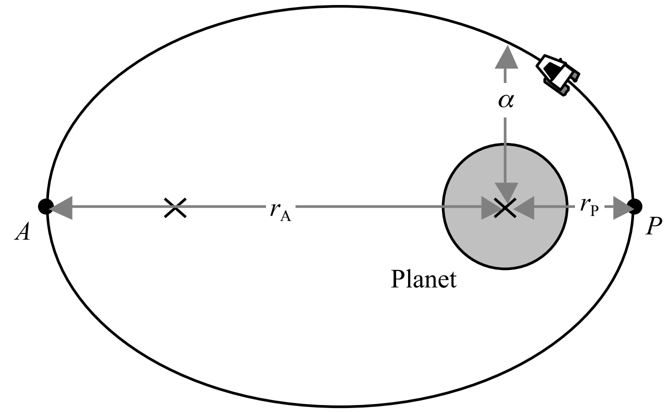
Homework: Energy and angular momentum for closed orbits Name ________________________

1. Consider again the orbit of the shuttlecraft shown in section II of the tutorial, reproduced below. In this problem we develop some quantitative relationships among the semi-major axis a, eccentricity ε, latus rectum α, apogee distance r_A, and perigee distance r_P.

   a. Each focus of an ellipse is located a distance εa away from the geometric "center" of the ellipse. Use this fact to express r_A and r_P in terms of a and ε. Show all work.

   b. Show that the latus rectum α can be expressed in terms of a and ε as follows: $\alpha = a(1 - \varepsilon^2)$.

      (*Hint:* For any point along the ellipse, how must the sum of the distances from that point to each focus relate to *a?* Apply this relationship to the point at which the latus rectum is measured.)



Polar view of orbit *(not to scale)*

   c. Combine your results from parts a and b in order to express r_A and r_P in terms of α and ε (rather than in terms of a and ε, as you did in part a of this problem). Show all work.

2. In this problem we will make use of the fact that for closed orbits the latus rectum α of the orbit is equal to $\alpha = l^2/GM$, where G is the universal gravitational constant, l is the angular momentum *per unit mass* of the orbiting body, and M is the (much larger) mass of the body at one focus of the orbit.

   The following questions refer to the shuttlecraft orbit shown in Problem 1 above.

   a. Use your knowledge of Kepler's laws and your results in Problem 1 to determine expressions for (i) the shuttle's speed v_P at perigee, and (ii) the shuttle's speed v_A at apogee. Write each expression in terms of G, M, ε, and the corresponding distance (r_P or r_A).

   b. Use your results from part a of this problem to calculate the quantities listed below. In your work, assume that the planet has approximately the same radius and mass as Earth (*i.e.,* $R_{planet} \approx R_E = 6.38 \times 10^6$ m, and $GM \approx GM_E = 3.98 \times 10^{14}$ N-m²/kg). In addition, suppose that the orbit of the shuttle has perigee $r_P = 2R_E$ and apogee $r_A = 8R_E$.

      i. the eccentricity ε of the original (elliptical) orbit

      ii. the latus rectum α of the orbit

      iii. the speed v_P at perigee in the orbit

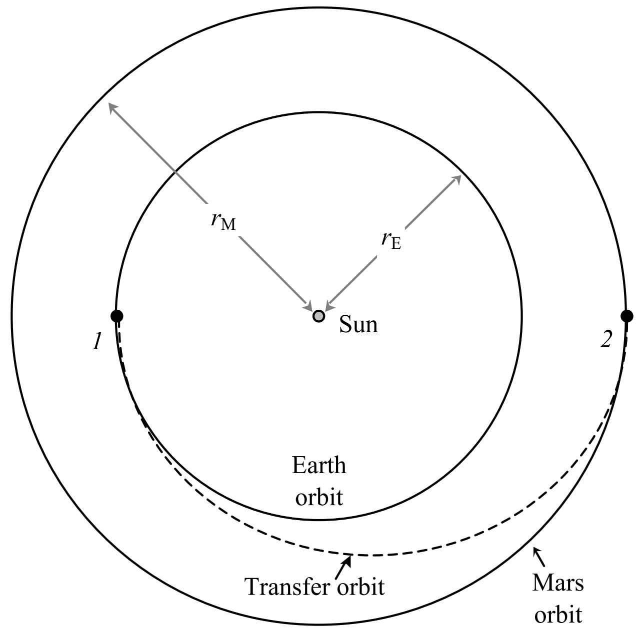
      iv. the speed v_A at apogee in the orbit

      v. the minimum speed your shuttlecraft must attain at point P (*e.g.,* by quickly firing your thrusters as you pass that point) in order to leave the planet

      (*Hint:* What minimum value of eccentricity ε is required for the shuttle's orbit in this case?)

3. In interplanetary exploration, elliptical transfer orbits, like the one shown from Earth to Mars, allow space probes to reach the intended destination using a minimum of fuel. (Such orbits are called *Hohmann transfer orbits.*)

   Note: Assume the orbits of Earth and Mars about the Sun are circular with radii 1.000 AU and 1.524 AU, respectively. Ignore any effects due to the rotation of the Earth.



   a. Describe *qualitatively* the maneuvers required for the probe upon entering and leaving the transfer orbit. That is, must the probe *increase* or *decrease* its speed (i) upon entering the transfer orbit at point *1?* (ii) upon entering Mars orbit at point *2?* Explain.

   b. Determine the following quantities:

   - the semimajor axis, eccentricity, and latus rectum of the transfer orbit
   - the time required for the probe to travel from point *1* to point *2* along the transfer orbit

   (*Note:* When measuring time in units of earth-years and distance in AU's, the quantity $4\pi/GM_{\text{sun}}$ has a value of unity.)

   c. It is said that interplanetary expeditions can be launched at only certain times, or "launch opportunities." In light of your results from part b above, explain what is meant by this statement.

   d. Make your results from part (a) quantitative (rather than just qualitative) by calculating:

   - the change in speed (in m/s) required for the probe to enter the transfer orbit at point *1*
   - the change in speed (in m/s) required for the probe to enter Mars orbit upon reaching point *2*

4. Consider again the pairs of orbits you compared in parts A and B in section III of the tutorial.

   a. In part A of section III you considered two orbits in which a shuttlecraft would have the same total energy. Determine the angle formed by the orbits at a point where they intersect. Clearly show all work.

   (*Hint:* Consider the ratio of angular momenta $\left|\vec{L}_{elliptical}\right| / \left|\vec{L}_{circular}\right|$ evaluated at the intersection point.)

   b. For any closed orbit of semi-major axis *a,* show that the speed *v(r)* of the shuttlecraft at any point along its orbit around the planet (mass *M*) is expressed as the following function of *r:*

   $$v(r) = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)},$$

   c. In part B in section III you considered two orbits in which a shuttlecraft would have the same angular momentum. Determine the angle formed by the orbits at a point where they intersect.

   (*Hint:* Start this part the same way you did part a of this problem, although here the ratio of angular momenta for the orbits is now equal to 1. How can you use your result in part b of this problem to help you here?)

