GRAPHICAL INTERPRETATION OF DIFFERENTIAL EQUATIONS

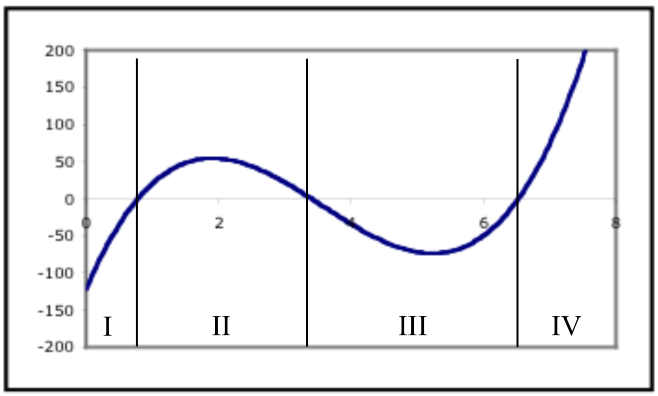
In mechanics, analytical techniques are often used to solve problems. However, graphical analysis of differential equations can also be used to yield a qualitative solution. This is especially useful for equations which you cannot solve analytically or to check your analytical solution. This tutorial discusses one such technique.

I. Developing terminology

In calculus, you learned that the second derivative is called the concavity. Another useful idea is if a function is *curving away* or *curving towards* the (horizontal) axis in particular region. You can use different analogies to determine this

- imagine the curve is a road and that you are driving down it from left to right. If your steering wheel is turned toward the axis, you are *curving toward*; if it is turned away, you are *curving away*.

- Imagine the concavity of the curve: if it's concave *toward* the horizontal axis, (if above or below), it's *curving towards*. If it's concave facing away from the axis, it's *curving away*.



	Curving Away or Towards?	Concavity + or -?	Function + or -?
Section I			
Section II			
Section III			
Section IV			

A. Consider the graph shown above.

1. For each section (I through IV), determine: (a) whether you are curving toward or away from the axis, (b) the concavity, and (c) the sign of the function. Fill in the table above.

2. Circle any inflection points and put a square around turning points on the graph above.

B. Use the table above to develop a rule that allows you to determine if you are curving toward or away if you know the concavity and sign of the function.

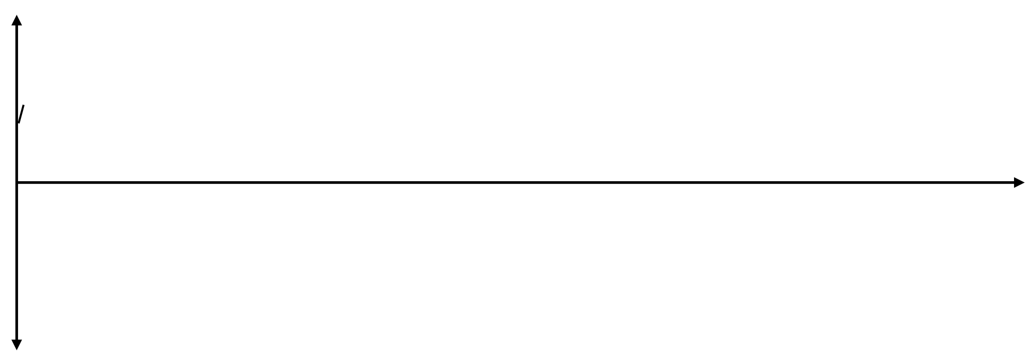
II. A simple system—the simple harmonic oscillator

For the moment, we're going to treat the simple harmonic oscillator purely mathematically and graphically. The SHO equation is $\frac{d^2x}{dt^2} = -\omega^2 x$. You already know the solution to this equation, but try to imagine you do not.

A. Using the rule you found in section I, determine if x curves toward or away from the axis, both, or neither. Explain your reasoning.

B. For what values of x are there inflection points? (*Hint:* Consider the mathematical definition of an inflection point.)

C. Using the relationship between the concavity and the value of the function, sketch the function until you have reached the axis. We have started you off (that little line along the axis…). Label your axes clearly.



D. Using your answer to part B, what happens to the graph as you cross the axis? Continue to sketch the graph for larger values of t.

E. Is your graph consistent with what you know about simple harmonic motion? Resolve any inconsistencies.

III. The damped harmonic oscillator

Again, we will treat the damped harmonic oscillator purely mathematically and graphically. The equation of motion describing the damped harmonic oscillator is $\frac{d^2x}{dt^2} = -2\gamma\frac{dx}{dt} - \omega_0^2 x$.

A. Developing tools to analyze and graph damped harmonic motion

It can be difficult to reason about equations that have three changing terms. However, we can break the equation down into smaller parts and study them individually. The equation can be rewritten to describe three special cases: turning points, inflection points, and axis crossings.

Consider the table below:

1. Discuss with your group the first line of equations. For each, prove to yourself that the condition leads to the equation.

Turning Point Equation		**Inflection Point Equation**		**Axis Crossing Equation**	
$\frac{dx}{dt} = 0$ so $\frac{d^2x}{dt^2} = -\omega_0^2 x$		$\frac{d^2x}{dt^2} = 0$ so $2\gamma\frac{dx}{dt} = -\omega_0^2 x$		$x = 0$ so $\frac{d^2x}{dt^2} = -2\gamma\frac{dx}{dt}$	
$\frac{d^2x}{dt^2}$	x	$\frac{dx}{dt}$	x	$\frac{d^2x}{dt^2}$	$\frac{dx}{dt}$
+		+		+	
–		–		–	

2. For each special case equation, fill in the remaining spaces in the table.

B. We can use the three special case equations to sketch a motion graph. The first step is to determine the order in which the inflection points, turning points, and axis crossings occur.

1. Imagine the oscillator is stretched to its maximum displacement and released from rest at $t = 0$.

   a. At the moment of release, which of the three special cases applies? How do you know?

   Use the appropriate special case equation to determine the sign of the concavity at $t = 0$.

b. Now consider how to continue sketching the motion graph by using mathematical reasoning about curvature and slope. For example, which of the three special cases occurs after $t = 0$?

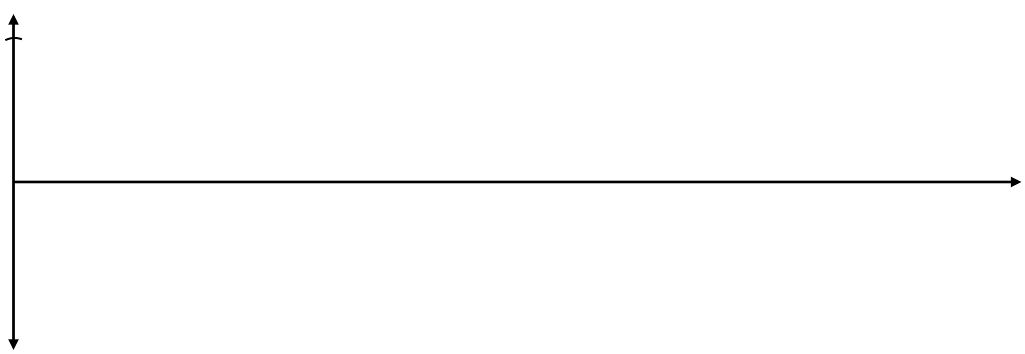
- Is it possible that a *turning point* happens next? Explain why or why not. (*Hint:* Can a turning point happen next if the concavity remains as it is, or must another special case point occur first?)

- Is it possible that an *inflection point* happens next, given the current values of the function and the slope? Use your results from the table on the preceding page to decide.

- Is it possible that an *axis crossing* happens next, given the current values of the slope and concavity? Use your results from the table on the preceding page to decide.

c. After completing part b above, decide with your partners which <u>two</u> special case points happen next. Be sure that your group agrees on the reasoning you used to arrive at your results.

d. Use what you have developed thus far to sketch the graph from the moment of release until the first minimum. Label your axes clearly.



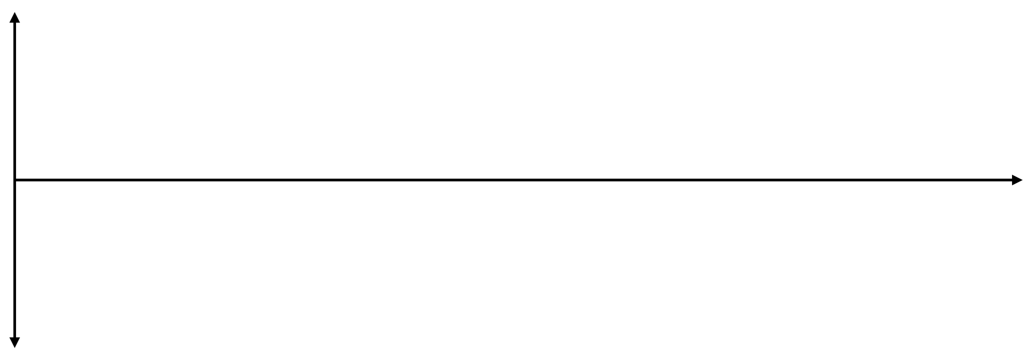
2. What are the <u>next three</u> special points? Write them in order below, and then add them to your graph above. Be sure that your group agrees on the reasoning you used to arrive at your answer.

C. In the graph on the previous page, you were guided to reason that the graph crossed the *t*-axis. The mathematical equation doesn't require this to actually happen. Consider what would happen if the graph did *not* cross the *t*-axis after the first inflection point.

1. Could another inflection point occur next? Explain how you arrived at your answer.

2. What about another turning point? Explain how you arrived at your answer.

3. Use these new assumptions to draw another graph below. Label your axes clearly.



D. Is it possible to determine only from the form of the differential equation of motion whether the object will cross the *t*-axis or not?

1. What is the difference in the motion of the object in each case?

2. Is all damped harmonic motion harmonic? Explain.

