INSTRUCTOR NOTES
Harmonic motion in two dimensions

Emphasis

This tutorial is designed to extend student understanding of simple harmonic motion in one dimension to two or more dimensions. Students are guided to recognize the effect of various factors on the *x-y* trajectory of two-dimensional oscillators, both isotropic and non-isotropic.

Prerequisites

Students should have completed the tutorial *Simple harmonic motion* beforehand. In addition, it is recommended, though not required, that students have had lectures on 2-D oscillators.

TUTORIAL PRETEST

The pretest asks students to interpret information about different 2-D harmonic oscillators and to use that information to draw a possible *x-y* trajectory for each one. The pretest begins by describing a (frictionless) 2-D oscillator in words and mathematically, by means of the expression for the (vector) net force on the oscillator:

$$\vec{F}_{net}(x, y) = \left(-k_x x\,\hat{i}\right) + \left(-k_y y\,\hat{j}\right)$$

For several different cases the students are specifically told the ratio k_y/k_x of force constants and the initial conditions of motion for each of four different cases. In two cases the oscillator is isotropic $(k_x = k_y)$ and in the other two cases the force constants differ by a factor of 4 $(k_y = 4k_x)$. Thus the students are expected to draw *x-y* trajectories that show the appropriate ratio of oscillation frequencies along the *x*- and *y*-axes ($\omega_y = \omega_x$ for the isotropic cases, $\omega_y = 2\omega_x$ for the non-isotropic cases). In all cases the students are asked to explain the reasoning.

After lecture instruction students often have difficulty recognizing that only the mass and force constant affect the frequency $\omega = (k/m)^{1/2}$ of a simple harmonic oscillator. Some students seem to believe that the amplitude also affects frequency (see notes from the tutorial *Simple harmonic motion*). Expect to see evidence of such difficulties on this pretest. For example, some students may draw the trajectory for the isotropic oscillator in part a with equal amplitudes in the *x*- and *y*-direction. Others may indicate that the non-isotropic oscillators in parts c and d (for which $k_y = 4k_x$) must have a larger amplitude in the *x*-direction than in the *y*-direction (by a factor of either 4 or 2) because the spring along the *y*-axis is stiffer than that along the *x*-axis.

In addition, many students may fail to recognize the need to (or how to) extend their thinking of 1-D oscillators to the 2-D case. For instance, some students may regard the algebraic expression for the net force (shown above) to mean that the oscillator always "seeks the origin." Such students may show incorrectly that the isotropic oscillator in part a passes through the origin. Other students—for either the isotropic or non-isotropic cases or both—will predict that the oscillators follow paths that "spiral toward" or "collapse toward" the origin, despite their prior experience with energy conservation in the 1-D case. (Such errors can and do arise even if the students had not previously covered phase space diagrams of damped oscillators.)

TUTORIAL SESSION

Equipment and handouts

Each group will need a whiteboard and set of markers, or a large sheet of paper. Each student will need a copy of the tutorial handout (no special handouts required).

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Discussion of tutorial worksheet

Section I: Frequencies of motion

Students begin by revisiting simple harmonic motion in one dimension. In part A they are expected to recall (optimally, from their work in the *Simple harmonic motion* tutorial) that the angular frequency ω of motion depends solely on mass and force constant by $\omega = (k/m)^{\frac{1}{2}}$. In part B they apply this relationship in order to compare the two force constants for both isotropic and non-isotropic oscillators. In part C students extend their thinking to consider an *x-y* trajectory (case #2 from the pretest) for which the frequencies are equal but the amplitudes along each axis are different. By considering how many oscillations occur along the *y*-axis for every one oscillation along the *x*-axis, students should recognize that the oscillator is isotropic and hence the force constants are equal to each other.

Section II: Trajectories of 2-D isotropic oscillators

In this section students explore the effect on the *x-y* trajectory of the phase offset between the oscillations in a 2-D isotropic oscillator. Students proceed in parts A and B by examining the same (elliptical) *x-y* trajectory from section I and considering various possible sets of initial conditions of motion for that trajectory. For each case, students determine which values of the phase angles φ and δ apply assuming the following forms of the solutions *x(t)* and *y(t)*:

$$x(t) = A_1 \cos(\omega_0 t + \varphi), \quad y(t) = A_2 \cos(\omega_0 t + \varphi + \delta)$$

Each set of initial conditions is carefully chosen so that the phase angles φ and δ will have the relatively "easy" values 0°, +90°, –90°, or 180°, but nonetheless watch carefully for sign errors whenever the correct answer is +90° or –90°. For example, when a sinusoidal curve is described as being "shifted to the right" by 90° compared to another, many students will incorrectly state that "to the right" means being shifted "ahead" by 90°. (Part of the tutorial *Simple harmonic motion* is designed to address this and related modes of incorrect reasoning.) When students complete the table in part B and summarize their results thus far in part C, check that students conclude that the phase difference δ between the *x*- and *y*-motions solely determines the direction in which the oscillator follows the trajectory, with $\delta = +90°$ yielding clockwise motion, $\delta = –90°$ counter-clockwise motion.

Part D of the tutorial presents several other trajectories for which the students must infer information about the appropriate value of the phase difference δ. Two of the trajectories are ellipses whose axes do not coincide with the *x*- and *y*-axes (similar to Case #3 from the pretest). If students do not know how to proceed, encourage them to extend the reasoning they used previously in part B, in which they sketched separate graphs of *x(t)* and *y(t)*. After analyzing the example trajectories in part D, students should be able to summarize their results in part E by describing qualitatively how varying δ from –180° to +180° affect both the shape of the trajectory and direction of motion of the isotropic oscillator.

TUTORIAL HOMEWORK

The homework gives students the opportunity to apply their results from the tutorial on 2-D isotropic oscillators. An additional problem guides students to extend their thinking to non-isotropic oscillators.

1. Students proceed from the expression of potential energy for a 2-D harmonic oscillator, $U(x, y) = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2$ to determine the component equations of motion and show that they are separable. Students must also explain why the solutions to those equations require two arbitrary phase angles *(φ and δ)*.

2. In this qualitative problem students critique a statement that articulates a common incorrect line of reasoning elicited by the pretest. In so doing, students show that different amplitudes

of oscillation along the *x*- and *y*-axes do not necessarily imply unequal force constants, but rather allow comparisons of potential or kinetic energy at different points along the trajectory.

3. Students analyze quantitatively two *x-y* trajectories of isotropic oscillators. They must use the given information to calculate $\omega_o, \varphi,$ and δ for each trajectory.

4. The final homework problem gives students the opportunity to investigate non-isotropic oscillators whose frequencies are commensurate. Students interpret three possible trajectories to deduce the commensurate relationship between the frequencies. Further, for one of the three trajectories, students must calculate the frequencies and force constants given the mass of the oscillator and the time to completely retrace the entire path.