Sum of a Sequence of Floors and Ceilings **December 1997**

1534. *Proposed by Donald Knuth, Stanford University, Stanford, California.*

Let m, n, and p be positive integers, and set

$$t_{m,p}(n) = \left\lceil \frac{\lfloor n/m \rfloor}{2p} \right\rceil, \qquad s_{m,p}(n) = t_{m,p}(0) + t_{m,p}(1) + \cdots + t_{m,p}(n-1).$$

Prove that $s_{m,p}(n)$ is a multiple of $t_{m,p}(n)$.

Solution by Matthias Beck, Akalu Tefera, and Melkamu Zeleke, Temple University, Philadelphia, Pennsylvania.

We prove a slightly more general result: Let $m, n, p \in N$ and define $T_{m,p}(n) := \lceil \lfloor n/m \rfloor / p \rceil$ and $S_{m,p}(n) := \sum_{j=0}^{n-1} T_{m,p}(j)$. Then $S_{m,p}(n)$ is a multiple of $T_{m,p}(n)$ if and only if $n \le m$ or at least one of the integers m, p, q is even, where $q := \lfloor (n-m)/(pm) \rfloor$.

To begin the proof, observe that if $n \le m$, then $S_{m,p}(n) = 0$, which is clearly a multiple of $T_{m,p}(n)$. Therefore, assume $n > m$. Then

$$\begin{aligned} T_{m,p}(n) &= \left\lceil \frac{\lfloor n/m \rfloor}{p} \right\rceil = \left\lfloor \frac{\lfloor n/m \rfloor + p - 1}{p} \right\rfloor = \left\lfloor \frac{\lfloor n/m - 1 \rfloor}{p} \right\rfloor + 1 \\ &= \left\lfloor \frac{n-m}{pm} \right\rfloor + 1 = q + 1 > 0, \end{aligned}$$

where we used the fact that, for $b \in N$, $\lfloor \lfloor a \rfloor / b \rfloor = \lfloor a/b \rfloor$. Furthermore,

$$S_{m,p}(n) = \sum_{j=0}^{n-1} T_{m,p}(j) = \sum_{j=0}^{n-1} \left(\left\lfloor \frac{j-m}{pm} \right\rfloor + 1 \right).$$

Now divide $[m, n-1]$ into subintervals of pm integers (plus the remaining final subinterval, which could be empty), each representing a constant contribution to $S_{m,p}(n)$. Thus, we have

$$\begin{aligned} S_{m,p}(n) &= \sum_{j=0}^{m-1} 0 + \sum_{k=0}^{q-1} \sum_{j=pmk+m}^{pm(k+1)+m-1} (k+1) + \sum_{j=pmq+m}^{n-1} (q+1) \\ &= \sum_{k=0}^{q-1} pm(k+1) + (n - pmq - m)(q+1) \\ &= \frac{pmq(q+1)}{2} + (n - pmq - m)(q+1) = \left(n - m - \frac{pmq}{2} \right) T_{m,p}(n). \end{aligned}$$

It follows that $T_{m,p}(n)$ divides $S_{m,p}(n)$ if and only if $n - m - pmq/2$ is an integer, which in turn holds if and only if pmq is even.