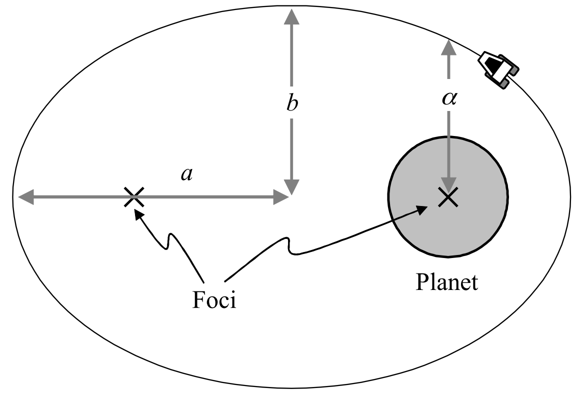
KEPLER'S FIRST LAW

I. Some "anatomy" of ellipses

Soon after launching from the *U.S.S. Enterprise,* you pilot your shuttlecraft into an elliptical orbit around an uncharted class-M planet, as shown at right. The foci of the orbit (each indicated by an "×"), semi-major axis *(a),* semi-minor axis *(b),* and latus rectum *(α)* are labeled on the diagram.



Polar view of orbit *(not to scale)*

In answering the following questions, use the following facts about ellipses:

- For each point along the ellipse, the sum of the distances from that point to each focus is $2a$.
- The foci of an ellipse are defined to be a distance εa away from its geometric "center."

A. Describe in words <u>and</u> sketches how the shape of an ellipse will change by: (i) keeping a constant and varying ε, (ii) keeping ε constant and varying a.

B. Show that the latus rectum α can be expressed in terms of a and ε as follows: $\alpha = a(1 - \varepsilon^2)$.

(*Hint:* Consider the point along the ellipse where each quantity is measured, and try to find right triangles with which to make use of the information given above.)

C. Any ellipse can also be expressed in polar coordinates with the equation shown below, where $r = r(\theta)$ is the distance measured from the center of the body being orbited (here, the planet):

$$r(\theta) = \frac{\alpha}{1 + \varepsilon \cos\theta} \qquad (1)$$

With your partners, convince yourselves that the above equation makes sense. For instance, what values of θ would correspond to: (i) the point of *farthest approach* for the orbiting body? (ii) the point of *closet approach?* (iii) the point(s) where the *latus rectum* of the orbit is measured?

II. Using Kepler's second law to prove Kepler's first law

Kepler's second law states that the angular momentum of an orbiting body, measured with respect to the center of the body it is orbiting, does not change.

For objects with constant mass, the quantity *angular momentum per unit mass,* $\vec{l} = \vec{r} \times \vec{v}$*,* is also a constant of the motion. The symbol l will be used to represent the magnitude of this quantity.

A. Recall that in polar coordinates position and velocity are expressed as the following:

$$\vec{r} = r\,\hat{e}_r\,; \quad \vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta \tag{2}$$

Using the above information, show that $l = r^2\dot{\theta}$. Discuss your reasoning with your partners.

B. In polar coordinates the radial equation of motion (using Newton's second law) can be written as:

$$ma_r = \left|\vec{F}^G(r)\right| \quad \Rightarrow \quad m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{GMm}{r^2} \tag{3}$$

Although we will not prove it here, it can be shown that Equation 3, which governs the functional dependence of r on θ *(i.e.,* $r = r(\theta)$*),* can be rewritten using a change of variables:

$$u(\theta) \equiv \frac{1}{r(\theta)} \quad \Rightarrow \quad \frac{d^2u}{d\theta^2} + u = \frac{GM}{l^2} \tag{4}$$

1. Use direct substitution to show that the function $u(\theta)$ shown below (as Equation 5) is a solution to the above differential equation (Equation 4).

$$u(\theta) = C\cos\theta + \frac{GM}{l^2} \tag{5}$$

(*Aside:* Do you notice any similarities between Equation 4—the differential equation that prescribes $u(\theta)$ as a function of θ—and the form of the differential equation of motion for a simple harmonic oscillator?)

2. One last step (for now, anyway…): Show that the above function $u(\theta)$ (Equation 5) can be rewritten as the polar equation of the elliptical orbit that appeared earlier (Equation 1). In doing so, show how the latus rectum α and eccentricity ε (which appear in Eq. 1) each depend on the constants C, l, G, and M that appear in Eq. 5.

   (*Note:* This result proves *Kepler's first law,* which states that all closed orbits are ellipses.)

C. Your findings in part B prove a crucial connection between the angular momentum per unit mass *(l)* of an orbiting body and the shape of its orbit. On the basis of your results above, answer the following:

1. If two differently shaped orbits correspond to the same value of *l,* which feature of the orbits must be identical to each other? Explain.

2. A circular orbit of diameter D and an elliptical orbit with semi-major axis $2D$ correspond to the same value of *l*. Determine the eccentricity of the elliptical orbit. Discuss your reasoning with your partners.

3. Which orbit must correspond to a larger value of *l:* a *circular orbit* or an *elliptical orbit* having the same semi-major axis as the circular one? Use a sketch of both orbits to support your answer.

