Applying Counting Techniques To Sudoku

Shelly Smith

June 15, 2009
How many ways...?

How many ways can we put the integers 1 through 5 in the spaces below if order matters?

____  ____  ____  ____  ____  ____
How many ways...?

How many ways can we put the integers 1 through 5 in the spaces below if order matters?

___ ___ ___ ___ ___ ___

▶ Permutations

4 2 1 5 3
How many ways...?

How many ways can we put the integers 1 through 5 in the spaces below if order matters?

___  ___  ___  ___  ___

- Permutations

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

\[4 \quad 2 \quad 1 \quad 5 \quad 3\]
How many ways can we put the integers 1 through 5 in the spaces below if order matters?

___  ___  ___  ___  ___

- Permutations
  \[5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120\]
  \[
  \begin{array}{c}
  \underline{4} \\
  \underline{2} \\
  \underline{1} \\
  \underline{5} \\
  \underline{3}
  \end{array}
  \]
- Permutations with fixed entries
  \[
  \begin{array}{c}
  \underline{2} \\
  \underline{2} \\
  \underline{2} \\
  \underline{2} \\
  \underline{2}
  \end{array}
  \]
How many ways can we put the integers 1 through 5 in the spaces below if order matters?

___ ___ ___ ___ ___

- **Permutations**
  
  \[
  5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120
  \]
  
  4 2 1 5 3

- **Permutations with fixed entries**
  
  \[
  4! = 24
  \]
  
  __ 2 __ __ __
Multiple fixed entries

- Inclusion-Exclusion

2 or 5 are fixed entries

___ 2 ___ ___ ___ or ___ ___ ___ ___ 5

2 5

4! + 4! − 3! = 42
Multiple fixed entries

- Inclusion-Exclusion: 2 or 5 are fixed entries

\[
|2 \cup 5| = |2| + |5| - |2 \cap 5| \\
= 4! + 4! - 3! \\
= 42
\]
Mixing it all up

A **derangement** is a permutation with **no** fixed entries. No digit is in its natural position.

\[
3\quad 1\quad 4\quad 5\quad 2
\]
A derangement is a permutation with no fixed entries. No digit is in its natural position.  \[3\ 1\ 4\ 5\ 2\]

Counting derangements is difficult, so we will count permutations with fixed entries and subtract the result from 5!. 
A derangement is a permutation with no fixed entries. No digit is in its natural position. 

\[ 3 \ 1 \ 4 \ 5 \ 2 \]

Counting derangements is difficult, so we will count permutations with fixed entries and subtract the result from $5!$.

\[
5! - |1 \cup 2 \cup 3 \cup 4 \cup 5| = 5! - (5 \cdot 4! - 10 \cdot 3! + 10 \cdot 2! - 5 \cdot 1! + 1 \cdot 0!)
\]
\[
= 120 - 5 \cdot 24 + 10 \cdot 6 - 10 \cdot 2 + 5 \cdot 1 - 1 \cdot 1
\]
\[
= 44
\]
Recall our first permutation.

\[
\begin{array}{cccccc}
4 & 2 & 1 & 5 & 3 \\
\end{array}
\]
Recall our first permutation.

How many permutations don’t have any digits in the same position as the first permutation?

\[
\begin{align*}
\text{where } r_i & \text{ is the number of ways to place } i \text{ digits in forbidden positions} \\
5! - r_1 4! + r_2 3! - r_3 2! + r_4 1! - r_5 0! & = 4 \cdot 2 \cdot 1 \cdot 5 \cdot 3 \\
1 \cdot 4 \cdot 2 \cdot 3 \cdot 5 & = 120
\end{align*}
\]
Recall our first permutation.

How many permutations don’t have any digits in the same position as the first permutation?

This is the same as the number of derangements: 44.
Recall our first permutation.

How many permutations don’t have any digits in the same position as the first permutation?

This is the same as the number of derangements: 44.

What if we want a third permutation with forbidden positions?
Permutations with forbidden positions

- Recall our first permutation.
- How many permutations don’t have any digits in the same position as the first permutation?
- This is the same as the number of derangements: 44.
- What if we want a third permutation with forbidden positions?

\[ 5! - r_14! + r_23! - r_32! + r_41! - r_50! \]

where \( r_i \) is the number of ways to place \( i \) digits in forbidden positions
A relation $R$ on a set $S$ is a subset of $S \times S$. We say that $a$ is related to $b$, $a \sim b$, if and only if $(a, b) \in R$.

A relation $R$ is an equivalence relation if and only if $R$ is reflexive, symmetric, and transitive.
A relation $R$ on a set $S$ is a subset of $S \times S$. We say that $a$ is related to $b$, $a \sim b$, if and only if $(a, b) \in R$.

A relation $R$ is an equivalence relation if and only if $R$ is reflexive, symmetric, and transitive.

**Reflexive.** For all $a \in S$, $a \sim a$. 

Shelly Smith

Applying Counting Techniques To Sudoku
A relation $R$ on a set $S$ is a subset of $S \times S$. We say that $a$ is related to $b$, $a \sim b$, if and only if $(a, b) \in R$.

A relation $R$ is an equivalence relation if and only if $R$ is reflexive, symmetric, and transitive.

**Reflexive.** For all $a \in S$, $a \sim a$.

**Symmetric.** For all $a, b \in S$, if $a \sim b$, then $b \sim a$. 
A relation $R$ on a set $S$ is a subset of $S \times S$. We say that $a$ is related to $b$, $a \sim b$, if and only if $(a, b) \in R$.

A relation $R$ is an equivalence relation if and only if $R$ is reflexive, symmetric, and transitive.

**Reflexive.** For all $a \in S$, $a \sim a$.

**Symmetric.** For all $a, b \in S$, if $a \sim b$, then $b \sim a$.

**Transitive.** For all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$. 

Applying Counting Techniques To Sudoku
### Sudoku terminology

**Sudoku**

**Block**

```
9 5 3 2 1 4 7 6 8
2 7 6 8 5 3 4 1 9
8 1 4 6 7 9 2 3 5
7 4 8 5 3 1 6 9 2
6 9 1 7 4 2 5 8 3
5 3 2 9 6 8 1 7 4
1 6 9 4 8 5 3 2 7
3 2 5 1 9 7 8 4 6
4 8 7 3 2 6 9 5 1
```

**Band**

```
9 5 3 2 1 4 7 6 8
2 7 6 8 5 3 4 1 9
8 1 4 6 7 9 2 3 5
7 4 8 5 3 1 6 9 2
6 9 1 7 4 2 5 8 3
5 3 2 9 6 8 1 7 4
1 6 9 4 8 5 3 2 7
3 2 5 1 9 7 8 4 6
4 8 7 3 2 6 9 5 1
```

**Stack**

```
9 5 3 2 1 4 7 6 8
2 7 6 8 5 3 4 1 9
8 1 4 6 7 9 2 3 5
7 4 8 5 3 1 6 9 2
6 9 1 7 4 2 5 8 3
5 3 2 9 6 8 1 7 4
1 6 9 4 8 5 3 2 7
3 2 5 1 9 7 8 4 6
4 8 7 3 2 6 9 5 1
```
Definition
(Russell and Jarvis) Let $A$ and $B$ be two Sudoku boards, then $A \sim B$ if and only if $A$ can be transformed into $B$ using one or more of the following operations:

- Permute the nine digits
- Permute the three stacks
- Permute the three bands
- Permute the three columns within a stack
- Permute the three rows within a band
- Reflect across a diagonal, vertical or horizontal axis
- $0^\circ, 90^\circ, 180^\circ$ or $270^\circ$ clockwise rotation
Greater/less Sudoku

Greater/less Sudoku
Greater/less Sudoku

Big Questions

▶ How can we create a Greater/less Sudoku puzzle?
▶ How many Greater/less Sudoku puzzles are there?
Greater/less Sudoku

Starting Small

- Start with one block.
  There are $9! = 362880$ ways to fill in the 9 digits, but only $2^{12} = 4096$ ways to choose the 12 inequalities.
Greater/less Sudoku

Starting Small

- Start with one block. There are $9! = 362880$ ways to fill in the 9 digits, but only $2^{12} = 4096$ ways to choose the 12 inequalities.
- Define an equivalence relation on the set of permutations, on the set on inequalities.
Greater/less Sudoku

Starting Small

- Start with one block.
  There are $9! = 362880$ ways to fill in the 9 digits, but only $2^{12} = 4096$ ways to choose the 12 inequalities.

- Define an equivalence relation on the set of permutations, on the set on inequalities.

- Combine blocks that will satisfy the conditions of Sudoku.
Starting Small

► Start with one block. There are $9! = 362880$ ways to fill in the 9 digits, but only $2^{12} = 4096$ ways to choose the 12 inequalities.

► Define an equivalence relation on the set of permutations, on the set on inequalities.

► Combine blocks that will satisfy the conditions of Sudoku.

► Start with smaller versions of Greater/less Sudoku.