Applying Counting Techniques To Sudoku

Shelly Smith

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7	4	8	5	3	1	6	9	2
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1	6	9	4	8	5	3	2	7
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June 15, 2009

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How many ways can we put the integers 1 through 5 in the spaces below if order matters?

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- Permutations with fixed entries

  $$4! = 24$$

  __ $\underline{2}$ __ __ __

Multiple fixed entries

- Inclusion-Exclusion 2 **or** 5 are fixed entries

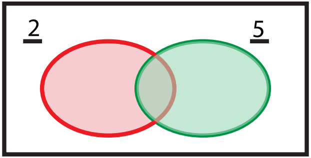
$\_\ 2\ \_\ \_\ \_$ **or** $\_\ \_\ \_\ \_\ 5$

Multiple fixed entries

- Inclusion-Exclusion $\quad$ 2 **or** 5 are fixed entries

$\_\ \underline{2}\ \_\ \_\ \_$ **or** $\_\ \_\ \_\ \_\ \underline{5}$

$$\begin{aligned}|\underline{2} \cup \underline{5}| &= |\underline{2}| + |\underline{5}| - |\underline{2} \cap \underline{5}| \\ &= 4! + 4! - 3! \\ &= 42\end{aligned}$$



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$$\begin{aligned} 5! - |\underline{1} \cup \underline{2} \cup \underline{3} \cup \underline{4} \cup \underline{5}| &= 5! - (5 \cdot 4! - 10 \cdot 3! + 10 \cdot 2! - 5 \cdot 1! + 1 \cdot 0!) \\ &= 120 - 5 \cdot 24 + 10 \cdot 6 - 10 \cdot 2 + 5 \cdot 1 - 1 \cdot 1) \\ &= 44 \end{aligned}$$

Permutations with forbidden positions

- Recall our first permutation.

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$$5! - r_1 4! + r_2 3! - r_3 2! + r_4 1! - r_5 0!$$

where r_i is the number of ways to place i digits in forbidden positions

Equivalence Relations

A relation R on a set S is a subset of $S \times S$.
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Symmetric. For all $a, b \in S$, if $a \sim b$, then $b \sim a$.

Transitive. For all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Sudoku

Block

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1	6	9	4	8	5	3	2	7
3	2	5	1	9	7	8	4	6
4	8	7	3	2	6	9	5	1

Band

9	5	3	2	1	4	7	6	8
2	7	6	8	5	3	4	1	9
8	1	4	6	7	9	2	3	5
7	4	8	5	3	1	6	9	2
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5	3	2	9	6	8	1	7	4
1	6	9	4	8	5	3	2	7
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Stack

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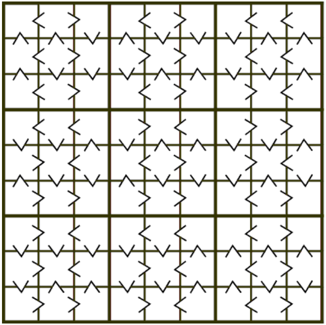
An Equivalence Relation on Sudoku Boards

Definition

(Russell and Jarvis) Let A and B be two Sudoku boards, then $A \sim B$ if and only if A can be transformed into B using one or more of the following operations:

- Permute the nine digits
- Permute the three stacks
- Permute the three bands
- Permute the three columns within a stack
- Permute the three rows within a band
- Reflect across a diagonal, vertical or horizontal axis
- 0°, 90°, 180° or 270° clockwise rotation

Greater/less Sudoku



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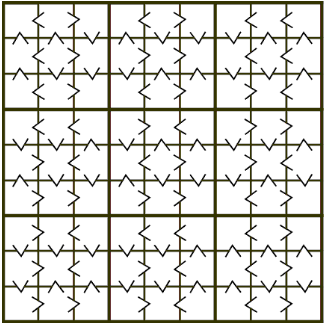


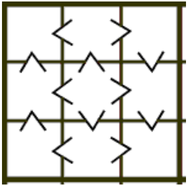
Big Questions

- How can we create a Greater/less Sudoku puzzle?
- How many Greater/less Sudoku puzzles are there?

Greater/less Sudoku

Starting Small

- Start with one block.
  There are $9! = 362880$ ways to fill in the 9 digits, but only $2^{12} = 4096$ ways to choose the 12 inequalities.



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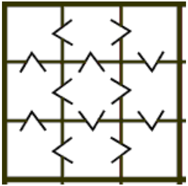
- Define an equivalence relation on the set of permutations, on the set on inequalities.



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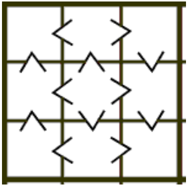
- Combine blocks that will satisfy the conditions of Sudoku.



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- Define an equivalence relation on the set of permutations, on the set on inequalities.
- Combine blocks that will satisfy the conditions of Sudoku.
- Start with smaller versions of Greater/less Sudoku.