A Multiple Integral Evaluation Inspired by the Multi-WZ Method[1]

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Abstract

We give an integral identity which was conjectured and proved by using the continuous version of the multi-WZ method.

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0. Introduction

There are relatively few known non-trivial evaluations of k-dimensional integrals, with *arbitrary* k. Celebrated examples are the Selberg and the Metha-Dyson integrals, as well as the Macdonald constant term ex-conjectures for the various root systems. They are all very important. See [AAR98] for a superb exposition of the various known proofs and of numerous intriguing applications.

At present, the (continuous version of the) WZ method [WZ92] is capable of mechanically proving these identities only for a fixed k. In principle for *any* fixed k (even, say, $k = 100000$), but in practice only for $k \leq 5$. However, by interfacing a human to the computer-generated output, the human may discern a pattern, and generalize the computer-generated proofs for $k = 1, 2, 3, 4$ to an arbitrary k.

Using this strategy, Wilf and Zeilberger [WZ92] gave a WZ-style proof of Selberg's integral evaluation. In this article we present a *new* multi-integral evaluation, that was *first* found by using the author's implementation of the continuous multi-WZ method which is called SMint[2]. Both the conjecturing part, and the proving part, were done by a close human-machine collaboration. Our proof hence may be termed *computer-assisted* but not yet *computer-generated*.

[1]This work will appear in the author's Ph.D. thesis.
[2]available from `http://www.math.temple.edu/~akalu/maplepack/SMint`

Now that the result is known and proved, it may be of interest to have a non-WZ proof, possibly by performing an appropriate change of variables, converting the multi-integral to a double integral. My advisor, Doron Zeilberger, is offering $100 for such a proof, provided it does not exceed the length of the present proof.

Since a key to the integral evaluation is the package SMint, first we give a brief description of the package.

1. A Brief Description of SMint

The "objects" of study in the continuous version of the multi-WZ theory are expressions of the kind

$$\int F(n, \mathbf{m}, \mathbf{y}, \mathbf{x}) d\mathbf{x}$$

and identities between them. In the above general integral-sum, n is a *discrete* variable, $\mathbf{m}$ is a *discrete* multi-variable, while x and y are *continuous* multi-variables, and F is *hypergeometric* in all its arguments.

For a given *hypergeometric* function $F(n, \mathbf{m}, \mathbf{y}, \mathbf{x})$, where $\mathbf{y} = (y_1, \ldots, y_k)$, we look for a recurrence operator $\sum_{i=0}^{I} a_i(n) E_n^i$, where $a_i(n)$ polynomial in n and E_n is the forward shift operator in n, and a k-tuple of *rational* functions(the *certificate*) $[R_1, \ldots, R_k]$ $(R_i = R_i(n, \mathbf{m}, \mathbf{y}, \mathbf{x}))$ such that the recurrence-differential operator

$$\sum_{i=0}^{I} a_i(n) E_n^i - \sum_{j=1}^{k} D_{x_j}.R_j$$

annihilates F,i.e.,

$$\sum_{i=0}^{I} a_i(n) F(n+i, \mathbf{m}, \mathbf{y}, \mathbf{x}) - \sum_{j=1}^{k} D_{x_j}(R_j F) = 0.$$

The existence of an operator of the above form is *guaranteed* by the fundamental theorem of the (continuous version of the) multi-WZ theory [WZ92].

Doron Zeilberger wrote a Maple implementation, TRIPLE_INTEGRAL[3], that performs the algorithm described in [WZ92] for the case of *three* continuous variables($k = 3$). But TRIPLE_INTEGRAL does not completely automate the method, for instance, it requires the user to guess the denominators of the R_i's.

[3] available from `http://www.math.temple.edu/~zeilberg/`

The author wrote two Maple packages Mint and SMint which improved and generalized Zeilberger's TRIPLE_INTEGRAL for *any* specific number of continuous variables so that it completely automates the continuous multi-WZ method. The package SMint is specially designed to handle identities which invlove pure multiple integrals where the integrand is *symmetric* w.r.t. the integration variables. The detailed technical description of Mint and SMint is available from the author's home page[4] and will also appear in a forthcoming paper [T99].

2. Notation

In the sequel, k is a positive integer, m and n are non-negative integers. The notations used in this article are defined as follows.

$$
\begin{aligned}
\mathbf{x} &:= (x_1, \ldots, x_k), \\
\hat{\mathbf{x}}_i &:= \begin{cases} (x_2, \ldots, x_k) & \text{for } i = 1 \\ (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) & \text{for } 1 < i < k \\ (x_1, \ldots, x_{k-1}) & \text{for } i = k \end{cases} \\
d\mathbf{x} &= dx_1 \cdots dx_k, \\
(y)_m &:= \begin{cases} 1 & \text{for } m = 0 \\ \prod_{i=0}^{m-1}(y+i) & \text{for } m > 0 \end{cases} \\
e_1(\mathbf{x}) &:= \sum_{i=1}^{k} x_i, \\
e_2(\mathbf{x}) &:= \sum_{1 \le i < j \le k} x_i x_j, \\
\Delta_n F(n, \mathbf{x}) &:= F(n+1, \mathbf{x}) - F(n, \mathbf{x}), \\
D_x &:= \frac{\partial}{\partial x}
\end{aligned}
$$

3. The Integral Evaluation

Theorem

$$
\int_{[0,+\infty)^k} (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})} d\mathbf{x} = \frac{m!(2m+n+k-1)!(k/2)_m}{(2m+k-1)!} \left(\frac{2(k-1)}{k}\right)^m T_k(m)
$$

[4] `http://www.math.temple.edu/~akalu/`

for any positive integer k, and for all non-negative integers m and n, where,

$$T_k(m) - T_k(m-1) = \frac{(k(k-2))^m((k-1)/2)_m}{(k-1)^{2m}(k/2)_m} T_{k-1}(m) \qquad k \geq 2,$$

$T_1(m) = 0$, $m \geq 0$, and $T_k(0) = 1$, $k \geq 2$.

4. Proof of the Integral Evaluation

If $k = 1$, then trivially, both sides of the integral equate to zero. Let $k > 1$ and $A_k(m,n)$ be the left side of the integral divided by

$$\frac{m!(2m+n+k-1)!(k/2)_m}{(2m+k-1)!} \left(\frac{2(k-1)}{k}\right)^m.$$

We want to show $A_k(m,n) = T_k(m)$, for all m, n in $\mathbb{Z}_{\geq 0}$. Let

$$F_k(m,n;\mathbf{x}) := \frac{(2m+k-1)!}{m!(2m+n+k-1)!(k/2)_m} \left(\frac{k}{2(k-1)}\right)^m (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})}$$

We construct[5]

$$R(u; v_1, \ldots, v_{k-1}) := \frac{u}{2m+n+k},$$

with the motive that

$$\text{(WZ 1)} \qquad \Delta_n F_k(m,n;\mathbf{x}) = -\sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m,n;\mathbf{x})].$$

Now, we verify (WZ 1),

$$\begin{aligned}
&\frac{F_k(m,n+1;\mathbf{x}) - F_k(m,n;\mathbf{x}) + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m,n;\mathbf{x})]}{F_k(m,n;\mathbf{x})} \\
= \;& \frac{F_k(m,n+1;\mathbf{x})}{F_k(m,n;\mathbf{x})} - 1 + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)] + R(x_i;\hat{\mathbf{x}}_i) D_{x_i}\left[\log\left(F_k(m,n;\mathbf{x})\right)\right] \\
= \;& \frac{e_1(\mathbf{x})}{2m+n+k} - 1 + \frac{k}{2m+n+k} + \\
& \sum_{i=1}^{k} \left(\frac{n}{e_1(\mathbf{x})}\frac{x_i}{2m+n+k} + \frac{m e_1(\hat{\mathbf{x}}_i)}{e_2(\mathbf{x})}\frac{x_i}{2m+n+k} - \frac{x_i}{2m+n+k}\right) \\
= \;& \frac{e_1(\mathbf{x})}{2m+n+k} - 1 + \frac{k}{2m+n+k} + \frac{n}{2m+n+k} + \frac{2m}{2m+n+k} - \frac{e_1(\mathbf{x})}{2m+n+k} \\
= \;& 0.
\end{aligned}$$

[5]The production of the rational function R and the corresponding recurrence-differential equation was done automatically by SMint for $k = 2, 3, 4$, and the output is available from http://www.math.temple.edu/~akalu/maplepack/rational1.output

Hence, by integrating both sides of (WZ 1) w.r.t. $x_1, \ldots, x_k$ over $[0,\infty)^k$, we get

$$A_k(m,n+1) - A_k(m,n) \equiv 0.$$

To complete the proof we show $A_k(m,0) = T_k(m)$ for all m in $\mathbb{Z}_{\geq 0}$.

To this end, set $A_k(m) := A_k(m,0)$ and $F_k(m;x) := F_k(m,0;\mathbf{x})$. Now, we construct[6],

$$R(u;v_1,\ldots,v_{k-1}) := \frac{((k-1)(m+1) + e_1(v_1,\ldots,v_{k-1}))u + e_2(v_1,\ldots,v_{k-1})}{(k-1)(m+1)(2m+k)}$$

with the motive that

$$\text{(WZ 2)} \qquad F_k(m+1;\mathbf{x}) - F_k(m;\mathbf{x}) = -\sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m;\mathbf{x})].$$

Verification of (WZ 2):

$$\begin{aligned}
&\frac{F_k(m+1;\mathbf{x}) - F_k(m;\mathbf{x}) + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)F_k(m;\mathbf{x})]}{F_k(m;\mathbf{x})} \\
= \;& \frac{F_k(m+1;\mathbf{x})}{F_k(m;\mathbf{x})} - 1 + \sum_{i=1}^{k} D_{x_i}[R(x_i;\hat{\mathbf{x}}_i)] + \sum_{i=1}^{k} R(x_i;\hat{\mathbf{x}}_i) D_{x_i}\left[\log\left(F_k(m;\mathbf{x})\right)\right] \\
= \;& \frac{ke_2(\mathbf{x})}{(m+1)(k-1)(2m+k)} - 1 + \sum_{i=1}^{k} \frac{(k-1)(m+1) + e_1(\hat{\mathbf{x}}_i)}{(m+1)(k-1)(2m+k)} + \\
& \sum_{i=1}^{k} \frac{(k-1)(m+1)x_i + e_2(\mathbf{x}))}{(m+1)(k-1)(2m+k)} \left(\frac{me_1(\hat{\mathbf{x}}_i)}{e_2(\mathbf{x})} - 1\right) \\
= \;& \frac{ke_2(\mathbf{x})}{(m+1)(k-1)(2m+k)} - 1 + \frac{k}{2m+k} + \frac{e_1(\mathbf{x})}{(m+1)(2m+k)} + \frac{2m}{2m+k} \\
& - \frac{e_1(\mathbf{x})}{2m+k} + \frac{me_1(\mathbf{x})}{(m+1)(2m+k)} - \frac{ke_2(\mathbf{x})}{(m+1)(k-1)(2m+k)} \\
= \;& 0.
\end{aligned}$$

Hence, by integrating both sides of (WZ 2) w.r.t. $x_1, \ldots, x_k$ over $[0,\infty)^k$, we obtain,

$$A_k(m+1) - A_k(m) = \frac{(k(k-2))^{m+1}((k-1)/2)_{m+1}}{(k-1)^{2(m+1)}(k/2)_{m+1}} A_{k-1}(m+1),$$

and noting that $A_k(0) = 1$, $k \geq 2$, $A_1(m) = 0$, it follows that $A_k(m) = T_k(m)$, for all m in $\mathbb{Z}_{\geq 0}$. Consequently, $A_k(m,n) = T_k(m)$ for all m, n in $\mathbb{Z}_{\geq 0}$. □

[6] The production of the rational function R and the corresponding recurrence-differential equation was done automatically by SMint for $k = 2,3,4,5$ and the output is available from `http://www.math.temple.edu/~akalu/maplepack/rational2.output`

By unfolding the recurrence equation for $T_k(m)$, we obtain the following identity.

Corollary

$$\int_{[0,+\infty)^k} (e_2(\mathbf{x}))^m (e_1(\mathbf{x}))^n e^{-e_1(\mathbf{x})} d\mathbf{x} = \frac{m!(2m+n+k-1)!(k/2)_m}{(2m+k-1)!} \left(\frac{2(k-1)}{k}\right)^m$$

$$\left(1 + \sum_{r=1}^{k-2} \sum_{1\le s_r \le \cdots \le s_1 \le m} \prod_{i=1}^{r} \frac{((k-i)^2-1)^{s_i}((k-i)/2)_{s_i}}{(k-i)^{2s_i}((k-i+1)/2)_{s_i}}\right)$$

5. Remarks

1. From the computational point of view, the recurrence form of the integral is *nicer* than its indefinite summation form (the above corollary),for the former requires $O(mk)$ calculations, whereas the latter requires $O(m^k)$ calculations. However, in both forms the evaluation of the right side of the integral is much faster (for specific m, n, and k) than the direct evaluation of the left side of our intergal. Hence both forms are indeed complete *answers* in the sense of Wilf [W82].

2. The present paper is an example of what Doron Zeilberger [Z98] calls *WZ Theory, Chapter 1 1/2*. Even though, at present, our proof, for general k, was human-generated, it seems that by using John Stembridge's [S95] Maple package for symmetric functions,*SF*, or an extension of it, it should be possible to write a new version of SMint that should work for *symbolic*, i.e. arbitrary, k, thereby fulfilling the hope raised in [Z98].

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