Preface

Mathematical Reasoning: Writing and Proof is designed to be a text for the first course in the college mathematics curriculum that focuses on the formal development of mathematics. The primary goals of the text are as follows:

- To help students learn how to read and understand mathematical definitions and proofs;
- To help students learn how to construct mathematical proofs;
- To help students learn how to write mathematical proofs according to accepted guidelines so that their work and reasoning can be understood by others; and
- To provide students with some mathematical material that will be needed for their further study of mathematics.

This type of course is becoming a standard part of the mathematics major at most colleges and universities. It is often referred to as a "transition course" from the calculus sequence to the upper level courses in the major. The transition is from the problem-solving orientation of calculus to the more abstract and theoretical upper-level courses. This is needed today because the principal goals of most calculus courses are developing students' understanding of the concepts of calculus and improving their problem-solving skills. Consequently, most students complete their study of calculus without seeing a formal proof or having constructed a proof of their own. This is in contrast to many upper-level mathematics courses, where the emphasis is on the formal development of abstract mathematical ideas, and the expectations are that students will be able to read and understand proofs and to construct and write coherent, understandable mathematical proofs.

Important Features of the Book

Mathematical Reasoning was written to assist students with the transition from calculus to upper level mathematics courses. Students should be able to use this text with a background of one semester of calculus. Following are some of the important ways this text will help with this transition.

1. **Emphasis on Writing in Mathematics**

   The issue of writing mathematical exposition is addressed throughout the book. Guidelines for writing mathematical proofs are incorporated into the text. These guidelines are introduced as needed and begin in Chapter 1. Appendix A contains a summary of all the guidelines for writing mathematical proofs that are introduced in the text. In addition, every attempt has been made to ensure that each proof presented in this text is written according to these guidelines in order to provide students with examples of well-written proofs.

2. **Instruction in the Process of Constructing Proofs**

   One of the primary goals of this book is to develop students' abilities to construct mathematical proofs. Another goal is to develop their abilities to write the proof in a coherent manner that conveys an understanding of the proof to the reader. These are two distinct skills.

   Instruction on how to write proofs begins in Section 1.2 and is developed further in Chapter 3. In addition, Chapter 5 is devoted to developing students' abilities to construct proofs using mathematical induction. Students are taught to organize their thought processes when attempting to construct a proof with a so-called know-show table. (See Sections 1.2 and 3.1.) Students use this table to work backward from what it is they are trying to prove while at the same time working forward from the assumptions of the problem.

3. **Emphasis on Active Learning**

   One of the underlying premises of this text is that the best way to learn and understand mathematics is to be actively involved in the learning process. However, it is unreasonable to expect students to go out and learn mathematics on their own. Students actively involved in learning mathematics need appropriate materials that will provide guidance and support in their learning of mathematics. This text provides these by incorporating two or three Preview Activities for each section and some activities within each section based on the material in that section. These activities can be done individually or in a collaborative learning setting, where students work in groups to brainstorm, make conjectures, test each others' ideas, reach consensus, and, it is hoped, develop sound mathematical arguments to support their work.

The Preview Activities at the beginning of each section should be completed by the students prior to the classroom discussion of the section. The purpose of the Preview Activities is to prepare students to participate in the classroom discussion of the section. Some Preview Activities will review prior mathematical work that is necessary for the new section. This prior work may contain material from previous mathematical courses or it may contain material covered earlier in this text. Other preview activities will introduce new concepts and definitions that will be used when that section is discussed in class.

In addition to the Preview Activities, each section of the text contains two or three activities related to the material contained in that section. These activities can be used for in-class group work or can be assigned as homework in addition to the exercises at the end of each section.

Content and Organization

Mathematical content is needed as a vehicle for learning how to construct and write proofs. The mathematical content for this text is drawn primarily from elementary number theory, including congruence arithmetic; elementary set theory; functions, including injections, surjections, and the inverse of a function; relations and equivalence relations; further topics in number theory such as greatest common divisors and prime factorizations; and cardinality of sets, including countable and uncountable sets. This material was chosen because it can be used to illustrate a broad range of proof techniques and it is needed as a prerequisite for many upper level mathematics courses.

The chapters in the text can roughly be divided into the following (possibly overlapping) classes:

- Constructing and Writing Proofs: Chapters 1, 3, and 5
- Content: Chapters 4, 6, 7, 8, and 9
- Logic: Chapter 2

The first chapter sets the stage for the rest of the book. It introduces the writing guidelines, discusses conditional statements, and begins instruction in the process of constructing a direct proof of a conditional statement. This is not meant to be a thorough introduction to methods of proof. Before this is done, it is necessary to introduce the students to the parts of logic that are needed to aid in the construction of proofs. This is done in Chapter 2.

Students need to learn some logic and gain experience in the traditional language and proof methods used in mathematics. Since this is a text that deals with constructing and writing mathematical proofs, the logic that is presented in Chapter 2 is intended to aid in the construction of proofs. The goals are to provide students with a thorough understanding of conditional statements, quantifiers, and logical equivalencies. Emphasis is placed on writing correct and useful negations of statements, especially those involving quantifiers. The logical equivalencies that are presented provide the logical basis for some of the standard proof techniques, such as proof by contrapositive, proof by contradiction, and proof using cases.

The standard methods for mathematical proofs are discussed in detail in Chapter 3. The mathematical content that is introduced to illustrate these proof methods is some elementary number theory, including congruence arithmetic. These concepts are used consistently throughout the text as a way to demonstrate ideas in direct proof, proof by contrapositive, proof by contradiction, proof by cases, and proofs using mathematical induction. This gives students a strong introduction to an important mathematical idea, while providing the instructor a consistent reference point and an example of how mathematical notation can greatly simplify a concept.

In Chapter 4, we take a break from introducing new proof techniques. Concepts of set theory are introduced, and the methods of proof studied in Chapter 3 are used to prove results about sets and operations on sets. The idea of an "element-chasing proof" is introduced in Section 4.2.

The three sections of Chapter 5 are devoted to proofs using mathematical induction. Again, the emphasis is not only on understanding mathematical induction but also on developing the ability to construct and write proofs that use mathematical induction.

The last four chapters are considered "mathematical content" chapters. Chapter 6 provides a thorough study of functions. The idea is to begin with a review of functions from previous courses so that students have a base from which to work. This notion of function is then extended to the general definition of function. Various proof techniques are employed in the study of injections, surjections, composition of functions, and inverses of functions.

Chapter 7 introduces the concepts of relations and equivalence relations. Section 7.4 is included to provide a link between the concept of an equivalence relation and the number theory that has been discussed throughout the text.

Chapter 8 continues the study of number theory. The highlights include problems dealing with greatest common divisors, prime numbers, the Fundamental Theorem of Arithmetic, and linear Diophantine equations.

Finally, Chapter 9 deals with further topics in set theory, focusing on cardinality, finite sets, countable sets, and uncountable sets.

A standard one-semester course in constructing and writing proofs should cover the first six chapters of the text and at least one of Chapter 7, Chapter 8, or Chapter 9. A class consisting of well-prepared and motivated students could cover two of the last three chapters. In addition, there are a few options that an instructor could choose to tailor the course to her or his needs. For example,

- Chapter 5 can be covered before Chapter 4 if it is desired to cover all methods of proof before beginning the "content" portion of the course. The only part of Chapter 5 that would need to be skipped is the material in Section 5.2 dealing with the cardinality of the power set. If desired, this material could be included when the power set is discussed in Chapter 4.

- Instructors who would like to cover topics in both Chapters 7 and 8 can omit a few selected sections from earlier chapters. Although it is an important and interesting section, Section 5.3 is not used in the remainder of the book. The same is true for Section 6.5. Finally, Section 3.5 can be skipped as long as the concept of a constructive proof is discussed during other parts of the course.

Supplementary Materials for the Instructor

The instructor's manual for this text includes suggestions on how to use the text, how to incorporate writing into the course, and how to use the preview activities and activities. The manual also includes solutions for all of the preview activities, activities, and exercises. In addition, for each section, there is a description of the purpose of each preview activity and how it is used in the corresponding section, and there are suggestions about how to use each activity in that section. The intention is to make it as easy as possible for the instructor to use the text in an active learning environment. These activities can also be used in a more traditional lecture-discussion course. In that case, some of the activities would be discussed in class or assigned as homework.

The instructor's manual is available by contacting the editorial offices at Prentice Hall in Upper Saddle River, NJ or by emailing a request to george_lobell@prenhall.com.

In addition, Adobe Acrobat (pdf) files are available to the instructor to assist the instructor in posting solutions to a course web page or distributing printed solutions to students. For each section, there is a file containing the solutions of the preview activities, and for each activity in the text, there is a file containing the solutions for that activity.

Instructors can contact the author through his email address for access to the files.

Acknowledgments

This text has benefited greatly from the comments of students who have used preliminary versions of the text and from ideas and insights of the reviewers of the text. I would like to thank the following reviewers:

> Frank Bäuerle, University of California, Santa Cruz; Michael C. Berg, Loyola Marymount University; Lifeng Ding, Georgia State University; Jeffrey Ehme, Spelman College; Christopher P. Grant, Brigham Young University; Joel Iiams, University of North Dakota; Robert Jajcay, Indiana State University; Iraj Kalantari, Western Illinois University; and William J. Keane, Boston College.

I would also like to express my sincere gratitude to my colleagues at Grand Valley State University who have used the text and have supported my work on this text. This includes Professors David Austin, Ed Aboufadel, Karen Brown, Will Dickinson, Philip Pratt, Steve Schlicker, Jody Sorensen, Clark Wells, and Pamela Wells. Special thanks go to Prof. Karen Novotny, who helped me immensely in writing the first draft of the text, Prof. Matt Boelkins, who carefully read the completed text and offered numerous helpful suggestions, and Prof. Larry King of the University of Michigan-Flint, who used an early version of the manuscript and made several useful suggestions for further development of the text.

Finally, a very special thank you goes to my wife, Karen, and my daughter, Laura, for their patience, understanding, and encouragement during the long time it took to develop this text.

Comments about the text and suggestions for improving it are welcome.

Ted Sundstrom
sundstrt@gvsu.edu