Regular Sierpinski Polyhedra[1]

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Introduction: In the paper titled Sierpinski N-Gons [2], Kevin Dennis and Steven Schlicker constructed Sierpinski n-gons in the plane for each positive integer n. In this paper, we extend these constructions to 3-space to build regular Sierpinski polyhedra.

Background: The process we will use to consturct Sierpinski polyhedra is the following (see [1] and [2] for more details). Let $\{x_1, x_2, \cdots, x_n\} \subset \mathbb{R}^3$ with $x_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$ the vertices of a regular polyhedron, A_0. For $r>0$, we define $\omega_i\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{1}{r}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{r-1}{r}\begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}$ for $1 \le i \le n$. Define $A_{m,i}(r) = \omega_i(A_{m-1}(r))$ and let $A_{m+1}(r) = \bigcup_{i=1}^{n} A_{m,i}(r)$.

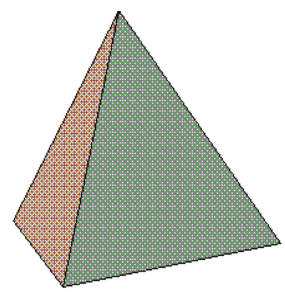


FIGURE 1

For example, let A_0 be the regular tetrahedron, as in figure 1, and let $r = 2$. Then, ω_i, when applied to A_0 contracts A_0 by a factor of 2 and then translates the image of A_0 so that the *i*th vertices of A_0 and the image of A_0 coincide. Then $A_{1,i}$ is the set of all points half way between any point in A_0 and x_i, or $A_{1,i}$ is a tetrahedron half the size of the original translated to the *i*th vertex of the original. A_0 and A_1 are shown in figures 1 and 2 respectively.

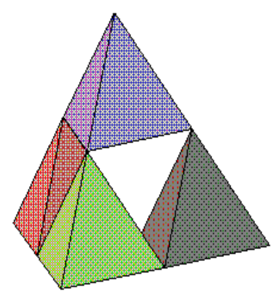


FIGURE 2
r=2

We can continue this procedure replacing A_0 with A_1. For i=1,2, 3,or 4, let $A_{2,i} = \omega_i(A_1)$ and let $A_2 = \bigcup_{i=1}^{n} A_{2,i}$. A_2 is shown in figure 3. Again, we can continue this procedure, each time replacing A_i with A_{i+1}. A_3 is shown in figure 4.

[1] This project was supported by a grant from the Summer Undergraduate Research Program at Grand Valley State University

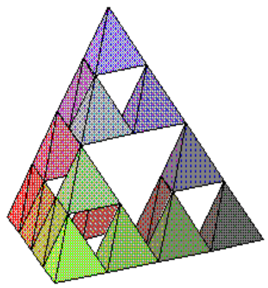


FIGURE 3
A_2 with $r=2$

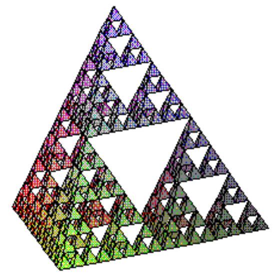


FIGURE 4
A_3 with $r=2$

Note that, for $r = 2$ the A_i consist of tetrahedra that just touch each other. For smaller values of r the tetrahedra overlap and for larger r the tetrahedra are disjoint. See figures 5 (A_1, r=1.5), and 6 (A_1, r=3).

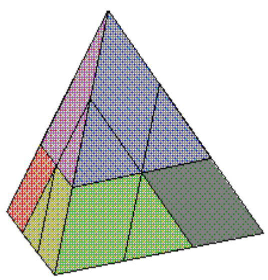


FIGURE 5
A_1 with $r = 1.5$

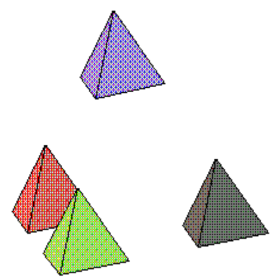


FIGURE 6
A_1 with $r = 3$

If we take the limit as i approaches infinity for the just touching value of r, the resulting figure is the Sierpinski tetrahedron. This algorithm for building the Sierpinski tetrahedron is called the deterministic algorithm.

Sierpinski polyhedra: In the above discussion, there seems to be no reason why we should restrict ourselves to looking at only the regular tetrahedron. Why not consider the other four regular polyhedra (hexahedron, octahedron, dodecahedron, icosahedron)? In what follows we will determine, for each regular polyhedra, the specific value of r that makes $A_m(r)$ just touching and we will determine the fractal dimension of each of the resulting Sierpinski polyhedra. As a consequence of Theorem 3, p. 184 from [1], the fractal dimension of a Sierpinski polyhedra with n vertices and a scale factor of r is $\frac{\ln(n)}{\ln(r)}$. See [2] for a proof. For example, the fractal dimension of the Sierpinski tetrahedron is $\frac{\ln(4)}{\ln(2)} = 2$.

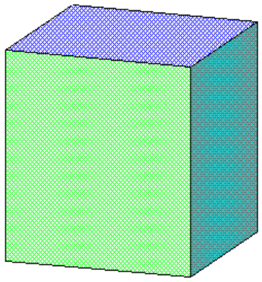


FIGURE 7

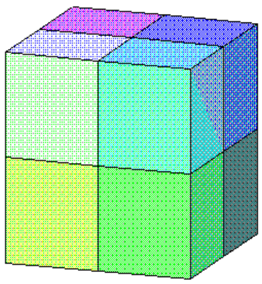


FIGURE 8

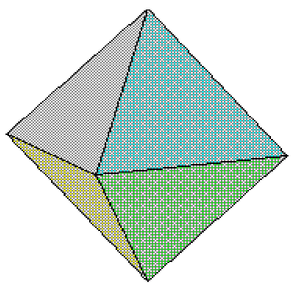


FIGURE 9

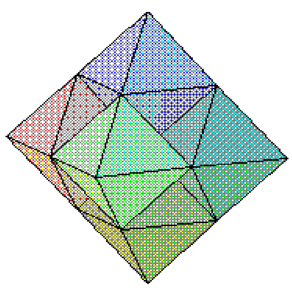


FIGURE 10

We begin construction of our Sierpinski polyhedra with the regular hexahedron, or more commonly, the cube. Figure 7 shows a regular hexahedron. By inspection, we can see that the cube has a scale factor of 2. This is illustrated in figure 8. It appears that the Sierpinski hexahedron is just the cube itself. The fractal dimension of the Sierpinski hexahedron is $\dfrac{\ln(8)}{\ln(2)} = 3$ as we expect.

Next, we look at the regular octahedron. Figure 9 shows a regular octahedron. By inspection, we note that each face of the Sierpinski octahedron will be a Sierpinski triangle and thus the regular octahedron has a scale factor of 2. This can be seen in figure 10. Thus, the fractal dimension of the regular octahedron is $\dfrac{\ln(6)}{\ln(2)} \approx 2.585$.

The scale factors for the regular dodecahedron and the regular icosahedron are more difficult to find, and we find those scale factors in the remainder of this paper.

Background information: The following regular pentagons which are inscribed in a circle of radius *r* will aid us in determining the scale factor for the dodecahedron and the icosahedron. Think of these pentagons as faces of a dodecahedron. In figure 11, we let the point *a* represent the center of the circle. Let d_1 be the length of any side of the face. We let $\overline{ac}$ be the perpendicular bisector of $\overline{eb}$. A little trigonometry shows

$$r = \frac{d}{2\sin\left(\frac{\pi}{5}\right)},\ s = r\cos\left(\frac{\pi}{5}\right),\ \text{and}\ s = \frac{d\cos\left(\pi/5\right)}{2\sin\left(\pi/5\right)}. \qquad (1)$$

In figure 12, we let h represent the length between two verticies which are not adjacent. We see that $h = 2d\cos\left(\frac{\pi}{5}\right)$.

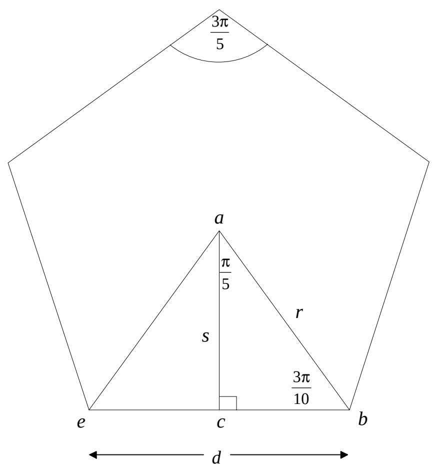


FIGURE 11

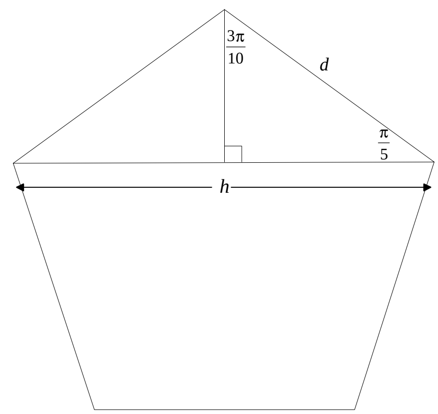
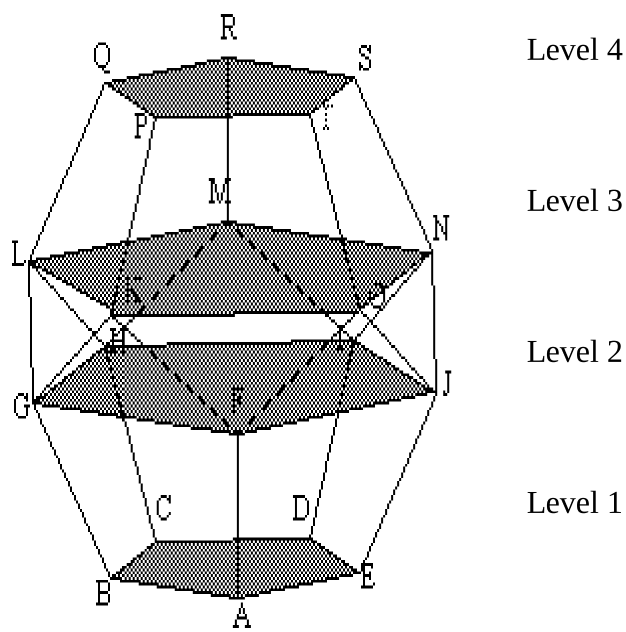


FIGURE 12

Regular dodecahedron: The regular dodecahedron has 12 faces all of which are regular pentagons. It has 20 vertices; three edges meet at each vertex. Figure 13 illustrates a regular dodecahedron.

FIGURE 13



The vertices of the dodecahedron can be thought of as four sets of five points each which lie on pentagons parallel to the *xy* plane. Each pentagon parallel to the *xy* plane is assigned a level as is shown in figure 13. We label the points (x_{ij}, y_{ij}, z_i) where i represents the level and j represents the location of that point on that level. More specifically, on level one, we will start with

$A = (x_{11}, y_{11}, z_1)$. We will then move in a clockwise direction when labeling the remaining points. Thus $B = (x_{12}, y_{12}, z_1)$, $C = (x_{13}, y_{13}, z_1)$, $D = (x_{14}, y_{14}, z_1)$, and $E = (x_{15}, y_{15}, z_1)$.

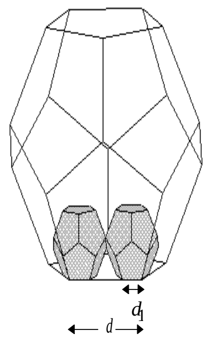


FIGURE 14

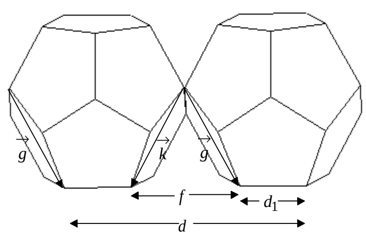
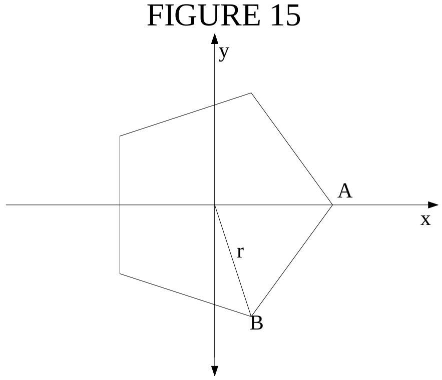


FIGURE 15



FUGURE 16

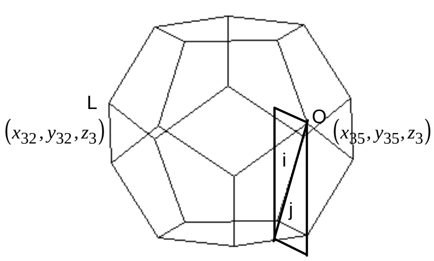


FIGURE 17

The remaining points on levels 2, 3, and 4 are labeled in a similar fashion.

We will now show how the scale factor can be found for the dodecahedron. In figure 14 we can see how two small dodecahedra will fit into a large dodecahdron in the just touching case.

We label the just touching dodecahedron as in figure 15. The scale factor in the just touching case will then be $\frac{d}{d_1}$.

To find $\frac{d}{d_1}$, we will find the angle between vectors $\vec{g}$ and $\vec{k}$. We will then be able to find *f*. Since all dodecahedra constructed are similar, it suffices to work with only the large dodecahedron. To find $\vec{g}$ and $\vec{k}$ we need to explicitly determine the points A, B, L, and O as labeled in figure 13. We will assume the base is on the *xy*-plane with center at (0,0) and is inscribed in a circle with radius *r*. The points B and A are illustrated in figure 16.

We can see from figure 16 that A = (*r*, 0, 0) and $B = \left(r\cos\left(-\frac{2\pi}{5}\right), r\sin\left(-\frac{2\pi}{5}\right), 0\right)$. Now we must look at the third level for our other two points. Triangle i, represented in figures 17 and 18, will aide us in finding L and O. We let the dihedral angle (the angle between two faces) of the decahedron be represented by θ. Therefore, angle qtO would be $\theta - \frac{\pi}{2}$. We can see

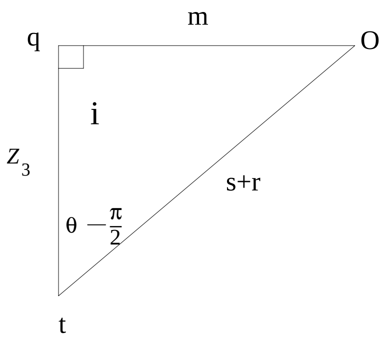


FIGURE 18

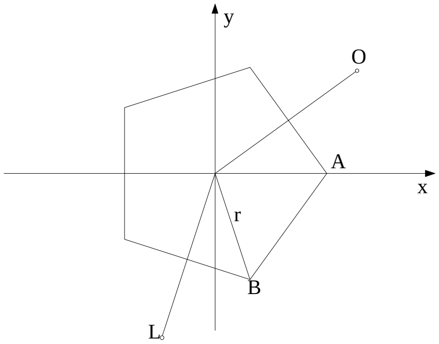


FIGURE 19

from figure 11 that $\left|\overline{tO}\right| = s + r$. Then,

$z_3 = (s+r)\cos\left(\theta - \frac{\pi}{2}\right)$. We let m be the length of the segment $\overline{qO}$. It follows that $m = (s+r)\sin\left(\theta - \frac{\pi}{2}\right)$. Note that s + m is the distance that the point O, as well as L, is away from the *z*-axis.

We now project the dodecahedron from figure 17 into the *xy*-plane. We see in figure 19 that

$$L = \left((s+m)\cos\left(-\frac{3\pi}{5}\right), (s+m)\sin\left(-\frac{3\pi}{5}\right), (s+r)\cos\left(\theta - \frac{\pi}{2}\right) \right)$$

and

$$O = \left((s+m)\cos\left(\frac{\pi}{5}\right), (s+m)\sin\left(\frac{\pi}{5}\right), (s+r)\cos\left(\theta - \frac{\pi}{2}\right) \right)$$N

ow that we know points B, A, L, and O, we can find the vectors $\vec{g}$ and $\vec{k}$ from figure 18. We know that

$$\vec{k} = A - O = \left\langle r - (s+m)\cos\left(\frac{\pi}{5}\right), -(s+m)\sin\left(\frac{\pi}{5}\right), -(s+r)\cos\left(\theta - \frac{\pi}{2}\right) \right\rangle$$ and

$$\vec{g} = B - L = \left\langle r\cos\left(-\frac{2\pi}{5}\right) - (s+m)\cos\left(-\frac{3\pi}{5}\right), r\sin\left(-\frac{2\pi}{5}\right) - (s+m)\sin\left(-\frac{3\pi}{5}\right), -(s+r)\cos\left(\theta - \frac{\pi}{2}\right) \right\rangle$$

Next, we use Maple to find the dot product of $\vec{g}$ and $\vec{k}$. We will let β be the angle between $\vec{g}$ and $\vec{k}$. From figure 12, we see that $\left|\vec{k}\right| = \left|\vec{g}\right| = h$ Then, since $\vec{k}\cdot\vec{g} = \left|\vec{k}\right|\left|\vec{g}\right|\cos(\beta)$, we know that $\cos(\beta) = \frac{\vec{k}\cdot\vec{g}}{h^2}$. We note from figure 16 that $\sin\left(\frac{\beta}{2}\right) = \frac{f/2}{h}$ and thus

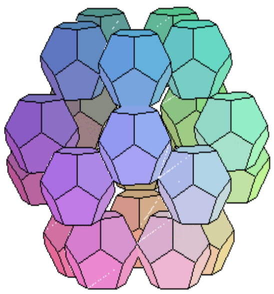
$$f = 2h\sin\left(\frac{\beta}{2}\right) = 2h\sqrt{\frac{1-\cos(\beta)}{2}}$$. So, $$d = 2d_1 + 2h\sqrt{\frac{1-\cos(\beta)}{2}} = 2d_1 + 2d_1\cos\left(\frac{\pi}{5}\right)\sqrt{\frac{1-\cos(\beta)}{2}}$$.

Finally, our scale factor is $\frac{d}{d_1}$. To determine a value for this scale factor, we substitute $\theta = \frac{3497}{5400}\pi$ for the dihedral angle [3, p. 53]. For $\frac{d}{d_1}$ Maple gives

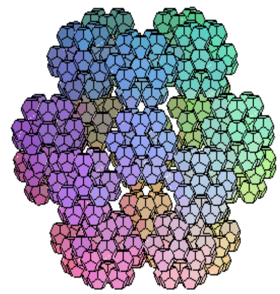
$$2+\frac{1}{8}\sqrt{14-6\sqrt{5}-4\cos\left(\frac{1903}{5400}\pi\right)\sqrt{5}+20\cos\left(\frac{1903}{5400}\pi\right)+30\cos\left(\frac{1903}{5400}\pi\right)^2+10\cos\left(\frac{1903}{5400}\pi\right)^2\sqrt{5}}\,\sqrt{5}$$
$$+\frac{1}{8}\sqrt{14-6\sqrt{5}-4\cos\left(\frac{1903}{5400}\pi\right)\sqrt{5}+20\cos\left(\frac{1903}{5400}\pi\right)+30\cos\left(\frac{1903}{5400}\pi\right)^2+10\cos\left(\frac{1903}{5400}\pi\right)^2\sqrt{5}}$$

or approximately 3.618107807.

The following figures illustrate an emerging Sierpinski dodecahedron.



A_1

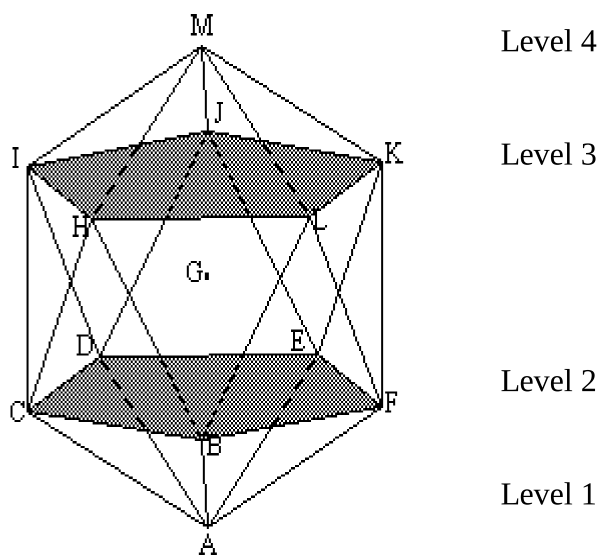


A_2

The fractal dimension of a Sierpinski dodecahedron is $\dfrac{\ln(20)}{\ln\left(d/d_1\right)} \approx 2.32958$.

Regular icosahedron: The regular icosahedron is another interesting polyhedron. It has 20 faces all of which are equilateral triangles. It has 12 vertices each with 5 edges meeting. Figure 20 shows a regular icosahedron. We will assume it is inscribed in the unit sphere.

FIGURE 20



In figure 20, we see that each vertex of the icosahedron lies on one of four "levels". The first level contains only one point, namely A = (0, 0, -1). As with level 1, level 4 contains only one vertex, namely M = (0, 0, 1). We will label the points $\left(x_{ij}, y_{ij}, z_i\right)$ as we did with the dodecahedron. Lastly, G = (0, 0, 0) is the center of the icosahedron. We note here that levels 2 and 3 are regular pentagons and are represented previously in figures 11, and 12. We let *r* be the radius of these pentagons.

We will now show how the scale factor can be found for the icosahedron. Figure 21 will show us how two icosahedra fit into a larger icosahedron when the smaller ones are just touching. We will view figure 21 as if we are looking down on the top of the icosahedron.

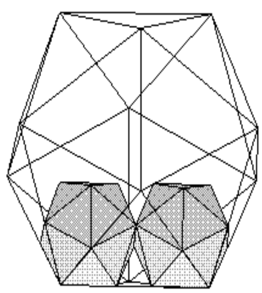


FIGURE 21

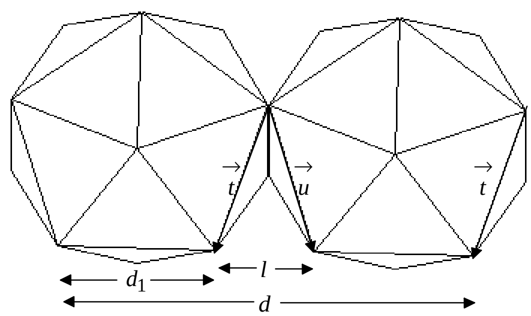


FIGURE 22

From figure 22, our scale factor is $\frac{d}{d_1}$. We can find $\frac{d}{d_1}$ by finding the angle between $\vec{u}$ and $\vec{t}$ which will allow us to find *l*. We know the small icosahedra are similar to the larger one. So, it suffices to find the corresponding points on the large icosahedron.

We need to determine the four points I, H, L, and K as labeled in figure 20. In order to do so, we first must find *r*. Let $\left(x_3, y_3, z_3\right)$ represent an arbitrary vertex on level three. Since our icosahedron is in the unit sphere, $x_3^2 + y_3^2 + z_3^2 = 1$ or $r^2 + z_3^2 = 1$. If follows directly that

$$z_3 = \sqrt{1 - r^2}\,. \tag{2}$$

Since *d* is the distance from (0, 0, 1) to $\left(x_3, y_3, z_3\right)$

$$d^2 = x_3^2 + y_3^2 + \left(z_3 - 1\right)^2. \tag{3}$$

Using (1), (2) and (3), we can use substitution and algebra to determine that

$r = \sqrt{\dfrac{4\sin^2\left(\frac{\pi}{5}\right)}{4\sin^4\left(\frac{\pi}{5}\right)}}$. Maple, gives us $r = 2\dfrac{\sqrt{3-\sqrt{5}}\sqrt{2}}{5-\sqrt{5}}$. (Note that Maple expresses r in radical form. To do this one can solve the equation sin(5x)=0 by expanding sin(5x) using the standard angle sum formulas for the sine and cosine (found in any book on trigonometry). One of the solutions to sin(5x)=0 will be $\sin\left(\frac{\pi}{5}\right)$.) Now, we can find I, H, L, and K. We know that each of these points has the same z-value, so we need only look at their x and y-values. Now we will project level 3 into the xy plane resulting in figure 23.

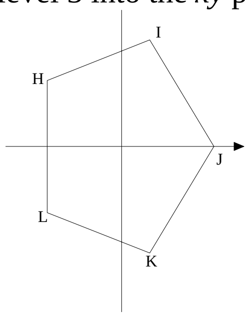


FIGURE 23

In figure 23, I = $\left(r\cos\left(\frac{2\pi}{5}\right), r\sin\left(\frac{2\pi}{5}\right)\right)$,

H = $\left(r\cos\left(\frac{4\pi}{5}\right), r\sin\left(\frac{4\pi}{5}\right)\right)$,

L = $\left(r\cos\left(\frac{6\pi}{5}\right), r\sin\left(\frac{6\pi}{5}\right)\right)$, and

K = $\left(r\cos\left(\frac{8\pi}{5}\right), r\sin\left(\frac{8\pi}{5}\right)\right)$. Then,

$\vec{u} = \text{L - K} = \left\langle r\cos\left(\frac{6\pi}{5}\right) - r\cos\left(\frac{8\pi}{5}\right), r\sin\left(\frac{6\pi}{5}\right) - r\sin\left(\frac{8\pi}{5}\right)\right\rangle$ and

$\vec{t} = \text{H - I} = \left\langle r\cos\left(\frac{4\pi}{5}\right) - r\cos\left(\frac{2\pi}{5}\right), r\sin\left(\frac{4\pi}{5}\right) - r\sin\left(\frac{2\pi}{5}\right)\right\rangle$.

Now we determine the cosine of the angle, β between $\vec{u}$ and $\vec{t}$ by using the dot product. We also note that the magnitude of both $\vec{u}$ and $\vec{t}$ is d_1. Thus, we have

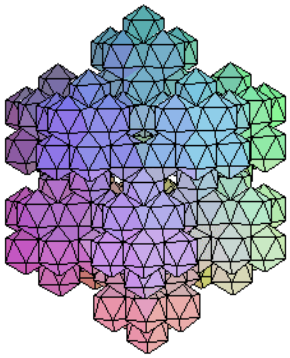
$$\vec{u}\cdot\vec{t} = \left|\vec{u}\right|\left|\vec{t}\right|\cos(\beta) = d_1^2\cos(\beta).$$

Now, $\sin\left(\frac{\beta}{2}\right) = \dfrac{l/2}{d_1}$ or $2d_1\sin\left(\frac{\beta}{2}\right) = l$. So

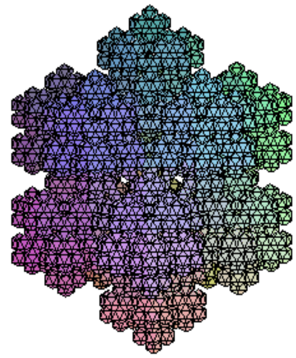
$$d = 2d_1 + l = 2d_1 + 2d_1\sin\left(\frac{\beta}{2}\right) = 2d_1 + 2d_1\sqrt{\frac{1-\cos(\beta)}{2}}.$$

Since $d_1 = 2r\sin\left(\frac{\pi}{5}\right)$, we can find $\cos(\beta)$. Maple gives us $\frac{3}{2} + \frac{\sqrt{5}}{2} \approx 2.618$ for the scale factor for the regular icosahedron.

The following figures illustrate an emerging Sierpinski icosahedron.



A_2



A_3

The fractal dimension of a Sierpinski icosahedron is $\dfrac{\log(12)}{\log\left(d/d_1\right)} \approx 2.581926$.

References

[1] Michael Barnsley, Fractals Everywhere, Academic Press Inc., San Diego, 1988

[2] Kevin Dennis and Steven Schlicker, *Sierpinski N-gons*, Pi Mu Epsilon Journal, spring 1995, V. 10, N. 2, p.81-90

[3] P. Pearce and S. Pearce, Polyhedra Primer, Van Nostrand Reinhold Co., 1978.