INSTRUCTOR NOTES
Conservative force fields

Emphasis

This tutorial guides students to recognize the meaning and equivalence of three different statements of a conservative force field: (a) the work done is path-independent; (b) the work done along any closed loop must be zero; (c) the vector curl of the force is zero at any location. The tutorial also serves as a "guided derivation" of the *z*-component of the curl in Cartesian coordinates.

Prerequisites

It is recommended that students are familiar with work, line integrals, and partial derivatives before this tutorial. It is also recommended, though not required, for students to have already completed the tutorial *Conservative forces and equipotential diagrams.*

TUTORIAL PRETEST

Note: This pretest is somewhat longer and more technical than most. As a result, consider assigning this pretest as a "take-home" pretest so that students might have time to make their responses clearer and more illustrative of their thinking. If you choose this approach, instruct students to abstain from using their textbook or notes to answer the questions, and tell them they may use 15 – 20 minutes to complete the pretest. In addition, if some students may not have had prior instruction on the curl of a vector field, announce to the class that those students who have not covered this topic previously should indicate this clearly on the pretest before submitting it.

On the pretest students are shown four different illustrations of vector fields mapped in the *x-y* plane. For each field they are asked (1) to state whether or not the curl of that field is zero at all locations and (2) to specify whether or not that field could represent a conservative force. For the first question, many students give responses, even after instruction on vector curl, suggesting the belief that the curl depends only on variations in direction of the field; these students will state incorrectly that case #3 represents a field with non-zero curl or that case #4 has zero curl ("the force lines do not change direction"). Other students will base their response on whether or not they believe a particle might follow a curved or straight trajectory through the field. For the second question, many students tend to associate this property with variations in magnitude of the force instead of the curl (or what they believe about the curl) of the force. For example, students might say incorrectly that case #2 or #4 are conservative because they can trace "field lines" connecting force vectors of equal magnitude, or that the "magnitude of the force … would not change if a particle were placed in its field." Alternatively, some students may correctly recognize that a conservative force must do work that is *path-independent* but incorrectly believe that it is sufficient for a force merely to be *position*-dependent, and in so doing state that all four cases represent conservative forces.

TUTORIAL SESSION

Equipment and handouts

Each group will need a whiteboard and set of markers, or a large sheet of paper. Each student will need a copy of the tutorial handout (no special handouts required).

Discussion of tutorial worksheet

Section I: Open and closed paths

The tutorial opens by reminding students that the work done on an object by a conservative force depends only upon the endpoints of the path taken by the object and not the specific path. The first section of the tutorial helps students understand that one important consequence of this property is that the work done by a conservative force around any closed loop must be zero.

Although many students have little trouble interpreting the quantity $\vec{F}\cdot d\vec{r}$ as the work done along the infinitesimal displacement $d\vec{r}$, some will incorrectly state that the integral $\int \vec{F}\cdot d\vec{r}$ (*e.g.,* for the path *C* shown on page 1) represents the "net force" on the object or "the sum of the forces along the path." These students may not have recognized that a line integral carries with it the implication that the path in question is being subdivided into a large number of displacement vectors $d\vec{r}$, and that the integral itself represents the addition of a very large number of quantities of the form $\vec{F}\cdot d\vec{r}$. The instruction for students to draw and label several other $d\vec{r}$ vectors along path *C* should help them recognize what quantities are being "added up" here.

Section II: Developing a test for conservative forces

This section of the tutorial helps students proceed from their result in section I—that the work done around any closed loop by a conservative force is zero—to devise a mathematical test by which to tell whether or not if a force is conservative. The overall goal for the remainder of the tutorial is to have students apply the zero-work condition for a small rectangular loop in the *x-y* plane and prove that, in the limit in which the loop becomes infinitesimally small:

$$\frac{Work\,around\,loop}{Area\,subtended\,by\,loop} \Rightarrow \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Bigg|_{(x,y)=(x_o,y_o)}$$

(The coordinates (x_o, y_o) correspond to the corner of the rectangle nearest the origin.) Only after they have reached this point are students told that the quantity they have just found to be zero is called the *z*-component of the *curl* of the force evaluated at (x_o, y_o).

Part C of section II will require the most time and effort on the part of the students, and as such the remainder of the discussion offered here focuses on what students are expected to do (and what they often do) in that part. The opening two questions in part C help "frame" both the thinking and the level of formalism that the students should use here. In particular, when students are asked to express mathematically the work done along leg *1* of the rectangular loop (see top of page 4), the question is intentionally left as a multiple-choice question. Students usually have little trouble identifying expression "a" as the correct answer, and they are often sufficiently careful when writing the other expressions for the work done along legs *2 – 4.* Nevertheless, monitor the students' work to make sure that they indicate the correct component of the force (denoted by the subscript on the *"F"*), the correct direction of displacement along each leg (denoted by the choice of overall sign), and the appropriate coordinates of the point at which the force is approximated. The full expression for the work done around the entire loop is:

$$\underbrace{F_x\left(x_o+\frac{\Delta x}{2},\,y_o\right)\Delta x}_{\text{Leg }1} + \underbrace{F_y\left(x_o+\Delta x,\,y_o+\frac{\Delta y}{2}\right)\Delta y}_{\text{Leg }2} - \underbrace{F_x\left(x_o+\frac{\Delta x}{2},\,y_o+\Delta y\right)\Delta x}_{\text{Leg }3} - \underbrace{F_y\left(x_o,\,y_o+\frac{\Delta y}{2}\right)\Delta y}_{\text{Leg }4}$$

Students are expected to then divide each term by $\Delta x\Delta y$, the area of the loop, and combine appropriate terms. The resulting expression for the work done around the loop divided by the area of the loop becomes the following (where the terms for legs *2* and *4* appear in the first grouping, and those for legs *1* and *3* appear in the second grouping):

$$\frac{Work}{Area} = \frac{F_y\left(x_o+\Delta x,\,y_o+\frac{\Delta y}{2}\right) - F_y\left(x_o,\,y_o+\frac{\Delta y}{2}\right)}{\Delta x} - \frac{F_x\left(x_o+\frac{\Delta x}{2},\,y_o+\Delta y\right) - F_x\left(x_o+\frac{\Delta x}{2},\,y_o\right)}{\Delta y}$$

Students then take the limit in which both Δx and Δy approach zero in order to obtain the expected difference in partial derivatives. As one might expect, students might be careless about signs, especially those that give rise to the second term, $-\partial F_x/\partial y$, of the target expression. However, a more serious error to watch for is the failure to recognize that each of the two groups of terms forms a partial derivative. Some students instead attempt to further "simply" their expression by "subtracting the arguments" of the functions; for example, rather than form $\partial F_y/\partial x$ from the first term, some will incorrectly rewrite the numerator of that term as follows:

$$\text{“} F_y\left(x_o+\Delta x,\, y_o+\frac{\Delta y}{2}\right) - F_y\left(x_o,\, y_o+\frac{\Delta y}{2}\right) = F_y\left(\Delta x, 0\right) \text{”}$$

To help students recognize the fallacy in this sort of argument, express doubt at their assertion and ask them a question such as, "For a function *f(x),* is it guaranteed that $f(x = 5)$ minus $f(x = 3)$ will be equal to $f(x = 2)$?" Usually this kind of question helps students confront their error.

For students who get stuck in part C, try asking them to recall and write down the definition of a partial derivative, or even a (full) derivative df/dx of a single variable function *f(x).* In so doing, students may better recognize what a derivative means and better discern how to build the desired partial derivatives from the terms they have collected. Some students may need more pointed questions, such as: What is the function being differentiated here? With respect to which variable? At what *x*- and *y*-coordinates is the derivative being evaluated? How can you tell?

The tutorial concludes with the introduction of the term "curl" and with questions that lead students to intuit that the curl of a force must be equal to zero at all locations in order to be conservative. Some of the homework problems (see below) guide students to prove this statement as well as help them construct and understand the full operational definition of the curl in Cartesian coordinates.

TUTORIAL HOMEWORK

The homework is designed to help students to "tie up" some loose ends from the tutorial and apply the reasoning developed in tutorial to new situations. *Note:* Problems 1 – 3 are highly recommended as homework following this tutorial.

1. This semi-quantitative problem gives students more practice with basic ideas of line integrals. The problem is similar to questions posed in section II.B of the tutorial, except here the students must consider the work done around a rectangular (not square) loop.
2. Students extend the reasoning from section II of the tutorial—in which they evaluated the work around a small rectangular loop in the *x-y* plane—to situations involving an *arbitrary* loop in the *x-y* plane. They must recognize and explain why the (*z*-component of the) curl must be zero at *all* locations in the *x-y* plane if a force is to be conservative.
3. Students extend the reasoning from section II of the tutorial to consider small loops in both the *y-z* and *x-z* planes, so as to construct the definitions of the *x*- and *y*-components of the curl (respectively) in Cartesian coordinates.
4. Students extend their work from tutorial by constructing the definition of the *z*-component of the curl in *cylindrical* coordinates.
5. Students revisit several of the vector force fields that appeared on the pretest and determine whether or not each one is a conservative force. Conservative forces can be identified by explaining qualitatively why the (*z*-component of the) curl is zero (*e.g.,* F_x may vary with x but does not vary with y, and F_y may vary with y but does not vary with x). Non-conservative forces should be identified by a closed path over which the work is non-zero.

