Results on Edge Covers of Certain Graph Families

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July 6, 2026

Abstract

An edge cover of a graph is a subset of the edges such that each vertex is incident to at least one edge in the subset. The edge cover polynomial is the generating polynomial that counts edge covers with a given number of edges. It is known that the numbers of edge covers of paths and cycles are the Fibonacci and Lucas numbers, respectively. In this paper, we investigate the edge cover number sequences and edge cover polynomials of ladder graphs and cycle stars. We also introduce a result that allows a graph to be decomposed into simpler components when counting edge covers.

1 Introduction

A *(simple) graph* G is an ordered pair (V, E) that is composed of a finite set of vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$ and a finite set of edges $E(G) = \{e_1, e_2, \ldots, e_m\}$ such that each edge e_i is a two-element subset of the vertex set. We will denote such as edge simply by $v_i v_j$ where v_i, v_j are the vertices in the two-element set. In this paper, we consider only simple graphs and will omit the word simple for convenience. Given edge $e = \{v, w\}$, we say that the vertices v, w are *incident* with edge e. The *degree* of a vertex v is the number of edges with which the vertex is incident. A degree 0 vertex is an *isolated vertex*, and a degree 1 vertex is a *pendant*.

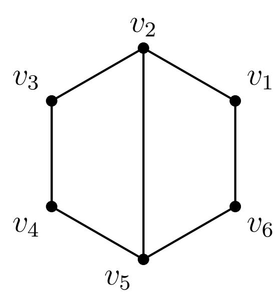

Figure 1: A graph with six vertices.

An *edge cover* of a graph G is a subset of $E(G)$ such that each vertex in $V(G)$ is incident to at least one edge in the subset. As an example, consider the graph in Fig. 1 with vertices

$v_1, v_2, \ldots, v_6$ and the edge set $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_2v_5\}$. It is easy to see that there are many subsets of $E(G)$ that would form an edge cover of G. For example, the set $\{v_1v_2, v_2v_3, v_4v_5, v_5v_6\}$ is an edge cover since each vertex v_i is an endpoint for at least one edge. The vertical edges $\{v_1v_6, v_2v_5, v_3v_4\}$ also form an edge cover. Finally, in any graph with no isolated vertices, $E(G)$ is also an edge cover. These are three of the 43 edge covers of the graph in Fig. 1.

In this paper, we focus on counting edge covers of graphs, both in total and according to the number of edges in the cover. The resulting edge cover sequences can provide new combinatorial interpretations of known integer sequences in OEIS [5] or generate new sequences. Two edge cover sequences known to correspond to famous integer sequences are those of path and cycle graphs. Let $\#G$ denote the total number of edge covers of G. For a path graph P_n with n vertices, it is known that $\#P_n = F_{n-1}$ where F_n are the Fibonacci numbers with initial values $F_0 = 0, F_1 = 1$. It is also known that for a cycle graph C_n with n vertices $n \geq 3$, $\#C_n = L_n$ where L_n are the Lucas numbers with initial values $L_0 = 2, L_1 = 1$ [1]. This approach of counting all edge covers differs from focusing on minimal/minimum edge covers as we will count all of the edge covers of a graph. Although our perspective is a purely combinatorial approach, counting edge covers also has applications in other disciplines such as Hausdorff metric geometry [3] and communication networks [4].

When we want to count edge covers grouped based on the number of edges, we will use the generating function of edge covers, the edge cover polynomial. Specifically, the *edge cover polynomial* of a graph G, denoted $E(G, x)$, is defined as

$$E(G, x) = \sum_{k=1}^{|E(G)|} e(G, k)x^k$$

where $e(G, k)$ is the number of edge covers of G with k edges. As an example, for C_4, there is one edge cover with 4 edges, four edge covers with 3 edges, and two edge covers with 2 edges, as shown in Fig. 2. Thus the edge cover polynomial of C_4 is

$$E(C_4, x) = x^4 + 4x^3 + 2x^2\,.$$

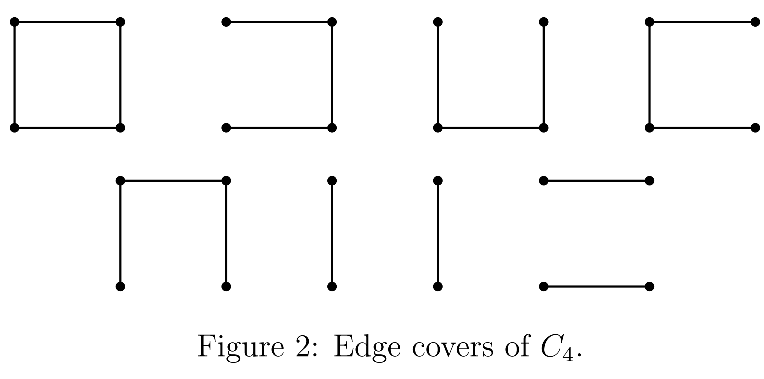

Figure 2: Edge covers of C_4.

The graph families we consider in this paper are ladder graphs and cycle star graphs. These graphs will be defined below in detail. Before these specific graphs, we introduce a new result, The Cutting-Open Lemma, that allows a graph to be decomposed into simpler components when counting edge covers. By using recurrence relations and properties of edge cover polynomials, we calculate the total number of edge covers for ladders and cycle stars along with formulas for their edge cover polynomials. In doing so, we obtain new sequences not in OEIS and provide new combinatorial interpretations to some known sequences. In the next section, we recall some known results on edge cover polynomials and prove a new decomposition result for edge covers.

2 Some known properties of edge covers

Some known results on the total number of edge covers and edge cover polynomials of graphs that we will be using are listed below.

Theorem 1. *Given a graph G with connected components $G_1, G_2, \ldots, G_k$, we have [1]*

$$\#G = \prod_{i=1}^{k} \#G_i \quad \textit{and} \quad E(G,x) = \prod_{i=1}^{k} E(G_i,x)\,.$$

Proposition 2. *The edge cover polynomial of path graphs satisfies the recurrence relation*

$$E(P_{n+1},x) = xE(P_n,x) + xE(P_{n-1},x)$$

for $n \geq 2$ where P_n is the path graph with n vertices. The initial values are $E(P_1,x) = 0$ and $E(P_2,x) = x$ [1].

We will now prove a new result that is useful for counting the edge covers of graphs due to how the larger graph often decomposes into smaller disconnected components. When considering all of the edge covers that do not contain a certain edge, we can simply view them as the edge covers of the graph formed by deleting the edge. It is harder to see intuitively what the edge covers of a graph that contain a certain edge correspond to. Fortunately there is a bijection between the set of edge covers of a graph that include an edge, and a graph that can be formed from "cutting up" the original graph and adding a few additional pendants to represent the chosen edge. The following lemma, developed by Alayont and Henning, describes this decomposition. This result is a generalization that extends two previous edge cover decomposition results, the *2-sum Lemma* and the *Pendant Lemma* introduced by the same authors in [2].

Lemma 3. *(Cutting-Open Lemma). Let G be a graph with $u, v \in V(G)$ and $uv \in E(G)$. Let w_i with $1 \leq i \leq k$ be the neighbors of u besides v, and y_j, with $1 \leq j \leq \ell$ be the neighbors of v not including u. Define the graph G' as*

$$G' = (G - \{u,v\}) \cup \{w_1u_1, u_1x_1, \ldots, w_ku_k, u_kx_k\} \cup \{y_1v_1, v_1z_1, \ldots, y_\ell v_\ell, v_\ell z_\ell\},$$

where all u_i, x_i, v_j, z_j are new vertices added to $V(G)$. Then the edge covers of G that include the edge uv are in bijection with the edge covers of G'.

Proof. The construction of G' can be visualized as follows. Given G and the edge uv in G, we remove the vertices u and v which removes all incident edges of u and v. This process essentially cuts the graph open at each endpoint, at u and v. For each edge w_iu incident with u, other than uv, we replace w_iu by w_iu_i and add the pendant edge u_ix_i. Similarly, for each edge y_jv incident with v, other than uv, we replace y_jv by y_jv_j and add the pendant edge v_jz_j. These pendant edges force the new vertices u_i and v_j to be covered, reflecting the fact that u

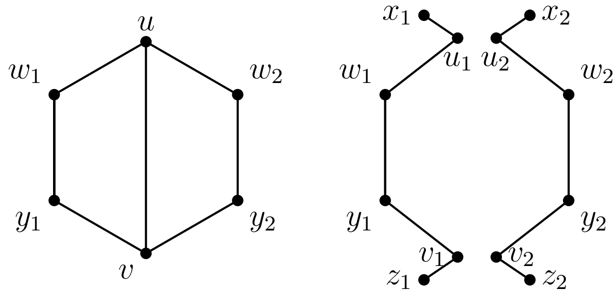

Figure 3: A graph G and the resulting graph G'.

and v were covered by uv in the original edge cover. A representation of this process is shown in Fig. 3.

We will show a bijection between edge covers of G including uv and those of G'. Let S be such an edge cover. We build an edge cover S' of G' corresponding to S as follows:

- Start with $S' = S$.
- Remove uv from S'.
- For each i, if $w_iu \in S$, then remove w_iu from S' and add w_iu_i.
- For each j, if $y_jv \in S$, then remove y_jv from S' and add y_jv_j.
- For each i, add u_ix_i, and for each j, add v_jz_j to S'.

First we show that this process produces an edge cover of G'. Note that all the pendant vertices x_i and z_j are covered by the pendant edges we added in the last step. These pendant edges also cover all the vertices u_i and v_j. The remaining vertices in G' are all the vertices in G but u, v. Consider one such vertex $a \in V(G) - \{u, v\}$. Since S is an edge cover of G, there is an edge e in S incident with a. If e is not incident with u or v, then e is still in S' by construction. If $e = w_iu$, then $a = w_i$ and $w_iu_i \in S'$ covers a. If $e = y_jv$, then $a = y_j$ and $y_jv_j \in S'$ covers a. Thus S' is an edge cover of G'.

To show that the mapping is onto, we construct an inverse. Let T be an edge cover of G'. Since each vertex x_i and z_j has degree one, every pendant edge u_ix_i and v_jz_j belongs to T. We construct an edge cover S of G as follows:

- Start with $S = T$.
- Remove the pendant edges u_ix_i and v_jz_j from S.
- For each i, replace each edge w_iu_i in S by w_iu.
- For each j, replace each edge y_jv_j in S by y_jv.
- Add the edge uv to S.

We first show that S is an edge cover of G. The vertices u and v are covered by the edge uv. Let $a \in V(G) - \{u, v\}$. Since T is an edge cover of G', there is an edge $e \in T$ incident with a. If e is not incident with any of the vertices u_i or v_j, then e remains in S. If $e = w_iu_i$, then

$a = w_i$, and the edge $w_i u \in S$ covers a. If $e = y_j v_j$, then $a = y_j$, and the edge $y_j v \in S$ covers a. Thus S is an edge cover of G containing uv.

It is straightforward to verify that this construction exactly reverses the previous one. Every edge $w_i u$ replaced by $w_i u_i$ is restored, every edge $y_j v$ replaced by $y_j v_j$ is restored, the added pendant edges are deleted, and the edge uv is reinserted. Hence the two constructions are inverse to one another, establishing the desired bijection. □

Corollary 4. *Let $E(G; uv, x)$ denote the edge cover polynomial of G restricted to edge covers that contain the edge uv, and let G' be defined as in the Cutting-Open Lemma. Then*

$$E(G; uv, x) = \frac{E(G', x)}{x^{k+\ell-1}}.$$

In particular, $\#(G; uv) = \#G'$ where $\#(G; uv)$ denotes the number of edge covers of G containing uv.

Proof. Because our mapping is a bijection between the edge covers of G containing uv and those of G', we know that $\#(G; uv) = \#G'$. Because the bijection replaces each edge $w_i u$ by $w_i u_i$ and each edge $y_j v$ by $y_j v_j$, the number of edges in the edge cover is unchanged except that the edge uv is removed and $k+\ell$ pendant edges $u_i x_i$ and $v_j z_j$ are added. Thus every edge cover of G' contains exactly $k + \ell - 1$ more edges than the corresponding edge cover of G. Hence,

$$E(G', x) = x^{k+\ell-1} E(G, x)\,,$$

or equivalently,

$$E(G, x) = \frac{E(G', x)}{x^{k+\ell-1}}\,.$$

□

3 Ladder Graphs

In this section we study the edge covers of ladder graphs. The *ladder graph* L_n is the Cartesian product of P_2 and P_n for $n \geq 1$. Note that $|V(L_n)| = 2n$ and $|E(L_n)| = 1 + 3(n-1)$.

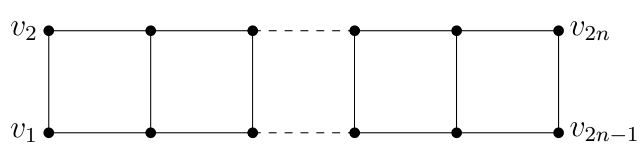

Figure 4: Ladder graph L_n.

Since each L_{n-1} is a subgraph of L_n, we can build edge covers of L_n recursively. This leads to the recurrence relations of the edge cover numbers and polynomials of L_n described below.

Theorem 5. *The number of edge covers for L_n, with initial values*

$$\#L_1 = 1, \qquad \#L_2 = 7, \qquad \#L_3 = 43,$$

satisfies the recurrence relation

$$\#L_n = 5\#L_{n-1} + 8\#L_{n-2} + 6\#L_{n-3} + 6\#L_{n-4} + \ldots + 6\#L_1\,,$$

or equivalently

$$\#L_n = 6\#L_{n-1} + 3\#L_{n-2} - 2\#L_{n-3}$$

for $n \geq 4$. Additionally, the edge cover polynomial $E(L_n, x)$ of L_n for $n \geq 4$ satisfies the recurrence relation

$$E(L_n, x) = x(x+1)(x+2)E(L_{n-1}, x) + x^3(x+2)E(L_{n-2}, x) - x^3(x+1)E(L_{n-3}, x)\,.$$

The corresponding initial polynomials are

$$E(L_1, x) = x, \qquad E(L_2, x) = x^4 + 4x^3 + 2x^2,$$

and

$$E(L_3, x) = x^7 + 7x^6 + 17x^5 + 15x^4 + 3x^3.$$

The sequence $1, 7, 43, 277, \ldots$, generated by the number of edge covers of L_n, is A286911, titled "Number of edge covers in the ladder graph P_2 x P_n." The coefficient rows of the edge cover polynomials $E(L_n, x)$, read in either increasing or decreasing powers of x, do not seem to appear in OEIS [5].

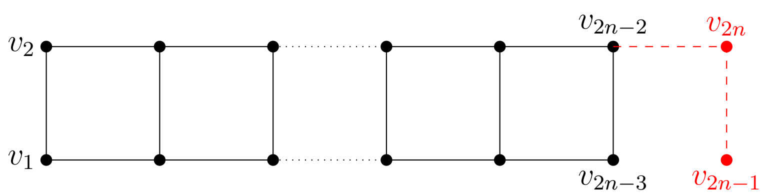

Figure 5: Extending an edge cover of L_{n-1} to an edge cover of L_n.

Proof. We will count the edge covers of L_n by building them from the edge covers of smaller ladder graphs by adding new edges for the remaining vertices. Given an edge cover of L_r with $r < n$, we will refer to the newly added edges as *end covers* of L_r. For example, in Fig. 5, the red-colored, dashed edges form an end cover of L_{n-1}.

For a given n, there are five end covers extending L_{n-1} to L_n, which are shown in Fig. 6.

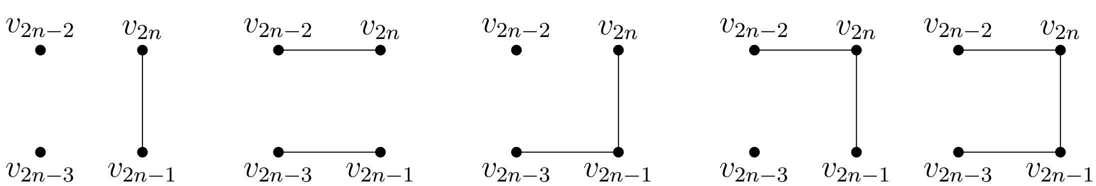

Figure 6: Five end covers of L_{n-1}.

These end covers provide us a way to extend edge covers of L_{n-1} to an edge cover of L_n. Therefore, they form $5\#L_{n-1}$ edge covers among the edge covers of L_n. In terms of the edge cover polynomials, we can also represent these edge covers as contributing

$$E(L_{n-1}, x)(x^3 + 3x^2 + x)$$

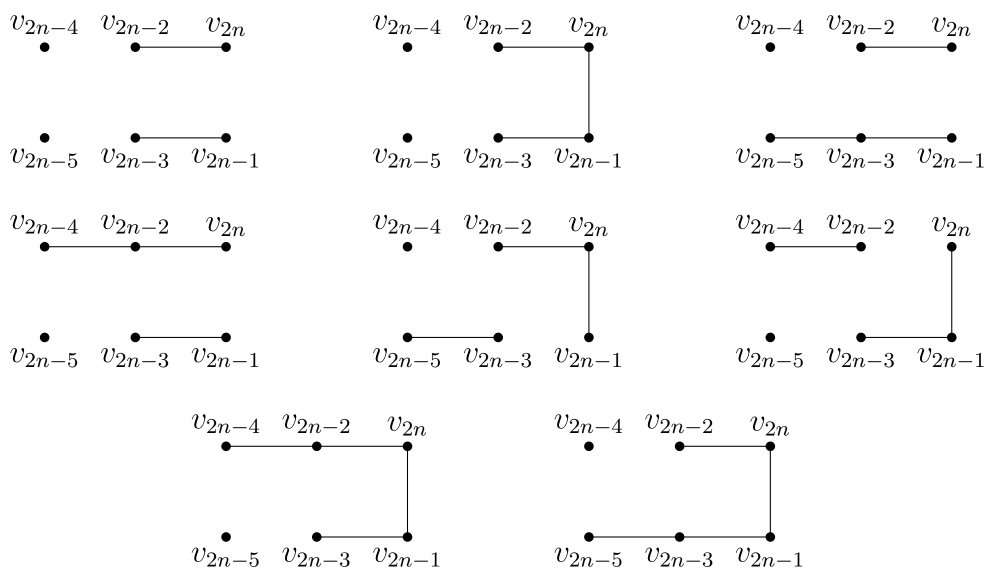

Figure 7: Eight end covers of L_{n-2}.

to $E(L_n, x)$. This is because one end cover has three edges, three have two edges, and one has one edge, so the corresponding factors are x^3, $3x^2$, and x.

Similarly, when extending L_{n-2} to L_n, we use the eight end covers shown in Fig. 7. To avoid double counting, we choose these end covers so that the resulting edge covers of L_n are not among those already counted from extensions of L_{n-1}.

These end covers contribute $8\#L_{n-2}$ edge covers among the edge covers of L_n. In terms of the edge cover polynomials, they contribute

$$E(L_{n-2}, x)(2x^4 + 5x^3 + x^2)$$

to $E(L_n, x)$. This is because two of these end covers have four edges, five have three edges, and one has two edges, so the corresponding factors are $2x^4$, $5x^3$, and x^2.

Next, we can extend L_{n-3} to L_n using the six different edge covers listed in Fig. 8. These six end covers are obtained from the last six end-cover cases for extending L_{n-2} to L_n by adding either $v_{2n-6}v_{2n-4}$ or $v_{2n-7}v_{2n-5}$. These cases contribute $6\#L_{n-3}$ edge covers to the edge covers of L_n and

$$E(L_{n-3}, x)(2x^5 + 4x^4)$$

to $E(L_n, x)$.

The end cover cases for L_{n-3} generalize to any L_r with $1 \leq r \leq n-3$. For each such r, there are six end covers obtained by continuing the pattern in L_{n-3} by adding an edge at the left end alternately at the top or bottom in each step. Hence these cases contribute a total of $6\#L_1 + 6\#L_2 + \cdots + 6\#L_{n-3}$ edge covers to $\#L_n$ and

$$E(L_1, x)(2x^{n+1} + 4x^n) + \cdots + E(L_{n-4}, x)(2x^6 + 4x^5) + E(L_{n-3}, x)(2x^5 + 4x^4)$$

to $E(L_n, x)$.

Putting the cases together, we obtain the equations

$$\#L_n = 5\#L_{n-1} + 8\#L_{n-2} + 6\#L_{n-3} + 6\#L_{n-4} + \ldots + 6\#L_1 \quad (1)$$

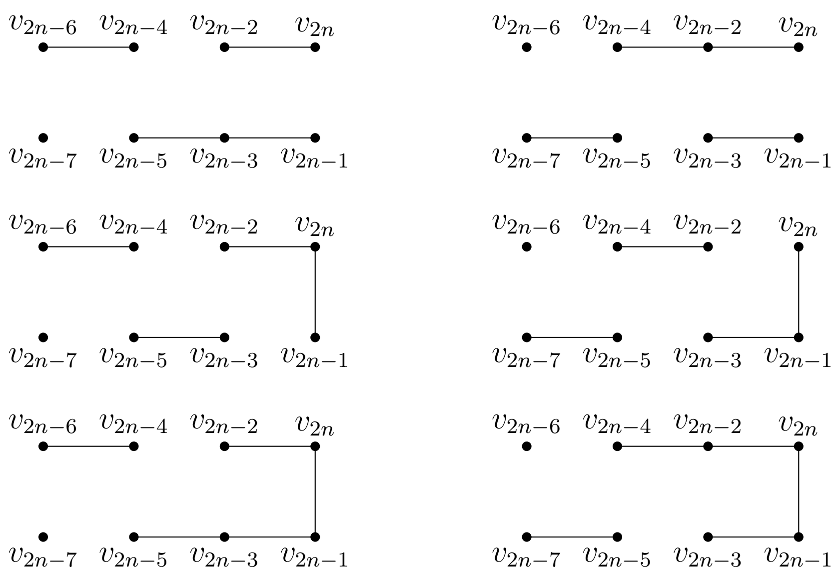

Figure 8: Six end covers of L_{n-3}.

and

$$E(L_n, x) = (x^3 + 3x^2 + x)E(L_{n-1}, x) + (2x^4 + 5x^3 + x^2)E(L_{n-2}, x) + (2x^5 + 4x^4)E(L_{n-3}, x) \\ + (2x^6 + 4x^5)E(L_{n-4}, x) + \cdots + (2x^{n+1} + 4x^n)E(L_1, x)\,. \quad (2)$$

Subtracting the corresponding equations for L_{n-1} from Eq. (1) gives

$$\#L_n = 6\#L_{n-1} + 3\#L_{n-2} - 2\#L_{n-3}\,.$$

Similarly, subtracting x times the corresponding polynomial equation for L_{n-1} from Eq. (2) gives

$$E(L_n, x) = (x^3 + 3x^2 + 2x)E(L_{n-1}, x) + (x^4 + 2x^3)E(L_{n-2}, x) - (x^4 + x^3)E(L_{n-3}, x)\,.$$

These equations match the recurrence relations available at [6]. □

4 Cycle Star Graphs

In this section we consider the edge covers of the graph family we call *cycle stars*. A *cycle star* graph is similar to a star graph, $K_{1,m}$, which has one degree m vertex and m pendants. This can be viewed as m copies of P_2 identified at one common vertex. In the cycle star graph, instead of joining paths, we join cycles of varying sizes at a center vertex. An example of a cycle star graph is the friendship graph F_n which is formed when we join n C_3's at a common vertex [7]. A *cycle star* graph, denoted $C_{n_1,n_2,\ldots,n_k}$, is formed by joining $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ at one vertex. An example $C_{3,4,4,5}$ is shown in Fig. 9. If all k cycles have length n, then we denote the cycle star graph by C_n^k. A friendship graph F_n with n triangles is C_3^n in our notation.

We start with a recurrence for the edge cover numbers of cycle star graphs.

Theorem 6. *For $k \geq 2$, the number of edge covers of cycle star graphs satisfies the recurrence*

$$\#C_{n_1,n_2,\ldots,n_k} = F_{n_k+1}\prod_{i=1}^{k-1} F_{n_i+2} + F_{n_k-1}\prod_{i=1}^{k-1} F_{n_i+2} + F_{n_k-2} \cdot \#C_{n_1,n_2,\ldots,n_{k-1}}\,.$$

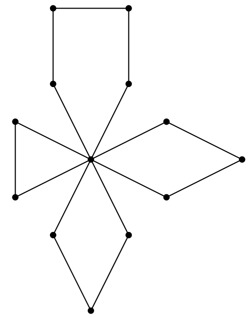

Figure 9: Cycle star $C_{3,4,4,5}$.

Proof. Let $C_{n_1,n_2,\ldots,n_k}$ be a cycle star with k cycles. Let u be the vertex that is common to all cycles. Let v be a neighbor of u in the cycle C_{n_k}, and let w be the other neighbor. We will count edge covers of $C_{n_1,n_2,\ldots,n_k}$ based on whether they contain uv or not.

Case 1: Suppose an edge cover does not contain uv. The graph reduces to $C_{n_1,n_2,\ldots,n_k} - uv$ shown in Fig. 10.

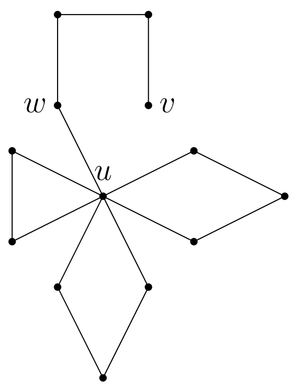

Figure 10: Cycle star $C_{3,4,4,5}$ with uv removed.

We now consider whether uw is included or not. If uw is not included, we have two disconnected pieces $C_{n_1,n_2,\ldots,n_{k-1}}$ and P_{n_k-1}. Therefore, the contribution to the edge cover count from this case is

$$\#C_{n_1,n_2,\ldots,n_{k-1}} \cdot \#P_{n_k-1} = F_{n_k-2} \cdot \#C_{n_1,n_2,\ldots,n_{k-1}} .$$

If uw is included, then we apply the Cutting-Open Lemma, Lemma 3 to obtain a disjoint union of k paths as shown in Fig. 11. The paths are $P_{n_1+3}, P_{n_2+3}, \ldots, P_{n_{k-1}+3}$ and P_{n_k}, respectively. Therefore, they contribute

$$F_{n_k-1} \cdot \prod_{i=1}^{k-1} F_{n_i+2}$$

edge covers to the total.

Case 2: Let us now consider the edge covers including the edge uv. As in the subcase with uw included above, we can apply the Cutting-Open Lemma and split the graph into disjoint paths. In this case, the paths are $P_{n_1+3}, P_{n_2+3}, \ldots, P_{n_{k-1}+3}$ and P_{n_k+2}, respectively. Therefore,

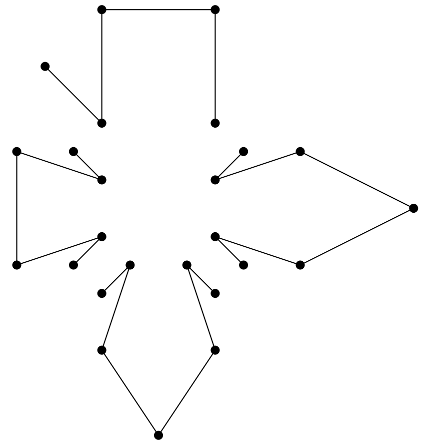

Figure 11: The graph obtained from $C_{3,4,4,5} - uv$ by applying the Cutting-Open Lemma to the edge uw.

they contribute

$$F_{n_k+1} \cdot \prod_{i=1}^{k-1} F_{n_i+2}$$

edge covers to the total.

Summing the contributions from the three cases gives the stated recurrence.

□

Corollary 7. *For $k \geq 2$ and cycle stars C_n^k with equal-length cycles, the recurrence becomes*

$$\#C_n^k = F_{n+2}^{k-1}(F_{n+1} + F_{n-1}) + F_{n-2}\#C_n^{k-1}\,.$$

Proof. Set $n_1 = n_2 = \cdots = n_k = n$ in Theorem 6. Then

$$\prod_{i=1}^{k-1} F_{n_i+2} = F_{n+2}^{k-1},$$

which gives the stated recurrence. □

The recurrence in Corollary 7 led us to several integer sequences. For example, fixing $n = 3$ gives $\#C_3^k = 5^k - 1$, producing the sequence $4, 24, 124, 624, \ldots$ for $k \geq 1$. This is A024049, and our result gives a new combinatorial interpretation of this sequence as the number of edge covers of friendship graphs. Similarly, fixing $n = 4$ gives $\#C_4^k = 8^k - 1$, producing $7, 63, 511, 4095, \ldots$, which is A024088. Fixing $n = 5$ gives $\#C_5^k = 13^k - 2^k$, producing $11, 165, 2189, 28545, \ldots$, which we did not find in OEIS at the time of writing [5]. On the other hand, fixing $k = 2$ and varying n gives $\#C_n^2 = 3F_{2n}$, producing $24, 63, 165, 432, \ldots$ for $n \geq 3$, which is A097134. Fixing $k = 3$ and varying n gives $124, 511, 2189, 9234, \ldots$, which we did not find in OEIS at the time of writing [5]. These examples suggest that the recurrence may have a simple closed form. In fact, we can prove the following formula for the edge cover numbers of all cycle stars.

Theorem 8. *For a cycle star graph $C_{n_1,n_2,\ldots,n_k}$, the edge cover number is given by*

$$\#C_{n_1,n_2,\ldots,n_k} = \prod_{i=1}^{k} F_{n_i+2} - \prod_{i=1}^{k} F_{n_i-2}\,.$$

Proof. We will proceed by induction on k. For $k = 1$, the cycle star becomes a single cycle C_{n_1}. The number of edge covers of a cycle graph is $L_{n_1} = F_{n_1+1} + F_{n_1-1} = F_{n_1+2} - F_{n_1-2}$.

Assume now that the claim is true for all cycle stars with k cycles, and we will show it is true for cycle stars with $k+1$ cycles. Using the recurrence relation for cycle star edge covers in Theorem 6 and the inductive assumption, we obtain

$$\begin{aligned}
\#C_{n_1,n_2,\ldots,n_{k+1}} &= F_{n_{k+1}+1}\prod_{i=1}^{k} F_{n_i+2} + F_{n_{k+1}-1}\prod_{i=1}^{k} F_{n_i+2} + F_{n_{k+1}-2}\cdot \#C_{n_1,n_2,\ldots n_k} \\
&= (F_{n_{k+1}+1} + F_{n_{k+1}-1})\prod_{i=1}^{k} F_{n_i+2} + F_{n_{k+1}-2}(\prod_{i=1}^{k} F_{n_i+2} - \prod_{i=1}^{k} F_{n_i-2}) \\
&= (F_{n_{k+1}+1} + F_{n_{k+1}-1} + F_{n_{k+1}-2})\prod_{i=1}^{k} F_{n_i+2} - \prod_{i=1}^{k+1} F_{n_i-2} \\
&= F_{n_{k+1}+2}\prod_{i=1}^{k} F_{n_i+2} - \prod_{i=1}^{k+1} F_{n_i-2} \\
&= \prod_{i=1}^{k+1} F_{n_i+2} - \prod_{i=1}^{k+1} F_{n_i-2}\,.
\end{aligned}$$

This completes the induction. □

Corollary 9. *For cycle stars C_n^k with equal-length cycles, we have*

$$\#C_n^k = F_{n+2}^k - F_{n-2}^k\,.$$

Proof. The result follows from Theorem 8 by letting $n_i = n$. □

For cycle stars with equal-length cycles, we further have a formula for the edge cover polynomials expressed in terms of path edge cover polynomials.

Theorem 10. *Given a cycle star C_n^k, we have*

$$E(C_n^k, x) = \left(\frac{E(P_{n+3}, x)}{x^2}\right)^k - E(P_{n-1}, x)^k\,.$$

Proof. We will prove the claim using induction on k. For $k = 1$, the cycle star becomes a single cycle C_n. Pick an edge e of the cycle. The edge covers can be split into two groups: those that do not include e and those that do. The edge covers without e are exactly the same as the edge covers of P_n and contribute $E(P_n, x)$ to the edge cover polynomial of C_n. For edge covers including e, we apply the Cutting-Open Lemma, Lemma 3 to obtain another path as shown in Fig. 12. Therefore, by Corollary 4, we find the contribution in this case to be

$$\frac{E(P_{n+2}, x)}{x}\,.$$

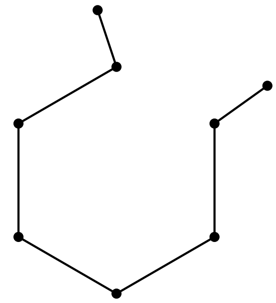

Figure 12: Cycle graph after cutting open at an edge.

Adding the contributions from the two cases and applying the recurrence for the path edge cover polynomials from Proposition 2, we obtain

$$\begin{aligned} E(C_n, x) &= \frac{E(P_{n+2}, x)}{x} + E(P_n, x) \\ &= \frac{E(P_{n+2}, x)}{x} + \frac{E(P_{n+1}, x)}{x} - E(P_{n-1}, x) \\ &= \frac{xE(P_{n+2}, x) + xE(P_{n+1}, x)}{x^2} - E(P_{n-1}, x) \\ &= \frac{E(P_{n+3}, x)}{x^2} - E(P_{n-1}, x)\,. \end{aligned} \tag{3}$$

This proves the claim for $k = 1$.

Consider now C_n^{k+1}. We use the same case decomposition as in the proof of Theorem 6 together with the polynomial version of the Cutting-Open Lemma, Corollary 4. If neither uv nor uw is included, we obtain a disconnected graph with P_{n-1} and C_n^k as two components. This contributes

$$E(P_{n-1}, x) \cdot E(C_n^k, x)\,.$$

If uv is not included but uw is included, then after cutting open, we obtain k paths with $n+3$ vertices each and one path with n vertices. The correction factor is x^{2k}, since u incident with $2k$ edges besides uw and w is incident with one edge besides uw. Therefore, in this case we obtain

$$\frac{E(P_{n+3}, x)^k \cdot E(P_n, x)}{x^{2k}}\,.$$

Finally, if uv is included, then after cutting open, we obtain k paths with $n+3$ vertices each and one path with $n+2$ vertices. The correction factor is x^{2k+1}, which gives us

$$\frac{E(P_{n+3}, x)^k \cdot E(P_{n+2}, x)}{x^{2k+1}}\,.$$

Summing all the contributions, we get

$$E(C_n^{k+1}, x) = E(P_{n-1}, x)E(C_n^k, x) + \frac{E(P_{n+3}, x)^k E(P_n, x)}{x^{2k}} + \frac{E(P_{n+2}, x)E(P_{n+3}, x)^k}{x^{2k+1}}\,.$$

Applying the inductive assumption gives

$$E(P_{n-1}, x)\left(\left(\frac{E(P_{n+3}, x)}{x^2}\right)^k - E(P_{n-1}, x)^k\right) + \frac{E(P_{n+3}, x)^k E(P_n, x)}{x^{2k}} + \frac{E(P_{n+2}, x)E(P_{n+3}, x)^k}{x^{2k+1}}\,.$$

Factoring out $\left(\frac{E(P_{n+3},x)}{x^2}\right)^k$, we obtain

$$\left(\frac{E(P_{n+3},x)}{x^2}\right)^k \left(E(P_{n-1},x) + E(P_n,x) + \frac{E(P_{n+2},x)}{x}\right) - E(P_{n-1},x)^{k+1}\,. \tag{4}$$

Note that from the recurrence of path polynomials given in Proposition 2 we deduce

$$E(P_{n-1},x) + E(P_n,x) = \frac{E(P_{n+1},x)}{x}\,.$$

Substituting this into Eq. (4), we obtain

$$\left(\frac{E(P_{n+3},x)}{x^2}\right)^k \left(\frac{E(P_{n+1},x)}{x} + \frac{E(P_{n+2},x)}{x}\right) - E(P_{n-1},x)^{k+1}\,. \tag{5}$$

By the same path-polynomial recurrence calculation used in Eq. (3), the expression in parentheses simplifies to $E(P_{n+3},x)/x^2$. Therefore, we get

$$E(C_n^{k+1}) = \left(\frac{E(P_{n+3},x)}{x^2}\right)^{k+1} - E(P_{n-1},x)^{k+1}\,,$$

which proves the claim. □

Acknowledgment

The second author would like to thank the MI-STEM Forward and GVSU OURS programs for partial financial support during this research project. The third author would like to thank the GVSU RISE Scholars Program for partial financial support. The first author acknowledges the use of generative AI tools for proofreading, language editing, LaTeX assistance, and suggestions on exposition and organization. All mathematical results and final decisions are the responsibility of the authors.

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