Emphasis

This tutorial is the first of a series of tutorials on orbital mechanics. In this tutorial students review the definition of angular momentum. They also prove Kepler's second law by showing that the angular momentum of an orbiting body is conserved.

Prerequisites

Students will need to be familiar with angular momentum as a vector quantity. Students will also be asked to apply basic concepts of velocity and acceleration in two dimensions. Lecture instruction on Kepler's second law, however, is not necessary to complete this tutorial.

TUTORIAL PRETEST

The pretest consists of two different problems. In the first problem students are shown the trajectories of three different objects moving with constant speed. For each case the students are asked to determine whether or not the angular momentum of the object is constant as measured with respect to a given reference point *O*. Although students may be expected to already understand angular momentum as the cross product $\vec{r} \times \vec{p}$, many will instead base their responses simply on whether or not they believe the product of the magnitudes of the position and magnitude vectors, $|\vec{r}||\vec{p}|$, remains constant. Many students use this line of reasoning to conclude incorrectly that in case 3—the linear trajectory—the angular momentum is initially large, decreases until the object attains its closest approach to point *O,* then increases thereafter.

The second problem is a two-part problem pertaining to the elliptical orbit of a comet around the sun. First, for each of several labeled points along the orbit, students are asked whether the comet moves with increasing speed, decreasing speed, or constant speed upon passing that point. Second, they are asked to rank four of the labeled points according to the angular momentum of the comet at those points. Students can answer the first part by determining whether the acceleration of the comet at each location (which always points toward the sun) has a component parallel or anti-parallel to the instantaneous velocity of the comet. The second part probes whether or not students recognize that angular momentum is constant throughout the orbit.

TUTORIAL SESSION

Equipment and handouts

Each group will need a whiteboard and set of markers, or a large sheet of paper. Each student will need a copy of the tutorial handout (no special handouts required).

Discussion of tutorial worksheet

Section I: Angular momentum

In the opening section of the tutorial students revisit the first pretest problem on angular momentum, in which three objects move with constant speed along trajectories of different shape. Students are reminded of the definition of angular momentum as the cross product between position and momentum vectors. The opening questions give students some practice applying this definition for specific locations along each trajectory as well as for different locations of the reference point with which angular momentum is measured. The students are then guided through the reasoning to answer the first pretest problem correctly.

For the case of the straight-line trajectory, the most difficult of the three examples, many students need the hint about considering the quantity $|\vec{r}|\sin\theta$ in order to conclude that angular momentum does not change. (The reasoning underlying this hint is left for students to discuss in Problem 1 of the tutorial homework.) When considering the angular momentum values corresponding to locations *R* and *T,* some students may stumble because of failing to recognize that the sine of an angle is equal to the sine of the supplement to that angle. The checkpoint at the end of this section is instrumental in reviewing students' explanations before proceeding.

Section II: Changes in angular momentum

In this section students are guided through a derivation of Kepler's second law by proving that the angular momentum of a point-like body (*e.g.,* a comet) orbiting another experiences no change in angular momentum when measured relative to the center of the (much more massive) body (*e.g.,* the sun) that the first body is orbiting. Parts A through C guide students to (1) evaluate the time derivative of the angular momentum, (2) interpret the result as the net torque, and (3) recognize that the zero net torque on the comet implies that its angular momentum remains constant. Students summarize these results in their own words in part D.

In preparation for the checkpoint at the end of the section, check that students use appropriate reasoning along the way. For instance, in part B, some students may misinterpret $d\vec{r}/dt$ as the time rate of change of the *length* of the position vector rather than as the full velocity vector. Remind such students that $d\vec{r}$ is a vector quantity and ask them to draw this vector on the diagram. (It may be necessary to ask students whether this vector is the sum or difference between two vectors; having recognized that it is the difference of two position vectors, students will be more likely to interpret it as the vector displacement and interpret $d\vec{r}/dt$ as the velocity.) If time allows, it may be useful at the checkpoint to ask students how their overall results would differ if the object under consideration were to experience a central force that is repulsive rather than attractive (*e.g.,* as in a Rutherford scattering experiment).

Section III: Application: Elliptical orbit of a comet

Students finish the tutorial by applying the results they obtained in the preceding section to the situation described in the second pretest problem. Specifically, students proceed from the knowledge that angular momentum is conserved to recognize that the speed of the comet is a maximum at the point of closest approach and a minimum at the point of farthest approach. They confirm this result by applying basic principles from two-dimensional kinematics; having noted at which points the tangential acceleration points along with or opposite from the instantaneous velocity, students should deduce that the comet always increases in speed as it approaches the sun and gradually decreases in speed as it recedes. The last two questions direct the students' attention once again to the fact that angular momentum is not always equal in magnitude to the product of the magnitudes of position and momentum.

TUTORIAL HOMEWORK

The homework gives students the opportunity to apply and extend their knowledge of angular momentum and Kepler's second law.

1. Students explain in their own words why the magnitude of a cross product can be treated as the magnitude of *either* vector in the cross product times the component of the other vector that is *perpendicular* to the first. Two subsequent questions give students practice in being versatile in applying this idea in the context of angular momentum.
2. This problem serves as a supplement to section II of the tutorial by leading students to recognize the meaning of the term *areal velocity* (in which Kepler's second law is often phrased) and the proportionality between areal velocity and angular momentum.
3. In this problem students work with mathematical expressions for position and velocity in polar coordinates. They prove that angular momentum per unit mass may be written $l = r^2\dot{\theta}$ and explain in words what it means for this quantity to be constant for an object moving in an orbit.

4. This multi-step qualitative problem motivates the important relationship between the angular momentum of an orbiting body and the latus rectum of the orbit. Successful completion of this problem requires students to synthesize several ideas from Newton's laws and two-dimensional kinematics as well as angular momentum.

