ANOTHER RECURRENCE RELATION FOR GENOCCHI NUMBERS

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Abstract. The two-dimensional rook theory can be generalized to three dimensions by assuming rooks attack along planes. With this generalization, there exists a family of three-dimensional boards whose rook numbers correspond to the unsigned Genocchi numbers of even index. Using this family and how rooks attack, we create a triangle generating the Genocchi numbers. We prove this triangle generation in two different ways.

Introduction

Using a generalization of rook polynomials to three dimensions, Alayont and Krzywonos describe a family of three-dimensional boards, called Genocchi boards, for which the numbers of ways of placing the maximum number of non-attacking rooks correspond to Genocchi numbers [1]. In this paper, we describe a triangle generation of Genocchi numbers using a recurrence relation for the number of rook placements on these Genocchi boards obtained from how one board embeds into one larger size board. In [1], the Genocchi boards are defined to be complementary to the triangle boards, another family of boards whose rook numbers correspond to central factorial numbers. Due to the two boards being complementary, the rook numbers of Genocchi boards can be expressed in terms of central factorial numbers. As a special case, the maximum rook numbers can also be expressed in terms of central factorial numbers. The correspondence between the Genocchi and central factorial numbers is then used to prove that the maximum rook numbers of the Genocchi boards are the Genocchi numbers [1]. In this paper, we provide two proofs for the recurrence relation generating the triangle for Genocchi numbers. The first is a combinatorial proof using the Genocchi boards. The second is an algebraic proof using the expression of rook numbers of Genocchi boards in terms of central factorial numbers.

We begin the paper with a background on the Genocchi and triangle boards in three dimensions. We then provide the recurrence relation for the rook numbers of Genocchi boards using the rook theory. In the last section, we include an algebraic proof of this relation using the expression of these rook numbers in terms of central factorial numbers.

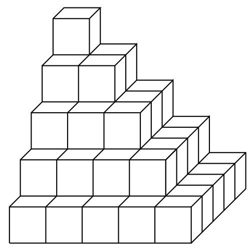
1. Rook Theory in Three and Higher Dimensions

In this section, we recall the generalized rook polynomial theory. A *board* in d dimensions is a subset of $[m_1] \times [m_2] \times \cdots \times [m_d]$ with cells corresponding to d-tuples $(i_1, i_2, \ldots, i_d)$ satisfying $1 \leq i_j \leq m_j$. A *full board* is the whole set $[m_1] \times [m_2] \times \cdots \times [m_d]$. In three and higher dimensions, rooks attack along hyperplanes which consist of cells with one fixed coordinate. In particular, in three dimensions, when we place a rook in cell (i_1, i_2, i_3), we can no longer place a rook in another cell with i_1 in the first coordinate, i_2 in the second coordinate, or i_3 in the third coordinate.

In three dimensions, we use *walls, slabs, layers* to denote the planes of cells with the first, second, third (respectively) coordinate fixed. Pictorially, we label layers from top to bottom. We continue to use the words rows and columns to refer to the rows and columns in each layer.

These are represented pictorially as rows and columns are represented in matrices. We say that the cells with fixed first and second coordinates form a *tower*.

In two dimensions, when rook numbers of triangle boards are considered, we obtain the sequence of Stirling numbers of the second kind. These triangle boards are generalized to three dimensions in [1] as follows. The size 1 triangle board is simply one cell. The size 2 triangle board is obtained by placing a 2×2 layer below the size 1 triangle board. Continuing recursively in a similar way, the size m board is obtained by adding an $m \times m$ layer at the bottom of a size $m-1$ triangle board. In terms of coordinates, the cells included in the size m triangle are (i, j, k) with $1 \le i, j \le k$ and $1 \le k \le m$. Size 5 triangle board in three dimensions is depicted below.



The rook numbers of these boards can be expressed in terms of central factorial numbers. Recall that the *central factorial numbers* are defined recursively by

$$T(r,s) = T(r-1, s-1) + s^2 T(r-1, s)$$

with $T(r, 1) = 1$ and $T(r, r) = 1$.

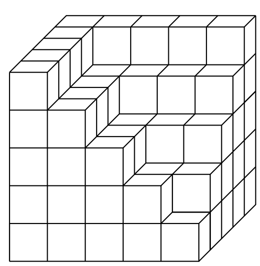
Theorem. [1] *The number of ways to place k rooks on a size n triangle board in three dimensions is equal to $T(n+1, n+1-k)$, where $0 \le k \le n$.*

Given a board B in d dimensions, we define the complement of B inside $[m_1] \times [m_2] \times \cdots \times [m_d]$ to be the set $[m_1] \times [m_2] \times \cdots \times [m_d] \backslash B$. We let $\bar{B}$ denote the complementary board.

Theorem (Complementary Board Theorem)**.** [1] *Let $\bar{B}$ be the complement of B inside $[m_1] \times [m_2] \times \cdots \times [m_d]$ and $R_B(x) = \sum_i r_i(B) x^i$ the rook polynomial of B. Then the number of ways to place $0 \le k \le \min_i m_i$ non-attacking rooks on B is*

$$r_k(\bar{B}) = \sum_{i=0}^{k} (-1)^i \binom{m_1 - i}{k-i} \binom{m_2 - i}{k-i} \cdots \binom{m_d - i}{k-i} (k-i)!^{d-1} r_i(B).$$

An interesting result is obtained when the complements of the three-dimensional triangle boards are considered. The picture below shows the complement of a size 4 triangle board inside $[5] \times [5] \times [5]$:



The number of ways to place the maximum number of rooks in this board is the unsigned 6th Genocchi number of even index (sequence A110501 in [4]). More generally, we let the complement of the size $n-1$ triangle board inside $[n] \times [n] \times [n]$ to be the size n *Genocchi board*, denoted by

Γ_n. By convention, the Genocchi board of size 1 consists of a single cell. We find that Γ_n has the following set definition in terms of the cell coordinates:

$$\Gamma_n = \{(i, j, k) : 1 \leq k \leq n \text{ and } 1 \leq i \text{ or } j \leq k\}.$$

The number of ways of placing n rooks on Γ_n is given by the $(n+1)$th unsigned Genocchi number of even index, which we denote by G_{n+1} [1]. This result is found using the complementary board theorem and the relationship

$$G_n = (-1)^n \sum_{j=1}^{n-1} (-1)^{j+1} j!^2 T(n-1, j)$$

between the Genocchi numbers and the central factorial numbers.

2. A recurrence relation for the rook numbers of Genocchi boards

Note that Γ_n includes the smaller board Γ_{n-1} by realizing Γ_{n-1} as

$$\Gamma_{n-1} = \{(i, j, k) : 2 \leq k \leq n \text{ and } 2 \leq i \text{ or } j \leq k\}.$$

Recall that we number the layers from top to bottom. The difference $\Gamma_n \backslash \Gamma_{n-1}$ consists of one outside slab corresponding to the first row of each layer in Γ_n and one outside wall corresponding to the first column of each layer in Γ_n. These two outside planes share a common tower at the corner.

Using this relationship between the two consecutive size Genocchi boards, we can obtain a recursive formula for the rook numbers of the Genocchi boards as given in the following theorem.

Theorem. *Let $r_k(\Gamma_n)$ denote the k-rook number of the size n Genocchi board. Then $r_k(\Gamma_n)$ satisfies the recurrence relation*

$$r_k(\Gamma_n) = r_k(\Gamma_{n-1}) + (2(n-k)+1)(n-k+1)r_{k-1}(\Gamma_{n-1}) + (n-k+2)(n-k+1)^3 r_{k-2}(\Gamma_{n-1})$$

where by convention we let $r_k(\Gamma_n) = 0$ for $k < 0$.

Proof: Consider the number of ways of placing k rooks in Γ_n. There are three possible cases: when all k rooks lie in Γ_{n-1} realized inside Γ_n, or when there are exactly $k-1$ rooks in Γ_{n-1}, or when there are exactly $k-2$ rooks in Γ_{n-1}.

In the first case, there are $r_k(\Gamma_{n-1})$ possible placements.

In the second case, there are $r_{k-1}(\Gamma_{n-1})$ possible placements for the $k-1$ rooks. Once these rooks are placed, the planes corresponding to these $k-1$ rooks in all three coordinates become unavailable on the outside wall and slab. After removing $k-1$ layers from the initial n layers, we have $n-k+1$ layers available for the last rook. On each layer, the outside wall and slab together contain $2n-1$ initial cells, corresponding to the first row and first column. However, by placing the $k-1$ rooks in Γ_{n-1}, $2k-2$ cells among these will become disqualified, leaving $2n-2k+1$ positions. Therefore, there are $(2(n-k)+1)(n-k+1)r_{k-1}(\Gamma_{n-1})$ ways to place k rooks in Γ_n so that $k-1$ rooks are in Γ_{n-1}.

The last case is similar to the second case. We have $r_{k-2}(\Gamma_{n-1})$ ways to place $k-2$ rooks in Γ_{n-1}. Note that we cannot place any of the remaining 2 rooks on the corner of the outside slab and wall, but that one rook must be in the slab and the other in the wall. Once the $k-2$ rooks are placed, there are $n-k+2$ layers and $n-1-(k-2)$ columns available on the slab corresponding to the first row. Hence there are $(n-k+2)(n-k+1)$ ways to place the rook on that slab. When we move to the wall corresponding to the first column, there are $n-k+1$ layers and $n-1-(k-2)$ rows available, which gives us $(n-k+1)^2$ ways to place the rook in that wall. Therefore there are a total of $(n-k+2)(n-k+1)^3 r_{k-2}(\Gamma_{n-1})$ ways to place the k in Γ_n so that $k-2$ rooks are in Γ_{n-1}.

Putting together all these three cases gives the recurrence relation in general. When $k < 2$, one or more of these cases do not exist, however, the recurrence relation still holds due to the fact that $r_k(\Gamma_n) = 0$ for $k < 0$. □

The resulting triangle consisting of $r_k(\Gamma_n)$'s is as follows:

$n \backslash k$	0	1	2	3	4	5	6
1	1	1					
2	1	7	3				
3	1	22	69	17			
4	1	50	507	959	155		
5	1	95	2227	13964	18077	2073	
6	1	161	7252	106720	468053	445671	38227

We can see from the triangle that $r_n(\Gamma_n)$'s correspond to the unsigned Genocchi numbers.

3. Algebraic proof of the recurrence relation

We now provide an algebraic proof of the recurrence relation using the complementary board theorem.

Lemma. *Let Γ_n denote the Genocchi board of size n. Then*

$$r_k(\Gamma_n) = \sum_{i=0}^{k} (-1)^i \binom{n-i}{k-i}^3 (k-i)!^2 T(n, n-i).$$

where $T(r,s)$ are the central factorial numbers.

Proof: Recall that the Genocchi board of size n is the complement of the triangle board of size $n-1$ inside $[n] \times [n] \times [n]$. The proof then follows immediately by using the complementary board theorem and the theorem on the rook numbers of triangle boards. □

Corollary: $r_n(\Gamma_n) = \sum_{i=0}^{n-1} (-1)^i (n-i)!^2 T(n, n-i)$.

Proof: From the lemma,

$$r_n(\Gamma_n) = \sum_{i=0}^{n} (-1)^i \binom{n-i}{n-i}^3 (n-i)!^2 T(n, n-i).$$

However, the term for $i = n$ vanishes due to $T(n, 0)$ not being defined. □

We now prove the recurrence relation on $r_k(\Gamma_n)$ using cases. For cases where $2 \le k \le n-1$, all three terms in the recurrence relation survives, so we prove those cases all together.

The case $k = 0$ is trivial since the recurrence relation becomes

$$1 = r_0(\Gamma_n) = r_0(\Gamma_{n-1}) = 1.$$

The case $k = 1$ reduces to

$$r_1(\Gamma_n) = r_1(\Gamma_{n-1}) + (2n-1) \cdot n \cdot r_0(\Gamma_{n-1}) = r_1(\Gamma_{n-1}) + (2n-1)n$$

which follows immediately since r_1 of any board is equal to the number of cells in that board.

The case $k = n$ reduces to proving

$$r_n(\Gamma_n) = r_{n-1}(\Gamma_{n-1}) + 2r_{n-2}(\Gamma_{n-1}).$$

Using the lemma and its corollary, this is equivalent to showing

$$\sum_{i=0}^{n-1}(-1)^i(n-i)!^2T(n,n-i) = \sum_{i=0}^{n-2}(-1)^i(n-1-i)!^2T(n-1,n-1-i)+ \\ 2\sum_{i=0}^{n-2}(-1)^i(n-1-i)^3(n-2-i)!^2T(n-1,n-1-i).$$

Noting that $(n-1-i)^3(n-2-i)!^2$ can be rewritten as $(n-1-i)(n-1-i)!^2$, we obtain

$$\sum_{i=0}^{n-1}(-1)^i(n-i)!^2T(n,n-i) = \sum_{i=0}^{n-2}(-1)^i(n-1-i)!^2T(n-1,n-1-i)+ \\ 2\sum_{i=0}^{n-2}(-1)^i(n-1-i)(n-1-i)!^2T(n-1,n-1-i).$$

Combining the two summations on the right hand side, we reduce the equality to

$$\sum_{i=0}^{n-1}(-1)^i(n-i)!^2T(n,n-i) = \sum_{i=0}^{n-2}(-1)^i(2(n-i)-1)(n-1-i)!^2T(n-1,n-1-i). \tag{1}$$

Using the recurrence relation on central factorial numbers, we have

$$T(n,n-i) = T(n-1,n-1-i) + (n-i)^2T(n-1,n-i),$$

where the first term vanishes for $i = n-1$ and the second term vanishes for $i = 0$. Therefore, the left hand side of the equality (3) can be rewritten as

$$\sum_{i=0}^{n-2}(-1)^i(n-i)!^2T(n-1,n-1-i) + \sum_{i=1}^{n-1}(-1)^i(n-i)^2(n-i)!^2T(n-1,n-i)$$

By making a change of variable in the second summation, we obtain

$$\sum_{i=0}^{n-2}(-1)^i(n-i)!^2T(n-1,n-1-i) - \sum_{i=0}^{n-2}(-1)^i(n-1-i)^2(n-1-i)!^2T(n-1,n-1-i) \tag{2}$$

Note that

$$(n-i)!^2-(n-1-i)^2(n-1-i)!^2 = (n-1-i)!^2((n-i)^2-(n-1-i)^2) = (n-1-i)!^2(2(n-i)-1).$$

Therefore, (2) can be rewritten as

$$\sum_{i=0}^{n-2}(n-1-i)!^2(2(n-i)-1)T(n-1,n-1-i)$$

which is exactly the same as the right hand side of (3). Therefore, the recurrence relation holds for $k = n$ as well.

In the case of $2 \le k \le n-1$, using the lemma, the recurrence relation translates into

$$\begin{aligned}\sum_{i=0}^{k}(-1)^i\binom{n-i}{k-i}^3(k-i)!^2T(n,n-i) &= \sum_{i=0}^{k}(-1)^i\binom{n-1-i}{k-i}^3(k-i)!^2T(n-1,n-1-i)\\ &+(2(n-k)+1)(n-k+1)\sum_{i=0}^{k-1}(-1)^i\binom{n-1-i}{k-1-i}^3(k-1-i)!^2T(n-1,n-1-i)\\ &+(n-k+2)(n-k+1)^3\sum_{i=0}^{k-2}(-1)^i\binom{n-1-i}{k-2-i}^3(k-2-i)!^2T(n-1,n-1-i)\end{aligned}$$

On the right hand side, we can combine the terms in the summations for $0 \le i \le k-2$. By rewriting each combination in terms of $\binom{n-1-i}{k-1-i}$, we find that the general term is

$$(3)\qquad (-1)^i\left(\frac{(n-k)^3}{k-i}+(2(n-k)+1)(n-k+1)+(n-k+2)(k-1-i)\right)\cdot\binom{n-1-i}{k-1-i}^3\cdot(k-1-i)!^2T(n-1,n-1-i)$$

On the left hand side, by using the recurrence relation for central factorial numbers as before, we obtain

$$\sum_{i=0}^{k}(-1)^i\binom{n-i}{k-i}^3(k-i)!^2T(n-1,n-1-i)+\sum_{i=1}^{k}(-1)^i\binom{n-i}{k-i}^3(k-i)!^2(n-i)^2T(n-1,n-i)$$

As before, by making a change of variable in the second summation, we can combine the two summations. The term corresponding to $0 \le i \le k-1$ in the combined summation then is

$$(4)\qquad (-1)^i\left(\frac{(n-i)^3}{k-i}-(n-i-1)^2\right)\binom{n-1-i}{k-1-i}^3(k-1-i)!^2T(n-1,n-1-i)$$

Therefore, to prove the recurrence relation all we need is to show the equality of (**??**) and (4). This follows algebraically by showing that the rational expressions are equal, which is a straightforward calculation.

For $i = k-1, k$, the remaining terms on both sides are

$$(-1)^kT(n-1,n-1-k)+(-1)^{k-1}(n-k+1)^3T(n-1,n-k)$$

on the left hand side, and

$$(-1)^kT(n-1,n-1-k)+(-1)^{k-1}(n-k)^3T(n-1,n-k)+(2(n-k)+1)(n-k+1)(-1)^{k-1}T(n-1,n-k)$$

on the right hand side. It can be easily checked that these two expressions are equal.

Therefore, this finishes the algebraic proof that the recurrence relation holds.

References

[1] Feryal Alayont, Nicholas Krzywonos, “Rook Polynomials in Three and Higher Dimensions”, to appear in *Involve*

[2] Michael Domaratzki, “Combinatorial Interpretations of a Generalization of the Genocchi Numbers”’, *Journal of Integer Sequences*, Volume 7 (2004), Article 04.3.6.

[3] Guo-Niu Han, “Escaliers Évalués et Nombres Classiques”, *Publ. I.R.M.A. Strasbourg, Actes 24e Séminaire Lotharingien*, 461/S24 (1993), 77-85.

[4] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/

[5] Benjamin Zindle, "Rook Polynomials for Chessboards of Two and Three Dimensions", School of Mathematical Sciences, Rochester Institute of Technology, New York, 2007.

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