) &= 3^n - 1 \\ \#(X^{2,n} \oplus \{S_1[a_1], S_1[a_2]\}) &= 3^n \qquad \text{for all } n \geq 1.\end{aligned}$$

Note that for $n = 1$, the configurations lie in $\mathbb{R}$, as they are all string configurations. For $n = 2$, the configurations contain P_2, and so lie in $\mathbb{R}^2$. For $n \geq 3$, we will show that the configurations can lie in $\mathbb{R}^3$. To see this, let $h(A, B) = r$. Then let us array the points in B to form a regular n-gon in the plane, with the n-gon capable of being inscribed in a circle of radius $\frac{\sqrt{2}r}{2}$. Then we place a_1 on the axis of the polygon at an elevation of $\frac{\sqrt{2}r}{2}$, and a_2 on the axis at an elevation of $-\frac{\sqrt{2}r}{2}$. Thus, we do indeed find that $d_E(a_i, b_j) = r$ for all i and j. Hence, $X^{2,n}$ can exist in $\mathbb{R}^3$, and the reader may imagine that adjoining the S_1's to a_1 and a_2 does not add to the dimension necessary to house the configuration.

References

[1] N. J. A. Sloane, (2006), The On-Line Encyclopedia of Integer Sequences, published electronically at `www.research.att.com/~njas/sequences/`.

[2] James R. Munkres. Topology: Second Edition. Prentice Hall, Upper Saddle River, 2000.

[3] Chantel Blackburn, Kristina Lund, Steve Schlicker, Patrick Sigmon, Alex Zupan. The Prime 19 is Missing in the Hausdorff Metric Geometry, *Submitted.*

[4] Titu Andreescu, Zuming Feng. A Path to Combinatorics for Undergraduates: Counting Strategies. Birkhäuser. Boston. 2004.

[5] Kristina Lund, Steven Schlicker, Patrick Sigmon. Fibonacci Numbers in the Hausdorff Metric Geometry, *Submitted.*

[6] Tom Apostol. Mathematical Analysis: A Modern Approach to Advanced Calculus. Addison-Wesley Publishing Company, Inc. Reading, Massachusetts. 1960.

[7] Agnieszka Bogdewicz. Some Properties of Hyperspaces, *Demonstratio Mathematica*, XXXIII: 135-149, 2000.

[8] Christopher Bay, Amber Lembcke, and Steven Schlicker. When Lines Go Bad in Hyperspace. *Demonstratio Mathematica*, No. 3, Volume 38 (2005), p. 689-701.

[9] Andrew Browder. Mathematical Analysis: an Introduction. Springer-Verlag, New York, 1996.

[10] Jeff Henrikson. Completeness and Total Boundedness of the Hausdorff Metric. *MIT Undergraduate Journal of Mathematics*, Volume 1 (1999), p. 69-80.

A Appendix of SPACK Sieve Results

The table below holds those integers k for which we found configurations such that $\#(X) = k$. Note that this list is by no means exhaustive. At present, the table contains the results only for configurations $X = A \cup B$ in which $|A| \cdot |B| \leq 25$. Gaps in the table, particularly for small numbers, represent strong candidates for SPACK-0 numbers.

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 84, 85, 86, 88, 89, 90, 92, 93, 94, 96, 98, 99, 100, 102, 103, 104, 105, 106, 107, 108, 110, 111, 112, 113, 114, 116, 117, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167, 168, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 188, 189, 191, 192, 193, 194, 196, 198, 199, 200, 201, 203, 204, 205, 206, 207, 208, 210, 211, 212, 213, 214, 215, 216, 217, 219, 220, 221, 222, 224, 225, 226, 227, 228, 231, 232, 233, 234, 235, 237, 239, 240, 241, 242, 243, 244, 245, 246, 247, 248, 249, 250, 252, 254, 256, 258, 260, 261, 264, 265, 266, 267, 268, 270, 272, 273, 276, 277, 278, 279, 280, 281, 282, 285, 286, 287, 288, 289, 290, 292, 293, 294, 296, 298, 300, 301, 302, 303, 304, 305, 306, 308, 309, 310, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 324, 326, 328, 330, 332, 333, 334, 335, 336, 337, 338, 339, 342, 343, 344, 345, 346, 348, 351, 352, 354, 356, 357, 359, 360, 362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 376, 377, 378, 379, 381, 383, 384, 386, 387, 388, 389, 390, 391, 392, 393, 394, 396, 398, 399, 400, 402, 404, 405, 406, 407, 408, 410, 411, 412, 414, 415, 417, 418, 420, 422, 423, 424, 427, 428, 429, 431, 432, 434, 436, 438, 439, 440, 441, 443, 444, 447, 448, 450, 451, 453, 454, 455, 456, 457, 459, 460, 461, 464, 465, 466, 467, 468, 469, 470, 471, 474, 475, 477, 478, 480, 483, 484, 485, 486, 487, 488, 489, 491, 492, 493, 494, 496, 499, 500, 501, 504, 505, 506, 507, 508, 510, 511, 512, 513, 514, 516, 517, 520, 521, 522, 523, 526, 527, 528, 529, 530, 531, 534, 536, 537, 539, 540, 543, 544, 547, 549, 550, 551, 552, 553, 555, 556, 558, 560, 561, 562, 564, 567, 569, 570, 572, 574, 577, 578, 579, 580, 582, 584, 585, 586, 587, 588, 589, 590, 591, 594, 595, 596, 600, 603, 605, 606, 608, 609, 610, 611, 612, 613, 618, 621, 623, 624, 625, 626, 630, 632, 633, 636, 639, 640, 642, 644, 645, 647, 648, 651, 654, 656, 657, 659, 662, 663, 666, 668, 669, 672, 673, 674, 675, 676, 678, 680, 684, 685, 686, 688, 690, 692, 693, 696, 697, 698, 699, 701, 702, 704, 709, 711, 712, 713, 714, 715, 716, 717, 718, 719, 720, 722, 723, 725, 726, 727, 728, 729, 732, 735, 738, 740, 741, 744, 745, 750, 752, 754, 756, 762, 767, 768, 773, 776, 777, 778, 784, 786, 792, 798, 800, 801, 802, 805, 806, 807, 810, 811, 812, 813, 816, 818, 819, 820, 822, 825, 828, 832, 833, 834, 837, 840, 842, 844, 845, 846, 849, 852, 855, 856, 857, 860, 861, 862, 864, 867, 870, 872, 875, 876, 878, 879, 881, 882, 885, 888, 889, 890, 892, 894, 896, 897, 899, 900, 902, 903, 908, 910, 911, 912, 917, 918, 922, 924, 930, 932, 934, 936, 939, 942, 945, 948, 951, 952, 953, 954, 956, 957, 958, 960, 963, 965, 966, 967, 968, 969, 970, 972, 975, 977, 978, 979, 982, 983, 985, 987, 988, 990, 992, 993, 994, 996, 1000, 1001, 1002, 1007, 1008, 1016, 1017, 1018, 1020, 1024, 1025, 1026, 1029, 1030, 1032, 1033, 1034, 1036, 1038, 1039, 1040, 1041, 1044, 1046, 1047, 1048, 1053, 1054, 1056, 1060, 1061, 1062, 1064, 1068, 1069, 1071, 1072, 1074, 1076, 1077, 1078, 1080, 1082, 1083, 1085, 1086, 1089, 1090, 1092, 1093, 1094, 1095, 1096, 1101, 1104, 1109, 1110, 1112, 1115, 1116, 1118, 1119, 1120, 1122, 1126, 1128, 1132, 1135, 1136, 1137, 1138, 1140, 1143, 1146, 1149, 1150, 1151, 1153, 1155, 1156, 1160, 1161, 1162, 1167, 1168, 1170, 1172, 1173, 1176, 1177, 1178, 1185, 1187, 1189, 1190, 1192, 1193, 1194, 1195, 1197, 1200, 1201, 1206, 1207, 1212, 1214, 1215, 1216, 1218, 1219, 1221, 1223, 1224, 1226, 1229, 1232, 1233, 1234, 1239, 1240, 1242, 1243, 1245, 1248, 1251, 1256, 1259, 1262, 1263, 1264, 1265, 1268, 1269, 1272, 1273, 1287, 1288, 1289, 1290, 1295, 1296, 1301, 1305, 1310, 1311, 1312, 1313, 1314, 1317, 1318, 1320, 1321, 1323, 1329, 1336, 1341, 1342, 1345, 1347, 1352, 1353, 1356, 1358, 1359, 1370, 1373, 1378, 1381, 1382, 1383, 1384, 1385, 1386, 1390, 1392, 1395, 1398, 1400, 1402, 1404, 1408, 1417, 1422, 1424, 1425, 1426, 1428, 1432, 1434, 1439, 1440, 1443, 1447, 1450, 1451, 1452, 1453, 1455, 1456, 1457, 1458, 1460, 1464, 1471, 1472, 1473, 1480, 1488, 1489, 1490, 1492, 1494, 1497, 1504, 1507, 1509, 1524, 1533, 1536, 1537, 1544, 1546, 1551, 1556, 1561, 1562, 1563, 1568, 1571, 1576, 1578, 1584, 1585, 1586, 1590, 1591, 1594, 1597, 1600, 1602, 1604, 1606, 1607, 1610, 1611, 1612, 1617, 1619, 1620, 1621, 1624, 1632, 1635, 1637, 1638, 1640, 1644, 1647, 1649, 1651, 1656, 1657, 1660, 1661, 1662, 1665, 1671, 1673, 1674, 1680, 1683, 1689, 1691, 1692, 1695, 1698, 1699, 1700, 1707, 1710, 1711, 1712, 1716, 1720, 1722, 1727, 1728, 1729, 1731, 1734, 1735, 1740, 1750, 1760, 1764, 1767, 1777, 1779, 1780, 1784, 1786, 1787, 1788, 1798, 1799, 1800, 1812, 1817, 1818, 1824, 1826, 1829, 1834, 1836, 1839, 1844, 1855, 1856, 1860, 1861, 1862, 1868, 1870, 1871, 1872, 1873, 1878, 1888, 1890, 1896, 1910, 1911, 1912, 1922, 1924, 1927, 1934, 1935, 1936, 1938, 1941, 1943, 1950, 1952, 1956, 1957, 1962, 1967, 1976, 1977, 1978, 1988, 1994, 1995, 2000, 2010, 2012, 2013, 2015, 2032, 2034, 2040, 2041, 2050, 2054, 2055, 2057, 2066, 2069, 2072, 2077, 2078, 2079, 2080, 2088, 2094, 2096, 2097, 2102, 2103, 2104, 2105, 2112, 2118, 2124, 2130, 2133, 2134, 2136, 2139, 2142, 2146, 2149, 2150, 2151, 2154, 2157, 2159, 2160, 2161, 2162, 2169, 2172, 2174, 2175, 2177, 2178, 2180, 2181, 2183, 2185, 2186, 2187, 2190, 2192, 2196, 2202, 2204, 2210, 2211, 2212, 2220, 2232, 2235, 2240, 2246, 2250, 2253, 2257, 2266, 2268, 2271, 2276, 2277, 2281, 2282, 2286, 2288, 2289, 2290, 2292, 2295, 2301, 2304, 2319, 2320, 2322, 2328, 2334, 2335, 2352, 2356, 2361, 2369, 2370, 2373, 2374, 2376, 2377, 2380, 2384, 2385, 2388, 2394, 2400, 2401, 2404, 2406, 2412, 2416, 2417, 2418, 2423, 2424, 2426, 2429, 2432, 2433, 2439, 2440, 2442, 2445, 2448, 2450, 2451, 2457, 2460, 2462, 2463, 2466, 2472, 2479, 2481, 2487, 2493, 2494, 2502, 2511, 2517, 2520, 2523, 2526, 2530, 2537, 2538, 2539, 2540, 2544, 2547, 2548, 2552, 2553, 2556, 2562, 2565, 2568, 2578, 2583, 2585, 2589, 2592, 2604, 2612, 2626, 2628, 2629, 2634, 2636, 2638, 2640, 2646, 2651, 2655, 2664, 2670, 2673, 2679, 2680, 2682, 2685, 2689, 2690, 2691, 2700, 2705, 2706, 2715, 2716, 2718, 2724, 2733, 2736, 2740, 2742, 2751, 2755, 2757, 2758, 2767, 2779, 2784, 2791, 2792, 2793, 2802, 2808, 2818, 2820, 2827, 2832, 2838, 2840, 2841, 2843, 2844, 2846, 2850, 2852, 2857, 2863, 2868, 2879, 2880, 2883, 2890, 2896, 2899, 2901, 2904, 2907, 2912, 2914, 2916, 2926, 2928, 2929, 2931, 2955, 2961, 2968, 2972, 2976, 2979, 2985, 2990, 2991, 3000, 3003, 3006, 3009, 3023, 3024, 3029, 3032, 3033, 3035, 3057, 3060, 3072, 3074, 3075, 3084, 3085, 3087, 3090, 3095, 3096, 3099, 3102, 3106, 3108, 3112, 3120, 3129, 3135, 3136, 3138, 3141, 3147, 3150, 3153, 3156, 3159, 3170, 3180, 3190, 3195, 3196, 3207, 3214, 3218, 3222, 3224, 3232, 3240, 3246, 3257, 3263, 3268, 3270, 3278, 3281, 3285, 3290, 3295, 3300, 3305, 3307, 3314, 3318, 3321, 3328, 3329, 3333, 3336, 3339, 3340, 3346, 3348, 3354, 3366, 3368, 3369, 3372, 3384, 3397, 3402, 3405, 3408, 3419, 3423, 3430, 3435, 3439, 3441, 3445, 3451, 3456, 3471, 3480, 3481, 3504, 3510, 3512, 3513, 3525, 3528, 3534, 3535, 3537, 3544, 3555, 3560, 3564, 3567, 3576, 3580, 3588, 3600, 3601, 3602, 3608, 3640, 3646, 3651, 3656, 3674, 3688, 3691, 3692, 3700, 3711, 3712, 3713, 3717, 3726, 3727, 3736, 3740, 3741, 3746, 3747, 3756, 3760, 3762, 3765, 3771, 3776, 3791, 3795, 3799, 3808, 3819, 3830, 3834, 3848, 3852, 3857, 3861, 3867, 3870, 3879, 3880, 3888, 3898, 3901, 3902, 3904, 3907, 3908, 3911, 3915, 3924, 3933, 3936, 3942, 3951, 3953, 3954, 3956, 3960, 3961, 3968, 3978, 3984, 4008, 4024, 4035, 4042, 4050, 4055, 4069, 4080, 4082, 4096, 4117, 4119, 4120, 4136, 4146, 4147, 4166, 4175, 4224, 4235, 4240, 4256, 4266, 4269, 4271, 4273, 4278, 4280, 4288, 4290, 4310, 4324, 4347, 4352, 4356, 4363, 4366, 4371, 4388, 4392, 4401, 4402, 4406, 4431, 4456, 4480, 4496, 4512, 4524, 4541, 4560, 4572, 4580, 4587, 4592, 4602, 4643, 4672, 4694, 4724, 4736, 4755, 4768, 4770, 4778, 4783, 4826, 4830, 4831, 4832, 4834, 4842, 4849, 4851, 4852, 4857, 4878, 4886, 4898, 4902, 4918, 4935, 4939, 4942, 4944, 4946, 4956, 4971, 4986, 4995, 5006, 5010, 5013, 5018, 5019, 5022, 5040, 5041, 5049, 5056, 5064, 5067, 5073, 5076, 5079, 5082, 5085, 5091, 5097, 5102, 5118, 5120, 5142, 5146, 5161, 5169, 5171, 5184, 5186, 5190, 5196, 5218, 5238, 5240, 5241, 5246, 5250, 5262, 5273, 5296, 5299, 5312, 5325, 5346, 5349, 5358, 5361, 5375, 5379, 5391, 5400, 5403, 5410, 5432, 5438, 5447, 5448, 5451, 5458, 5462, 5463, 5480, 5496, 5505, 5508, 5515, 5552, 5556, 5562, 5574, 5580, 5592, 5593, 5610, 5619, 5633, 5634, 5646, 5648, 5654, 5660, 5664, 5672, 5680, 5691, 5697, 5704, 5708, 5712, 5724, 5730, 5736, 5745, 5754, 5758, 5772, 5782, 5797, 5808, 5814, 5820, 5823, 5825, 5826, 5829, 5847, 5850, 5874, 5878, 5886, 5897, 5925, 5928, 5931, 5936, 5958, 5986, 5996, 5997, 6012, 6021, 6024, 6035, 6036, 6042, 6054, 6079, 6082, 6093, 6095, 6096, 6117, 6144, 6145, 6163, 6177, 6188, 6225, 6227, 6244, 6246, 6256, 6266, 6272, 6276, 6290, 6291, 6297, 6306, 6325, 6336, 6342, 6365, 6375, 6378, 6392, 6399, 6408, 6416, 6417, 6420, 6443, 6448, 6450, 6453, 6462, 6468, 6480, 6486, 6498, 6504, 6513, 6521, 6537, 6539, 6558, 6559, 6560, 6561, 6564, 6568, 6582, 6588, 6625, 6641, 6648, 6650, 6694, 6703, 6714, 6723, 6749, 6764, 6777, 6780, 6795, 6800, 6810, 6826, 6828, 6842, 6864, 6880, 6896, 6903, 6924, 6951, 6960, 6968, 6969, 6989, 6992, 7013, 7031, 7041, 7053, 7054, 7056, 7110, 7120, 7122, 7131, 7153, 7161, 7164, 7178, 7185, 7192, 7193, 7200, 7203, 7211, 7217, 7233, 7239, 7246, 7266, 7281, 7284, 7302, 7316, 7338, 7347, 7350, 7386, 7398, 7428, 7431, 7434, 7435, 7446, 7450, 7453, 7464, 7465, 7480, 7503, 7509, 7511, 7515, 7521, 7531, 7540, 7544, 7568, 7578, 7591, 7613, 7642, 7644, 7645, 7670, 7696, 7704, 7721, 7748, 7761, 7776, 7779, 7798, 7801, 7802, 7831, 7833, 7858, 7859, 7860, 7866, 7881, 7887, 7896, 7905, 7912, 7921, 7950, 7966, 7969, 7971, 7983, 7992, 8016, 8048, 8072, 8082, 8094, 8097, 8099, 8136, 8148, 8160, 8169, 8187, 8190, 8198, 8201, 8232, 8234, 8264, 8271, 8281, 8301, 8317, 8319, 8328, 8355, 8388, 8437, 8439, 8454, 8464, 8476, 8480, 8498, 8505, 8506, 8507, 8512, 8520, 8532, 8550, 8571, 8576, 8610, 8637, 8640, 8660, 8696, 8744, 8760, 8780, 8794, 8832, 8848, 8871, 8876, 8911, 8912, 8915, 8937, 8938, 8961, 8991, 8992, 9018, 9023, 9048, 9082, 9088, 9099, 9105, 9111, 9120, 9146, 9153, 9168, 9177, 9180, 9186, 9195, 9197, 9204, 9207, 9216, 9222, 9231, 9234, 9281, 9321, 9324, 9326, 9333, 9339, 9350, 9366, 9421, 9432, 9439, 9443, 9457, 9503, 9544, 9553, 9582, 9599, 9622, 9660, 9677, 9704, 9712, 9715, 9729, 9784, 9896, 10105, 10116, 10118, 10122, 10125, 10137, 10152, 10160, 10177, 10256, 10260, 10284, 10338, 10376, 10608, 10678, 10848, 10890, 10897, 10898, 10944, 10952, 10960, 11114, 11143, 11152, 11169, 11191, 11240, 11272, 11320, 11351, 11378, 11408, 11481, 11510, 11516, 11568, 11575, 11584, 11610, 11637, 11655, 11703, 11717, 11718, 11738, 11745, 11766, 11799, 11810, 11817, 11826, 11834, 11853, 11871, 11887, 11889, 11895, 11917, 11950, 11960, 12024, 12033, 12070, 12122, 12225, 12272, 12300, 12311, 12318, 12328, 12351, 12362, 12398, 12432, 12442, 12473, 12482, 12528, 12544, 12546, 12576, 12585, 12650, 12690, 12699, 12756, 12762, 12790, 12870, 12896, 12912, 12922, 12959, 12984, 13021, 13024, 13029, 13106, 13111, 13155, 13167, 13284, 13289, 13335, 13368, 13376, 13384, 13386, 13395, 13428, 13434, 13504, 13506, 13518, 13596, 13623, 13656, 13680, 13698, 13707, 13728, 13750, 13752, 13841, 13887, 13891, 13896, 13912, 13962, 14004, 14017, 14124, 14127, 14201, 14250, 14255, 14272, 14313, 14340, 14401, 14411, 14413, 14490, 14496, 14551, 14562, 14588, 14642, 14713, 14802, 14829, 14979, 14983, 14984, 15007, 15017, 15030, 15047, 15084, 15086, 15111, 15133, 15138, 15168, 15174, 15192, 15223, 15296, 15321, 15346, 15384, 15404, 15426, 15445, 15469, 15480, 15590, 15722, 15731, 15761, 15807, 15808, 15848, 15873, 15918, 15924, 15927, 15959, 16032, 16051, 16081, 16177, 16206, 16252, 16266, 16280, 16322, 16363, 16454, 16479, 16497, 16502, 16564, 16569, 16593, 16604, 16622, 16665, 16674, 16703, 16778, 16779, 16807, 16836, 16849, 16857, 16920, 16941, 16992, 17061, 17090, 17111, 17136, 17224, 17297, 17304, 17316, 17370, 17373, 17375, 17382, 17458, 17475, 17489, 17529, 17532, 17553, 17581, 17655, 17671, 17703, 17752, 17825, 17828, 17832, 17841, 17856, 17865, 17892, 17914, 17946, 17970, 17987, 18006, 18008, 18052, 18074, 18081, 18125, 18207, 18242, 18261, 18288, 18312, 18345, 18387, 18458, 18493, 18513, 18562, 18577, 18615, 18624, 18642, 18657, 18723, 18738, 18768, 18792, 18804, 18867, 18921, 18931, 18957, 18969, 18991, 19002, 19016, 19083, 19143, 19221, 19314, 19341, 19354, 19377, 19383, 19438, 19488, 19632, 19641, 19681, 19682, 19683, 19692, 19694, 19706, 19708, 19731, 19746, 19764, 19867, 19926, 19941, 19992, 20013, 20053, 20064, 20109, 20118, 20151, 20175, 20226, 20276, 20289, 20354, 20409, 20415, 20424, 20426, 20465, 20476, 20504, 20505, 20538, 20619, 20668, 20688, 20720, 20746, 20800, 20880, 21041, 21043, 21060, 21124, 21182, 21285, 21288, 21343, 21366, 21439, 21447, 21450, 21466, 21528, 21594, 21693, 21729, 21780, 22011, 22044, 22065, 22110, 22113, 22154, 22332, 22344, 22371, 22388, 22441, 23045, 23512, 23864, 23904, 23922, 23942, 23986, 23994, 24003, 24088, 24154, 24440, 25024, 25088, 25102, 25152, 25418, 25544, 25570, 26024, 26367, 26368, 26414, 26656, 26756, 26779, 26947, 26974, 27027, 27200, 27254, 27271, 27298, 27342, 27378, 27419, 27459, 27513, 27520, 27621, 27675, 27680, 27729, 27793, 27824, 27840, 27843, 27940, 28001, 28216, 28341, 28506, 28626, 28858, 29000, 29059, 29144, 29160, 29205, 29282, 29480, 29481, 29570, 29748, 29800, 29948, 30033, 30084, 30230, 30318, 30470, 30531, 30532, 30856, 30870, 30885, 30888, 31155, 31244, 31432, 31526, 31656, 31712, 31851, 31957, 32127, 32184, 32403, 32456, 32510, 32584, 32670, 32847, 33399, 33496, 33499, 33579, 33688, 33935, 34282, 34323, 34411, 34625, 34649, 34903, 34938, 35075, 35186, 35227, 35324, 35361, 35370, 35478, 35700, 35715, 35776, 36254, 36440, 36593, 36780, 36787, 36833, 36980, 37019, 37305, 37350, 37504, 37569, 37616, 37812, 37948, 38051, 38240, 38294, 38472, 38481, 38798, 39011, 39043, 39077, 39225, 39546, 39617, 39698, 39806, 39906, 39960, 40149, 40196, 40311, 40507, 40635, 40807, 40819, 40833, 40881, 41084, 41171, 41319, 41364, 41503, 41532, 41607, 41614, 41623, 41720, 41757, 41768, 41841, 41987, 42050, 42072, 42282, 42331, 42547, 42696, 42699, 42785, 42840, 43139, 43245, 43287, 43326, 43385, 43426, 43664, 43765, 43770, 43781, 43883, 43980, 44004, 44112, 44154, 44212, 44466, 44471, 44628, 45021, 45199, 45228, 45426, 45438, 45716, 45723, 45797, 45877, 45904, 46090, 46749, 46959, 46989, 46992, 47135, 47260, 47311, 47416, 47541, 47578, 47589, 47670, 48114, 48156, 48273, 48389, 48631, 48906, 49023, 49173, 49208, 49301, 49450, 49692, 49707, 49793, 49843, 49851, 49935, 50178, 51368, 51412, 51437, 52699, 52941, 52950, 53112, 53139, 53175, 53191, 56008, 57152, 58216, 58984, 59047, 59048, 59049, 59578, 59906, 61567, 61776, 62096, 62480, 62542, 62560, 63775, 63807, 63856, 64017, 64098, 64341, 64466, 64503, 64665, 64705, 64720, 65018, 65289, 65784, 65912, 66552, 66914, 67497, 68156, 69135, 69813, 70654, 71505, 72273, 72752, 72986, 73233, 73672, 75104, 75195, 76264, 76736, 77442, 77782, 79020, 80025, 80600, 80656, 81055, 81763, 82240, 82351, 82458, 82782, 83001, 83028, 83423, 84392, 84911, 85056, 85167, 85490, 86541, 86648, 87511, 87769, 88064, 88233, 88705, 88892, 89224, 89392, 89418, 89690, 89979, 90353, 90513, 90549, 91471, 91521, 91896, 92188, 92381, 92490, 93311, 93410, 93809, 94520, 94698, 94967, 95727, 96306, 96713, 97154, 97527, 97650, 97874, 98136, 98505, 98586, 98669, 99666, 100031, 100642, 101271, 101360, 101783, 101991, 102076, 102378, 102786, 103555, 103695, 103938, 104025, 104605, 105183, 105282, 105615, 105786, 105939, 106313, 106507, 106617, 107739, 107994, 109878, 110000, 110530, 110555, 111534, 113832, 113907, 114393, 114493, 115465, 116192, 116920, 116947, 117572, 118122, 118206, 118656, 131546, 133232, 135472, 140032, 140114, 140800, 142906, 145387, 147682, 148286, 148592, 148834, 149563, 149806, 150291, 150777, 152187, 153668, 156195, 158572, 163845, 174735, 175720, 177145, 177146, 180640, 181160, 184406, 187760, 189512, 191131, 191734, 191971, 192476, 192943, 193266, 195059, 195547, 196377, 198656, 198952, 202139, 202624, 205987, 207083, 208048, 209051, 210209, 212418, 213540, 215864, 216443, 217958, 218779, 220988, 221011, 221468, 225163, 226709, 228135, 228380, 229415, 229801, 231165, 231261, 234762, 238009, 238196, 238914, 244652, 246061, 246947, 247283, 247495, 254326, 255660, 256782, 263375, 264625, 316928, 323728, 333009, 335386, 349304, 350032, 351489, 358431, 408656, 412880, 417856, 427137, 430312, 431536, 446744, 447784, 450631, 450904, 451602, 455639, 458329, 463602, 469664, 473290, 473881, 490129, 490535, 496335, 496826, 499999, 503667, 512154, 512233, 515450, 531439, 533291, 536321, 552940, 554910, 555758, 573225, 592260, 693601, 709682, 735104, 816985, 819170, 978064, 996787, 1014458, 1025512, 1050824, 1053739, 1068475, 1072867, 1073354, 1084508, 1119364, 1162157, 1166719, 1168591, 1206482, 1247989, 2145368, 2248976, 2300576, 2338345, 2445394, 2459696, 2542687, 2633546, 5049626, 5366288, 5566147, 5745121, 10924399, 11784608, 24997921

B Proof that 57 is SPACK-3

We know that there exists a configuration X_0 which lies in 3 dimensions such that $\#(X_0) = 57$. We would like to show that there exists no configuration X in 1 or 2 dimensions such that $\#(X) = 57$, thereby establishing 57 as the smallest SPACK-3 number.

In [3] the authors prove that if an infinite configuration Y exists such that $\#(Y) = a$, then there exists a finite configuration Y' such that $\#(Y') = a$ as well. Thus, in looking to prove that there exist no configurations in 1 and 2 dimensions such that $\#(X) = 57$, it suffices to examine finite configurations.

Let us suppose that there exists a finite configuration X such that $\#(X) = 57$ and that X exist in 1 or two dimensions. Then X contains a largest subconfiguration X', where "largest" is defined in the following sense. If X contains no polygon subconfigurations, then X' is the longest string subconfiguration in X. If X contains one or more polygon subconfigurations, then X can be written at $X' \oplus \{$ some collection of adjoined string configurations $\}$. With X' thus defined, we see that X' must also lie in 1 or 2 dimensions, and that $\#(X') \leq \#(X)$ by Rule 1 in Section 5.4. We shall refer to X' as the base configuration for X. On order to show that no 1 or 2 dimensional configuration X can exist such that $\#(X) = 57$, we will consider all base configurations X' in 1 and 2 dimensions such that $\#(X') \leq 57$. We will then build on these X' by adjoining string configurations. We will adjoin an increasing number of points to each base configuration in all combinations until $\#(X) > 57$ for every configuration derived in this manner. Then by again utilizing Rule 1 from Section 5.4, we will know that adjoining of further points cannot yield a configuration such that $\#(X) = 57$, thereby completing the proof.

Note that we need not consider adjoining strings which would alter the base configuration. For example, when looking at configurations which can be derived from the base configuration S_5 we need not consider $S_5 \oplus S_3[3]$, since this configuration can be rewritten as $S_6 \oplus S_2[3]$, and will therefore be considered when we examine the base configuration S_6. Also, we need not explicitly consider configurations in which we adjoin a point at a location adjacent to another endpoint, since $\#(X)$ will not change, by Rule 2.

Due to space constraints and the length of this proof, we will simple list all configurations which can be based upon a given base configuration, omitting pictures. We will begin our proof by considering those base configurations which are just string configurations. Note that it is not possible to adjoin a point to S_1, S_2, S_3, or S_4 in a way that will simultaneously increase $\#(X)$ and not lengthen the base configuration, so we start with the base configuration S_5. For all of the string configurations we assume that the point 1 is taken to be one of the endpoints, and all successive points are labeled with increasing subscripts as we traverse down the string configuration.

Base configuration: S_5

$$
\begin{aligned}
\#(S_5) = 3 \\
\#(S_5 \oplus S_1[3]) = 4 \\
\#(S_5 \oplus S_2[3]) = 7 \\
\#(S_5 \oplus \{S_2[3], S_1[3]\}) = 8 \\
\#(S_5 \oplus \{S_2[3], S_2[3]\}) = 15 \\
\#(S_5 \oplus \{S_2[3], S_2[3], S_1[3]\}) = 16 \\
\#(S_5 \oplus \{S_2[3], S_2[3], S_2[3]\}) = 31 \\
\#(S_5 \oplus \{S_2[3], S_2[3], S_2[3], S_1[3]\}) = 32 \\
\#(S_5 \oplus \{S_2[3], S_2[3], S_2[3], S_2[3]\}) = 63
\end{aligned}
$$

Base configuration: S_6

$$
\begin{aligned}
\#(S_6) = 5 \\
\#(S_6 \oplus S_1[3]) = 6 \\
\#(S_6 \oplus \{S_1[3], S_1[4]\}) = 8 \\
\#(S_6 \oplus S_2[3]) = 11 \\
\#(S_6 \oplus \{S_2[3], S_1[3]\}) = 12 \\
\#(S_6 \oplus \{S_2[3], S_1[4]\}) = 14 \\
\#(S_6 \oplus \{S_2[3], S_1[3], S_1[4]\}) = 16 \\
\#(S_6 \oplus \{S_2[3], S_2[4]\}) = 25 \\
\#(S_6 \oplus \{S_2[3], S_2[3]\}) = 23 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_1[3]\}) = 24 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_1[4]\}) = 30 \\
\#(S_6 \oplus \{S_2[3], S_2[4], S_1[3]\}) = 28 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_1[3], S_1[4]\}) = 32 \\
\#(S_6 \oplus \{S_2[3], S_2[4], S_1[3], S_1[4]\}) = 32 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[3]\}) = 47 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[4]\}) = 53 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[3], S_1[3]\}) = 48 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[3], S_1[4]\}) = 62 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[4], S_1[3]\}) = 56 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[4], S_1[4]\}) = 60 \\
\#(S_6 \oplus \{S_2[3], S_2[3], S_2[3], S_2[3]\}) = 95
\end{aligned}
$$

Base configuration: S_7

$$\#(S_7) = 8$$
$$\#(S_7 \oplus S_1[3]) = 10$$
$$\#(S_7 \oplus S_1[4]) = 9$$
$$\#(S_7 \oplus \{S_1[3], S_1[4]\}) = 12$$
$$\#(S_7 \oplus \{S_1[3], S_1[5]\}) = 12$$
$$\#(S_7 \oplus S_2[3]) = 18$$
$$\#(S_7 \oplus S_2[4]) = 17$$
$$\#(S_7 \oplus \{S_1[3], S_1[4], S_1[5]\}) = 16$$
$$\#(S_7 \oplus \{S_2[3], S_1[3]\}) = 20$$
$$\#(S_7 \oplus \{S_2[3], S_1[4]\}) = 21$$
$$\#(S_7 \oplus \{S_2[3], S_1[5]\}) = 22$$
$$\#(S_7 \oplus \{S_2[4], S_1[3]\}) = 22$$
$$\#(S_7 \oplus \{S_2[4], S_1[4]\}) = 18$$
$$\#(S_7 \oplus S_3[3]) = 26$$
$$\#(S_7 \oplus \{S_2[3], S_1[3], S_1[4]\}) = 24$$
$$\#(S_7 \oplus \{S_2[3], S_1[3], S_1[5]\}) = 24$$
$$\#(S_7 \oplus \{S_2[3], S_1[4], S_1[5]\}) = 28$$
$$\#(S_7 \oplus \{S_2[4], S_1[3], S_1[4]\}) = 24$$
$$\#(S_7 \oplus \{S_2[3], S_1[3], S_1[5]\}) = 28$$
$$\#(S_7 \oplus \{S_3[4], S_1[3]\}) = 34$$
$$\#(S_7 \oplus \{S_3[4], S_1[4]\}) = 27$$
$$\#(S_7 \oplus \{S_2[3], S_2[3]\}) = 38$$
$$\#(S_7 \oplus \{S_2[3], S_2[4]\}) = 39$$
$$\#(S_7 \oplus \{S_2[3], S_2[5]\}) = 40$$
$$\#(S_7 \oplus \{S_2[4], S_2[4]\}) = 35$$
$$\#(S_7 \oplus \{S_3[4] \oplus S_1[1]\}) = 34$$
$$\#(S_7 \oplus \{S_2[3], S_1[3], S_1[4], S_1[5]\}) = 32$$
$$\#(S_7 \oplus \{S_2[4], S_1[3], S_1[4], S_1[5]\}) = 32$$
$$\#(S_7 \oplus \{S_3[4], S_1[3], S_1[4]\}) = 36$$
$$\#(S_7 \oplus \{S_3[4], S_1[3], S_1[5]\}) = 44$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[3]\}) = 40$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[4]\}) = 45$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[5]\}) = 46$$
$$\#(S_7 \oplus \{S_2[3], S_2[4], S_1[3]\}) = 44$$
$$\#(S_7 \oplus \{S_2[3], S_2[4], S_1[4]\}) = 42$$
$$\#(S_7 \oplus \{S_2[3], S_2[4], S_1[5]\}) = 50$$
$$\#(S_7 \oplus \{S_2[3], S_2[5], S_1[3]\}) = 44$$
$$\#(S_7 \oplus \{S_2[3], S_2[5], S_1[4]\}) = 49$$
$$\#(S_7 \oplus \{S_2[4], S_2[4], S_1[3]\}) = 46$$
$$\#(S_7 \oplus \{S_2[4], S_2[4], S_1[4]\}) = 36$$

$$\#(S_7 \oplus \{S_3[4] \oplus S_1[1], S_1[3]\}) = 44$$
$$\#(S_7 \oplus \{S_3[4] \oplus S_1[1], S_1[4]\}) = 36$$
$$\#(S_7 \oplus \{S_3[4], S_2[3]\}) = 60$$
$$\#(S_7 \oplus \{S_3[4], S_2[4]\}) = 53$$
$$\#(S_7 \oplus \{S_3[4], S_2[2]\}) = 60$$
$$\#(S_7 \oplus \{S_3[4], S_1[3], S_1[4], S_1[5]\}) = 48$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[3], S_1[4]\}) = 48$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[3], S_1[5]\}) = 48$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[4], S_1[5]\}) = 60$$
$$\#(S_7 \oplus \{S_2[3], S_2[4], S_1[3], S_1[4]\}) = 48$$
$$\#(S_7 \oplus \{S_2[3], S_2[4], S_1[3], S_1[5]\}) = 56$$
$$\#(S_7 \oplus \{S_2[3], S_2[4], S_1[4], S_1[5]\}) = 56$$
$$\#(S_7 \oplus \{S_2[3], S_2[5], S_1[3], S_1[4]\}) = 56$$
$$\#(S_7 \oplus \{S_2[3], S_2[5], S_1[3], S_1[5]\}) = 48$$
$$\#(S_7 \oplus \{S_2[4], S_2[4], S_1[3], S_1[4]\}) = 48$$
$$\#(S_7 \oplus \{S_2[4], S_2[4], S_1[3], S_1[5]\}) = 60$$
$$\#(S_7 \oplus \{S_3[4] \oplus S_1[1], S_1[3], S_1[5]\}) = 56$$
$$\#(S_7 \oplus \{S_3[4] \oplus S_1[1], S_2[4]\}) = 70$$
$$\#(S_7 \oplus \{S_3[4], S_2[4], S_1[4]\}) = 54$$
$$\#(S_7 \oplus \{S_3[4], S_3[4]\}) = 80$$
$$\#(S_7 \oplus \{S_2[3], S_2[3], S_1[3], S_1[4], S_1[5]\}) = 64$$
$$\#(S_7 \oplus \{S_3[4], S_2[4], S_1[3], S_1[4]\}) = 72$$
$$\#(S_7 \oplus \{S_3[4], S_2[4], S_1[3], S_1[5]\}) = 92$$

Base configuration: S_8

$$\#(S_8) = 13$$
$$\#(S_8 \oplus S_1[3]) = 16$$
$$\#(S_8 \oplus S_1[4]) = 15$$
$$\#(S_8 \oplus \{S_1[3], S_1[4]\}) = 20$$
$$\#(S_8 \oplus \{S_1[3], S_1[5]\}) = 18$$
$$\#(S_8 \oplus \{S_1[3], S_1[6]\}) = 20$$
$$\#(S_8 \oplus \{S_1[4], S_1[5]\}) = 18$$
$$\#(S_8 \oplus S_2[3]) = 29$$
$$\#(S_8 \oplus S_2[4]) = 28$$
$$\#(S_8 \oplus \{S_1[3], S_1[4], S_1[5]\}) = 24$$
$$\#(S_8 \oplus \{S_1[3], S_1[4], S_1[6]\}) = 24$$
$$\#(S_8 \oplus \{S_1[3], S_2[3]\}) = 32$$
$$\#(S_8 \oplus \{S_2[3], S_1[4]\}) = 35$$
$$\#(S_8 \oplus \{S_2[3], S_1[5]\}) = 33$$
$$\#(S_8 \oplus \{S_2[3], S_1[6]\}) = 36$$
$$\#(S_8 \oplus \{S_2[4], S_1[3]\}) = 36$$
$$\#(S_8 \oplus \{S_2[4], S_1[4]\}) = 30$$
$$\#(S_8 \oplus \{S_2[4], S_1[5]\}) = 33$$
$$\#(S_8 \oplus \{S_2[4], S_1[6]\}) = 34$$
$$\#(S_8 \oplus S_3[4]) = 43$$
$$\#(S_8 \oplus \{S_1[3], S_1[4], S_1[5], S_1[6]\}) = 32$$
$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[4]\}) = 40$$
$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[5]\}) = 36$$
$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[6]\}) = 40$$
$$\#(S_8 \oplus \{S_2[3], S_1[4], S_1[5]\}) = 42$$
$$\#(S_8 \oplus \{S_2[3], S_1[4], S_1[6]\}) = 42$$
$$\#(S_8 \oplus \{S_2[3], S_1[5], S_1[6]\}) = 44$$
$$\#(S_8 \oplus \{S_2[4], S_1[3], S_1[4]\}) = 40$$
$$\#(S_8 \oplus \{S_2[4], S_1[3], S_1[5]\}) = 42$$
$$\#(S_8 \oplus \{S_2[4], S_1[3], S_1[6]\}) = 44$$
$$\#(S_8 \oplus \{S_2[4], S_1[4], S_1[5]\}) = 36$$
$$\#(S_8 \oplus \{S_2[4], S_1[4], S_1[6]\}) = 36$$
$$\#(S_8 \oplus \{S_2[4], S_1[5], S_1[6]\}) = 44$$
$$\#(S_8 \oplus \{S_3[4], S_1[3]\}) = 56$$
$$\#(S_8 \oplus \{S_3[4], S_1[4]\}) = 45$$
$$\#(S_8 \oplus \{S_3[4], S_1[5]\}) = 51$$
$$\#(S_8 \oplus \{S_3[4], S_1[6]\}) = 52$$
$$\#(S_8 \oplus \{S_2[3], S_2[3]\}) = 61$$
$$\#(S_8 \oplus \{S_2[3], S_2[4]\}) = 64$$
$$\#(S_8 \oplus \{S_2[3], S_2[5]\}) = 62$$
$$\#(S_8 \oplus \{S_2[3], S_2[6]\}) = 65$$
$$\#(S_8 \oplus \{S_2[4], S_2[4]\}) = 58$$
$$\#(S_8 \oplus \{S_2[4], S_2[5]\}) = 61$$

$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[4], S_1[5]\}) = 48$$
$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[4], S_1[6]\}) = 48$$
$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[5], S_1[6]\}) = 48$$
$$\#(S_8 \oplus \{S_2[3], S_1[4], S_1[5], S_1[6]\}) = 56$$
$$\#(S_8 \oplus \{S_2[4], S_1[3], S_1[4], S_1[5]\}) = 48$$
$$\#(S_8 \oplus \{S_2[4], S_1[3], S_1[4], S_1[6]\}) = 48$$
$$\#(S_8 \oplus \{S_2[4], S_1[3], S_1[5], S_1[6]\}) = 56$$
$$\#(S_8 \oplus \{S_2[4], S_1[4], S_1[5], S_1[6]\}) = 48$$
$$\#(S_8 \oplus \{S_3[4], S_1[3], S_1[4]\}) = 60$$
$$\#(S_8 \oplus \{S_3[4], S_1[3], S_1[5]\}) = 66$$
$$\#(S_8 \oplus \{S_3[4], S_1[3], S_1[6]\}) = 68$$
$$\#(S_8 \oplus \{S_3[4], S_1[4], S_1[5]\}) = 54$$
$$\#(S_8 \oplus \{S_3[4], S_1[4], S_1[6]\}) = 54$$
$$\#(S_8 \oplus \{S_3[4], S_1[5], S_1[6]\}) = 68$$
$$\#(S_8 \oplus \{S_2[3], S_1[3], S_1[4], S_1[5], S_1[6]\}) = 64$$
$$\#(S_8 \oplus \{S_3[4] \oplus S_1[1], S_1[3]\}) = 72$$

Base configuration: S_9

$$\#(S_9) = 21$$
$$\#(S_9 \oplus S_1[3]) = 26$$
$$\#(S_9 \oplus S_1[4]) = 24$$
$$\#(S_9 \oplus S_1[5]) = 25$$
$$\#(S_9 \oplus \{S_1[3], S_1[4]\}) = 32$$
$$\#(S_9 \oplus \{S_1[3], S_1[5]\}) = 30$$
$$\#(S_9 \oplus \{S_1[3], S_1[6]\}) = 30$$
$$\#(S_9 \oplus \{S_1[3], S_1[7]\}) = 32$$
$$\#(S_9 \oplus \{S_1[4], S_1[5]\}) = 30$$
$$\#(S_9 \oplus \{S_1[4], S_1[6]\}) = 27$$
$$\#(S_9 \oplus S_2[3]) = 47$$
$$\#(S_9 \oplus S_2[4]) = 45$$
$$\#(S_9 \oplus S_2[5]) = 46$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[5]\}) = 40$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[6]\}) = 36$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[7]\}) = 40$$
$$\#(S_9 \oplus \{S_1[3], S_1[5], S_1[6]\}) = 36$$
$$\#(S_9 \oplus \{S_1[3], S_1[5], S_1[7]\}) = 36$$
$$\#(S_9 \oplus \{S_1[4], S_1[5], S_1[6]\}) = 36$$
$$\#(S_9 \oplus \{S_2[3], S_1[3]\}) = 52$$
$$\#(S_9 \oplus \{S_2[3], S_1[4]\}) = 56$$
$$\#(S_9 \oplus \{S_2[3], S_1[5]\}) = 55$$
$$\#(S_9 \oplus \{S_2[3], S_1[6]\}) = 54$$
$$\#(S_9 \oplus \{S_2[3], S_1[7]\}) = 58$$
$$\#(S_9 \oplus \{S_2[4], S_1[3]\}) = 58$$
$$\#(S_9 \oplus \{S_2[4], S_1[4]\}) = 48$$
$$\#(S_9 \oplus \{S_2[4], S_1[5]\}) = 55$$
$$\#(S_9 \oplus \{S_2[4], S_1[6]\}) = 51$$
$$\#(S_9 \oplus \{S_2[4], S_1[7]\}) = 56$$
$$\#(S_9 \oplus \{S_2[5], S_1[3]\}) = 56$$
$$\#(S_9 \oplus \{S_2[5], S_1[4]\}) = 54$$
$$\#(S_9 \oplus \{S_2[5], S_1[5]\}) = 50$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[5], S_1[6]\}) = 48$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[5], S_1[7]\}) = 48$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[6], S_1[7]\}) = 48$$
$$\#(S_9 \oplus \{S_2[3], S_1[3], S_1[3]\}) = 64$$
$$\#(S_9 \oplus \{S_2[3], S_1[3], S_1[5]\}) = 60$$
$$\#(S_9 \oplus \{S_2[3], S_1[3], S_1[6]\}) = 60$$
$$\#(S_9 \oplus \{S_2[3], S_1[3], S_1[7]\}) = 64$$
$$\#(S_9 \oplus \{S_2[3], S_1[4], S_1[5]\}) = 70$$
$$\#(S_9 \oplus \{S_2[3], S_1[4], S_1[6]\}) = 63$$
$$\#(S_9 \oplus \{S_2[3], S_1[4], S_1[7]\}) = 70$$
$$\#(S_9 \oplus \{S_2[3], S_1[5], S_1[6]\}) = 66$$
$$\#(S_9 \oplus \{S_2[3], S_1[5], S_1[7]\}) = 66$$

$$\#(S_9 \oplus \{S_2[3], S_1[6], S_1[7]\}) = 72$$
$$\#(S_9 \oplus \{S_2[4], S_1[3], S_1[4]\}) = 64$$
$$\#(S_9 \oplus \{S_2[4], S_1[3], S_1[5]\}) = 70$$
$$\#(S_9 \oplus \{S_2[4], S_1[3], S_1[6]\}) = 66$$
$$\#(S_9 \oplus \{S_2[4], S_1[3], S_1[7]\}) = 72$$
$$\#(S_9 \oplus \{S_2[4], S_1[4], S_1[5]\}) = 60$$
$$\#(S_9 \oplus \{S_2[4], S_1[4], S_1[6]\}) = 54$$
$$\#(S_9 \oplus \{S_2[4], S_1[4], S_1[7]\}) = 60$$
$$\#(S_9 \oplus \{S_2[4], S_1[5], S_1[6]\}) = 66$$
$$\#(S_9 \oplus \{S_2[4], S_1[5], S_1[7]\}) = 66$$
$$\#(S_9 \oplus \{S_2[4], S_1[6], S_1[7]\}) = 68$$
$$\#(S_9 \oplus \{S_2[5], S_1[3], S_1[4]\}) = 72$$
$$\#(S_9 \oplus \{S_2[5], S_1[3], S_1[5]\}) = 60$$
$$\#(S_9 \oplus \{S_2[5], S_1[3], S_1[6]\}) = 66$$
$$\#(S_9 \oplus \{S_2[5], S_1[3], S_1[7]\}) = 68$$
$$\#(S_9 \oplus \{S_2[5], S_1[4], S_1[5]\}) = 60$$
$$\#(S_9 \oplus \{S_2[5], S_1[4], S_1[6]\}) = 63$$
$$\#(S_9 \oplus \{S_1[3], S_1[4], S_1[5], S_1[6]\}) = 64$$

Base configuration: S_{10}

$$\#(S_{10}) = 34$$
$$\#(S_{10} \oplus S_1[3]) = 42$$
$$\#(S_{10} \oplus S_1[4]) = 39$$
$$\#(S_{10} \oplus S_1[5]) = 40$$
$$\#(S_{10} \oplus \{S_1[3], S_1[4]\}) = 52$$
$$\#(S_{10} \oplus \{S_1[3], S_1[5]\}) = 48$$
$$\#(S_{10} \oplus \{S_1[3], S_1[6]\}) = 50$$
$$\#(S_{10} \oplus \{S_1[3], S_1[7]\}) = 48$$
$$\#(S_{10} \oplus \{S_1[3], S_1[8]\}) = 52$$
$$\#(S_{10} \oplus \{S_1[4], S_1[5]\}) = 48$$
$$\#(S_{10} \oplus \{S_1[4], S_1[6]\}) = 45$$
$$\#(S_{10} \oplus \{S_1[4], S_1[7]\}) = 45$$
$$\#(S_{10} \oplus \{S_1[5], S_1[6]\}) = 50$$
$$\#(S_{10} \oplus \{S_1[3], S_1[4], S_1[5]\}) = 64$$
$$\#(S_{10} \oplus \{S_1[3], S_1[4], S_1[6]\}) = 60$$
$$\#(S_{10} \oplus \{S_1[3], S_1[4], S_1[7]\}) = 60$$
$$\#(S_{10} \oplus \{S_1[3], S_1[4], S_1[8]\}) = 64$$
$$\#(S_{10} \oplus \{S_1[3], S_1[5], S_1[6]\}) = 60$$
$$\#(S_{10} \oplus \{S_1[3], S_1[5], S_1[7]\}) = 54$$
$$\#(S_{10} \oplus \{S_1[3], S_1[5], S_1[8]\}) = 60$$
$$\#(S_{10} \oplus \{S_1[3], S_1[5], S_1[7]\}) = 54$$
$$\#(S_{10} \oplus \{S_1[3], S_1[5], S_1[8]\}) = 60$$
$$\#(S_{10} \oplus \{S_1[4], S_1[5], S_1[6]\}) = 51$$
$$\#(S_{10} \oplus \{S_1[4], S_1[5], S_1[7]\}) = 45$$
$$\#(S_{10} \oplus \{S_1[4], S_1[5], S_1[8]\}) = 51$$
$$\#(S_{10} \oplus \{S_1[4], S_1[5], S_1[7], S_1[8]\}) = 63$$

Base configuration: S_{11}

$$\#(S_{11}) = 55$$
$$\#(S_{11} \oplus S_1[3]) = 68$$
$$\#(S_{11} \oplus S_1[4]) = 63$$
$$\#(S_{11} \oplus S_1[5]) = 65$$
$$\#(S_{11} \oplus S_1[6]) = 64$$

We now point out that if $l \geq 12$, then $\#(S_l) \geq \#(S_{12}) = 89 > 57$, so we have ruled out all 1 or 2 dimensional configurations built on base configurations with no polygonal subconfigurations. Next we consider those configurations build upon base configurations with only one polygonal subconfiguration, that is, when the base configuration is a polygonal configuration P_m. Since $\#(P_4) = 123 > 57$, we need only consider P_2, P_3, and P_4. When labeling the points of a configuration P_m, note that the base configuration initially has $2m$-fold symmetry, so it does not matter which point of P_m we choose to lable as point 1. After that, our labeling assumes that the points are labeling in increasing order proceeding either clockwise or counterclockwise around the polygon.

Base configuration: P_2

$$\#(P_2) = 7$$
$$\#(P_2 \oplus S_1[1]) = 8$$
$$\#(P_2 \oplus \{S_1[1], S_1[2]\}) = 10$$
$$\#(P_2 \oplus \{S_1[1], S_1[3]\}) = 9$$
$$\#(P_2 \oplus S_2[1]) = 15$$
$$\#(P_2 \oplus \{S_1[1], S_1[2], S_1[3]\}) = 12$$
$$\#(P_2 \oplus \{S_2[1], S_1[1]\}) = 16$$
$$\#(P_2 \oplus \{S_2[1], S_1[2]\}) = 18$$
$$\#(P_2 \oplus \{S_2[1], S_1[3]\}) = 17$$
$$\#(P_2 \oplus S_3[1]) = 23$$
$$\#(P_2 \oplus \{S_1[1], S_1[2], S_1[3], S_1[4]\}) = 16$$
$$\#(P_2 \oplus \{S_2[1], S_1[1], S_1[2]\}) = 20$$
$$\#(P_2 \oplus \{S_2[1], S_1[1], S_1[3]\}) = 18$$
$$\#(P_2 \oplus \{S_2[1], S_1[2], S_1[3]\}) = 22$$
$$\#(P_2 \oplus \{S_2[1], S_1[2], S_1[4]\}) = 21$$
$$\#(P_2 \oplus \{S_3[1], S_1[1]\}) = 24$$
$$\#(P_2 \oplus \{S_3[1], S_1[2]\}) = 28$$
$$\#(P_2 \oplus \{S_3[1], S_1[2]\}) = 26$$
$$\#(P_2 \oplus \{S_2[1], S_2[1]\}) = 31$$
$$\#(P_2 \oplus \{S_2[1], S_2[2]\}) = 33$$
$$\#(P_2 \oplus \{S_2[1], S_2[3]\}) = 32$$
$$\#(P_2 \oplus S_4[1]) = 38$$
$$\#(P_2 \oplus \{S_3[1] \oplus S_1[1]\}) = 30$$
$$\#(P_2 \oplus \{S_2[1], S_1[1], S_1[2], S_1[3]\}) = 24$$
$$\#(P_2 \oplus \{S_2[1], S_1[1], S_1[2], S_1[4]\}) = 24$$
$$\#(P_2 \oplus \{S_2[1], S_1[2], S_1[3], S_1[4]\}) = 28$$
$$\#(P_2 \oplus \{S_3[1], S_1[1], S_1[2]\}) = 30$$
$$\#(P_2 \oplus \{S_3[1], S_1[1], S_1[3]\}) = 27$$
$$\#(P_2 \oplus \{S_3[1], S_1[2], S_1[3]\}) = 34$$
$$\#(P_2 \oplus \{S_3[1], S_1[2], S_1[4]\}) = 33$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[1]\}) = 32$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[2]\}) = 38$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[3]\}) = 35$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[1]\}) = 36$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[3]\}) = 39$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1]\}) = 34$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[2]\}) = 40$$
$$\#(P_2 \oplus \{S_4[1], S_1[1]\}) = 40$$
$$\#(P_2 \oplus \{S_4[1], S_1[2]\}) = 46$$
$$\#(P_2 \oplus \{S_4[1], S_1[3]\}) = 43$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[1]\}) = 32$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[2]\}) = 36$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[3]\}) = 34$$

$$\#(P_2 \oplus \{S_3[1], S_2[1]\}) = 47$$
$$\#(P_2 \oplus \{S_3[1], S_2[2]\}) = 51$$
$$\#(P_2 \oplus \{S_3[1], S_2[3]\}) = 49$$
$$\#(P_2 \oplus \{S_3[1] \oplus S_2[1]\}) = 53$$
$$\#(P_2 \oplus \{S_4[1] \oplus S_1[1]\}) = 45$$
$$\#(P_2 \oplus \{S_4[1] \oplus S_1[2]\}) = 46$$
$$\#(P_2 \oplus S_5[1]) = 61$$
$$\#(P_2 \oplus \{S_2[1], S_1[1], S_1[2], S_1[3], S_1[4]\}) = 32$$
$$\#(P_2 \oplus \{S_3[1], S_1[1], S_1[2], S_1[3]\}) = 36$$
$$\#(P_2 \oplus \{S_3[1], S_1[1], S_1[2], S_1[4]\}) = 36$$
$$\#(P_2 \oplus \{S_3[1], S_1[2], S_1[3], S_1[4]\}) = 44$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[1], S_1[2]\}) = 40$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[1], S_1[3]\}) = 36$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[2], S_1[3]\}) = 46$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[2], S_1[4]\}) = 45$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[1], S_1[2]\}) = 40$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[1], S_1[3]\}) = 42$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[2], S_1[3]\}) = 44$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[3], S_1[4]\}) = 50$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1], S_1[2]\}) = 44$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1], S_1[3]\}) = 36$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1], S_1[4]\}) = 49$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[1], S_1[2]\}) = 40$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[1], S_1[3]\}) = 36$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[2], S_1[3]\}) = 44$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[2], S_1[4]\}) = 42$$
$$\#(P_2 \oplus \{S_4[1], S_1[1], S_1[2]\}) = 50$$
$$\#(P_2 \oplus \{S_4[1], S_1[1], S_1[3]\}) = 45$$
$$\#(P_2 \oplus \{S_4[1], S_1[2], S_1[4]\}) = 54$$
$$\#(P_2 \oplus \{S_4[1], S_1[2], S_1[3]\}) = 56$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[1]\}) = 48$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[2]\}) = 58$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[3]\}) = 53$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[1]\}) = 54$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[2]\}) = 56$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[3]\}) = 60$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[4]\}) = 61$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[1]\}) = 51$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[2]\}) = 62$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[3]\}) = 52$$

$$\#(P_2 \oplus \{(S_3[1] \oplus \{S_2[1]), S_1[1]\}\}) = 56$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[1]\}) = 56$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[2]\}) = 64$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[3]\}) = 60$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[1]\}) = 48$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[2]\}) = 54$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[3]\}) = 51$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[1]\}) = 48$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[2]\}) = 56$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[3]\}) = 52$$
$$\#(P_2 \oplus \{(S_4[1] \oplus \{S_1[1]), S_1[2]\}\}) = 60$$
$$\#(P_2 \oplus \{S_3[1], S_1[1], S_1[2], S_1[3], S_1[5]\}) = 48$$
$$\#(P_2 \oplus \{S_4[1], S_1[1], S_1[2], S_1[3]\}) = 60$$
$$\#(P_2 \oplus \{S_4[1], S_1[1], S_1[2], S_1[4]\}) = 60$$
$$\#(P_2 \oplus \{S_4[1], S_1[2], S_1[3], S_1[4]\}) = 72$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[1], S_1[2], S_1[3]\}) = 48$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[1], S_1[2], S_1[4]\}) = 48$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[2], S_1[3], S_1[4]\}) = 56$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[1], S_1[2], S_1[3]\}) = 48$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[1], S_1[2], S_1[4]\}) = 48$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[2], S_1[3], S_1[4]\}) = 60$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[2], S_1[3], S_1[4]\}) = 56$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[1], S_1[2], S_1[3]\}) = 48$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1], S_1[2], S_1[4]\}) = 56$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1], S_1[3], S_1[4]\}) = 48$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[1], S_1[2]\}) = 60$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[1], S_1[3]\}) = 54$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[2], S_1[3]\}) = 70$$
$$\#(P_2 \oplus \{S_3[1], S_2[1], S_1[2], S_1[4]\}) = 69$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[1], S_1[2]\}) = 60$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[1], S_1[3]\}) = 63$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[1], S_1[4]\}) = 66$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[2], S_1[3]\}) = 68$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[2], S_1[4]\}) = 66$$
$$\#(P_2 \oplus \{S_3[1], S_2[2], S_1[3], S_1[4]\}) = 78$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[1], S_1[2]\}) = 66$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[1], S_1[3]\}) = 54$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[2], S_1[3]\}) = 68$$
$$\#(P_2 \oplus \{S_3[1], S_2[3], S_1[2], S_1[4]\}) = 77$$

$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[1], S_1[2]\}) = 70$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[1], S_1[3]\}) = 63$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[2], S_1[3]\}) = 78$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_2[1]), S_1[2], S_1[4]\}) = 75$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[1], S_1[2]\}) = 60$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[1], S_1[3]\}) = 54$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[2], S_1[3]\}) = 66$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[1]), S_1[2], S_1[4]\}) = 63$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[1], S_1[2]\}) = 60$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[1], S_1[3]\}) = 54$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[2], S_1[3]\}) = 68$$
$$\#(P_2 \oplus \{(S_4[1] \oplus S_1[2]), S_1[2], S_1[4]\}) = 66$$
$$\#(P_2 \oplus \{(S_3[1] \oplus S_1[1]), S_1[1], S_1[2], S_1[3], S_1[4]\}) = 64$$
$$\#(P_2 \oplus \{S_2[1], S_2[1], S_1[1], S_1[2], S_1[3], S_1[4]\}) = 64$$
$$\#(P_2 \oplus \{S_2[1], S_2[2], S_1[1], S_1[2], S_1[3], S_1[4]\}) = 64$$
$$\#(P_2 \oplus \{S_2[1], S_2[3], S_1[1], S_1[2], S_1[3], S_1[4]\}) = 64$$

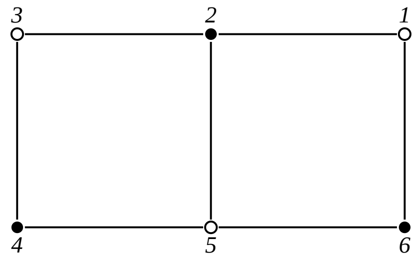
Base configuration: P_3

$$\#(P_3) = 18$$
$$\#(P_3 \oplus S_1[1]) = 21$$
$$\#(P_3 \oplus S_2[1]) = 39$$
$$\#(P_3 \oplus \{S_1[1], S_1[2]\}) = 26$$
$$\#(P_3 \oplus \{S_1[1], S_1[3]\}) = 24$$
$$\#(P_3 \oplus \{S_1[1], S_1[4]\}) = 25$$
$$\#(P_3 \oplus S_3[1]) = 60$$
$$\#(P_3 \oplus \{S_1[1], S_2[1]\}) = 42$$
$$\#(P_3 \oplus \{S_2[1], S_1[2]\}) = 47$$
$$\#(P_3 \oplus \{S_2[1], S_1[3]\}) = 45$$
$$\#(P_3 \oplus \{S_1[1], S_1[4]\}) = 46$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[3]\}) = 32$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[4]\}) = 30$$
$$\#(P_3 \oplus \{S_1[1], S_1[3], S_1[5]\}) = 27$$
$$\#(P_3 \oplus \{S_2[1], S_2[2]\}) = 86$$
$$\#(P_3 \oplus \{S_2[1], S_2[1]\}) = 81$$
$$\#(P_3 \oplus \{S_2[1], S_2[3]\}) = 84$$
$$\#(P_3 \oplus \{S_2[1], S_2[4]\}) = 85$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[2]\}) = 52$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[3]\}) = 48$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[4]\}) = 50$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[3]\}) = 58$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[4]\}) = 54$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[5]\}) = 55$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[6]\}) = 56$$
$$\#(P_3 \oplus \{S_2[1], S_1[3], S_1[4]\}) = 56$$
$$\#(P_3 \oplus \{S_2[1], S_1[3], S_1[5]\}) = 51$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[3], S_1[4]\}) = 40$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[3], S_1[5]\}) = 36$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[4], S_1[5]\}) = 36$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[2], S_1[3]\}) = 64$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[2], S_1[4]\}) = 60$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[2], S_1[5]\}) = 60$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[2], S_1[6]\}) = 64$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[3], S_1[4]\}) = 72$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[3], S_1[5]\}) = 66$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[3], S_1[6]\}) = 70$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[4], S_1[5]\}) = 66$$
$$\#(P_3 \oplus \{S_2[1], S_1[2], S_1[4], S_1[6]\}) = 63$$
$$\#(P_3 \oplus \{S_2[1], S_1[3], S_1[4], S_1[5]\}) = 68$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[3], S_1[4]\}) = 60$$
$$\#(P_3 \oplus \{S_2[1], S_1[1], S_1[3], S_1[5]\}) = 54$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[3], S_1[4], S_1[5]\}) = 48$$
$$\#(P_3 \oplus \{S_1[1], S_1[2], S_1[3], S_1[4], S_1[5], S_1[6]\}) = 64$$
$$\#(P_3 \oplus \{S_1[2] \oplus S_1[6]\}) = 51$$

Base configuration: P_4

$$\#(P_4) = 47$$
$$\#(P_4 \oplus S_1[1]) = 55$$
$$\#(P_4 \oplus S_2[1]) = 102$$
$$\#(P_4 \oplus \{S_1[1], S_1[2]\}) = 68$$
$$\#(P_4 \oplus \{S_1[1], S_1[3]\}) = 63$$
$$\#(P_4 \oplus \{S_1[1], S_1[4]\}) = 65$$
$$\#(P_4 \oplus \{S_1[1], S_1[5]\}) = 64$$

We have now ruled out all configurations built by adjoining points to base configurations with only one polygon subconfiguration. Now we must consider those configurations for which the base configuration has two or more polygonal subconfigurations. The reader may check that the only base configuration with more than one polygonal subconfiguration, which can lie in 1 or 2 dimensions, and with $\#(X) \leq 57$ is the one shown below. We shall call this configuration Q_2.



Base configuration: Q_2

$$\#(Q_2) = 43$$
$$\#(Q_2 \oplus S_1[1]) = 51$$
$$\#(Q_2 \oplus S_1[2]) = 46$$
$$\#(Q_2 \oplus \{S_1[1], S_1[2]\}) = 56$$
$$\#(Q_2 \oplus \{S_1[1], S_1[3]\}) = 60$$
$$\#(Q_2 \oplus \{S_1[1], S_1[4]\}) = 61$$
$$\#(Q_2 \oplus \{S_1[1], S_1[5]\}) = 54$$
$$\#(Q_2 \oplus \{S_1[1], S_1[6]\}) = 56$$
$$\#(Q_2 \oplus \{S_1[1], S_1[3]\}) = 60$$
$$\#(Q_2 \oplus \{S_1[2], S_1[5]\}) = 50$$
$$\#(Q_2 \oplus S_2[1]) = 94$$
$$\#(Q_2 \oplus S_2[5]) = 89$$
$$\#(Q_2 \oplus \{S_1[1], S_1[2], S_1[5]\}) = 60$$

We have now considered all finite configurations X in 1 and 2 dimensions such that $\#(X) \leq 57$. Having not found any finite configuration X_0 in 1 or 2 dimensions such that $\#(X_0) = 57$, we may therefore conclude that 57 is a SPACK-3 number.