MEROMORPHIC CONTINUATION OF MINIMAL PARABOLIC EISENSTEIN SERIES IN GL_n AND Sp_n

1. Definition and properties

This exposition follows the proof of meromorphic continuation of minimal parabolic Eisenstein series for GL_n and Sp_n as described in the Appendix 1 of [Langlands 1976]. We present the argument presented there in the case of $k = \mathbf{Q}$ using modern language.

We first start with $G = GL_n$. Let $P = P_{\min}$, the minimal parabolic in GL_n. Define φ_s corresponding to a tuple $s = (s_1, .., s_n)$ of complex numbers by

$$\varphi_s(g) = \varphi_s(nak) = a_1^{s_1} \cdots a_n^{s_n}$$

where $g = nak$ is the Iwasawa decomposition $G = NAK$, a_i are the diagonal entries of a. The Eisenstein series with this data is

$$E(s, g) = E(\varphi_s, g) = \sum_{\gamma \in P(\mathbf{Z}) \backslash G(\mathbf{Z})} \varphi_s(\gamma g) \delta_P^{1/2}(a)$$

where $\delta_P = \prod_i a_i^{n+1-2i}$ is the modulus function of P.

Theorem. *The series defining E is absolutely convergent for $s = (s_1, .., s_n)$ with $\mathrm{Re}(s_i - s_{i+1})$ sufficiently large for all $i < n$.*

Proof: Let $\eta_i(g)$ denote the determinant of the upper left $i \times i$ minor of $(gg^t)^{-1}$ where $g \in GL_n(\mathbf{R})$ with $n > i$. For $g = nak$ this function is

$$\eta_i(g) = a_1(g)^{-2} \cdots a_i(g)^{-2}$$

Observe that for $g \in GL_n(\mathbf{R})$,

$$\varphi_s(g) \delta_P^{1/2}(a) = \prod_{i=1}^{n} \eta_i(g)^{-\frac{s_i - s_{i+1} + 1}{2}}$$

with $s_{n+1} = \frac{n+1}{2}$, by convention.

We can use this observation to bound E by

$$|E(g)| = \left| \sum_{\gamma \in P(\mathbf{Z}) \backslash G(\mathbf{Z})} \prod_{i=1}^{n} \eta_i(\gamma g)^{-\frac{s_i - s_{i+1} + 1}{2}} \right| \leq \sum_{\gamma} \eta_n(g)^{-\frac{\sigma_n - \frac{n-1}{2}}{2}} \prod_{1}^{n-1} \eta_i(\gamma g)^{-\frac{\sigma_i - \sigma_{i+1} + 1}{2}}$$

where $\sigma_i = \mathrm{Re}(s_i)$. The last sum is dominated by

$$\eta_n(g)^{-\frac{\sigma_n - \frac{n-1}{2}}{2}} \prod_{i=1}^{n-1} \sum_{\gamma \in P_i(\mathbf{Z}) \backslash G(\mathbf{Z})} \eta_i(\gamma g)^{-\frac{\sigma_i - \sigma_{i+1} + \frac{1}{2}}{2}}$$

where P_i is the i^{th} maximal parabolic consisting of those matrices with 0's in the lower-left $i \times (n-i)$ block. Since η_i is determined left modulo $P_i(\mathbf{Z})$, each term in the previous sum can be obtained as the product of the $\eta_i(\gamma g)$'s corresponding to the image of γ in $P_i(\mathbf{Z}) \backslash G(\mathbf{Z})$. No two different cosets of $P(\mathbf{Z})$ will give the same set of coset representatives for all i, so each term in the sum over $P(\mathbf{Z})\backslash G(\mathbf{Z})$ is dominated by a different product. Since we have a finite product, it suffices to prove that each

$$\sum_{\gamma\in P_{i,n-i}(\mathbf{Z})\backslash G(\mathbf{Z})} \eta_i(\gamma g)^{-\frac{\sigma_i-\sigma_{i+1}+1}{2}}$$

is finite to prove convergence.

For fixed $g \in GL_n(\mathbf{R})$ there exists a norm ρ_i on $\wedge^i\mathbf{R}^n$ such that for $w_1, \cdots, w_i \in \mathbf{R}^n$

$$\rho_i(w_1 \wedge \cdots \wedge w_i) = \det(w^t(gg^t)^{-1}w)^{1/2}$$

where w is the $n \times i$ matrix with columns $w_1, \cdots, w_i$. Hence the sum

$$\sum_{\substack{v\in\wedge^i\mathbf{Z}^n \\ v \text{ primitive}}} \rho_i(v)^{-s}$$

is convergent for $\mathrm{Re}(s) > \binom{n}{i} = \dim \wedge^i\mathbf{R}^n$. This sum, for $s = \sigma_i - \sigma_{i+1} + 1$, dominates

$$\sum_{\gamma\in P_i(\mathbf{Z})\backslash G(\mathbf{Z})} \eta_i(\gamma g)^{-\frac{\sigma_i-\sigma_{i+1}+1}{2}}$$

This follows because we can associate a primitive vector to each $\gamma \in P_i(\mathbf{Z})\backslash G(\mathbf{Z})$ by taking the wedge of the first i columns of γ^{-1} and ρ_i evaluated at this wedge product equals $\eta_i(\gamma g)$. Hence the sum over $P_{i,n-i}(\mathbf{Z})\backslash G(\mathbf{Z})$ converges for $\sigma_i - \sigma_{i+1} > \binom{n}{i} - 1$. $\square$

2. Meromorphic continuation foror GL_n

For the functional equations we normalize the Eisenstein series. For $t \in \mathbf{C}$, set

$$Z(t) = t(t-1)\pi^{-t}\Gamma(t)\zeta(2t)$$

Then the normalized Eisenstein series is

$$E^*(s, g) = \prod_{i>j} Z(\frac{1 + s_j - s_i}{2})E(s, g)$$

Theorem. *The normalized Eisenstein series can be analytically continued to an entire function of $s = (s_1, \cdots, s_n) \in \mathbf{C}^n$ invariant under permutations of the s_i's. Moreover, the continuation is of polynomial growth in bounded tube domains in $\mathbf{C}^n$.*

Proof: The theorem will be proved by a two step induction.

In case $n = 1$, the theorem is obvious since $E^*(s, g) = g^s$.

Now let $n = 2$. We define the theta series

$$\theta(Y) = \sum_{v\in\mathbf{Z}^2} e^{-\pi v^t Y v}$$

for a positive definite matrix $Y \in GL_2(\mathbf{R})$. The functional equation of this series is obtained by Poisson summation.

$$\theta(Y) = (\det Y)^{-1/2}\theta(Y^{-1})$$

Now let $Y = (gg^t)^{-1}$ with $g \in GL_2(\mathbf{R})$. Then

$$\int_0^\infty (\theta(tY) - 1)t^{\frac{s_1-s_2+1}{2}} d^\times t = |\det g|^{-s_2+\frac{1}{2}}\pi^{-\frac{s_1-s_2+1}{2}}\Gamma(\frac{s_1 - s_2 + 1}{2})\zeta(s_1 - s_2 + 1)E(s_1, s_2, g)$$

On the other hand, using the functional equation of the theta series, we get

$$\begin{aligned}\int_0^\infty (\theta(tY)-1)\, t^{\frac{s_1-s_2+1}{2}} d^\times t &= \frac{2}{s_2-s_1-1} + \frac{2|\det g|}{s_1-s_2-1} \\ &\quad +|\det g| \int_1^\infty (\theta(tY^{-1})-1)\, t^{\frac{s_2-s_1+1}{2}} d^\times t \\ &\quad + \int_1^\infty (\theta(tY)-1)\, t^{\frac{s_1-s_2+1}{2}} d^\times t\end{aligned}$$

Both of the last integrals are entire because the theta functions (less constant terms) are of rapid decay. Therefore, this integral representation meromorphically continues the Eisenstein series to the whole plane. Furthermore, we get the functional equation, which is that

$$|\det g|^{-\frac{1}{2}} \int_0^\infty (\theta(tY)-1) t^s d^\times t$$

is invariant under

$$s \to 1-s \text{ and } Y \to Y^{-1}$$

This corresponds to the functional equation of the E^*:

$$|\det g|^{-s_2} E^*(s_1,s_2,g) = |\det (g^t)^{-1}|^{-s_1} E^*(s_2,s_1,(g^t)^{-1})$$

Using $(g^t)^{-1} = (\det g)^{-1} wgw^{-1}$, where w is the longest Weyl element in $GL_2(\mathbf{R})$

$$E^*(s_1,s_2,g) = |\det g|^{s_1+s_2} E^*(s_2,s_1,(\det g)^{-1}wgw^{-1}) = E^*(s_2,s_1,g)$$

To prove the claim about the growth of E^* we will use the Phragmén-Lindelöf principle. For $\mathrm{Re}(s_1-s_2)$ sufficiently large,

$$E(s_1,s_2,g) = (\det g)^{s_2-\frac{1}{2}} E(s_1-s_2+\frac{1}{2},0,g)$$

So we can consider $E(s,0,g)$ instead of $E(s_1,s_2,g)$ to bound E^*. For $\mathrm{Re}(s)=\sigma$ fixed and large enough

$$|E(s,0,g)| \le E(\sigma,0,g)$$

The normalizing factors are also of polynomial growth in the imaginary coordinate for fixed real part. Using the functional equation, similar bounds can be obtained for fixed $\mathrm{Re}(s)$ small. Since E^* is entire, now we can use Phragmén-Lindelöf principle to obtain the polynomial growth in bounded vertical strips.

Suppose now that the claim is proved for all E^* in dimensions less than n. We can express E when the dimension is n as an iterated sum consisting of smaller dimensional Eisenstein series

$$\begin{aligned}E(s,g) &= \sum_{\gamma\in P_i(\mathbf{Z})\backslash G(\mathbf{Z})} \sum_{\delta\in P(\mathbf{Z})\backslash P_i(\mathbf{Z})} \varphi_s(\delta\gamma g)\delta_P^{1/2}(\delta\gamma g) \\ &= \sum_\gamma \prod_1^i a_j(\gamma g)^{\frac{n-i}{2}} \prod_{i+1}^n a_j(\gamma g)^{-\frac{i}{2}} \sum_{\delta_1,\delta_2} \varphi\delta_{P_1}^{1/2}(\delta_1 m_1(\gamma g))\varphi\delta_{P_2}^{1/2}(\delta_2 m_2(\gamma g))\end{aligned}$$

where δ_1 and δ_2 range over $P_1(\mathbf{Z})\backslash GL_i(\mathbf{Z})$ and $P_2(\mathbf{Z})\backslash GL_{n-i}(\mathbf{Z})$ respectively, P_1 and P_2 being the minimal parabolics in the GL's. Rewrite

$$E(s,g) = |\det g|^{-\frac{i}{2}} \sum_{\gamma\in P_{i,n-i}(\mathbf{Z})\backslash G(\mathbf{Z})} E(s_1-\frac{n}{4},\cdots,s_i-\frac{n}{4},m_1(\gamma g))\, E(s_{i+1},\cdots,s_n,m_2(\gamma g))$$

Taking the normalizing factors into consideration we obtain a similar expression

$$\begin{aligned} E^*(s,g) &= |\det g|^{-\frac{i}{2}}\, Z_i(s) \\ &\times \sum_{\gamma \in P_{i,n-i}(\mathbf{Z})\backslash G(\mathbf{Z})} E^*(s_1 - \frac{n}{4}, \cdots, s_i - \frac{n}{4}, m_1(\gamma g))\, E^*(s_{i+1}, \cdots, s_n, m_2(\gamma g)) \end{aligned}$$

where

$$Z_i(s) = \prod_{j \le i, k > i} Z(\frac{1 + s_j - s_k}{2})$$

We showed that the original series defining $E^*(s,g)$ converges when $\sigma_i - \sigma_{i+1} \ge b$ for all $i < n$, for a sufficiently large constant b. By induction the theorem is true for the inside Eisenstein series for dimensions i and $n-i$, hence from the iterated expression of E^* the series is symmetric in the first i and last $n-i$ coordinates separately. Therefore, for a permutation π leaving $\{1, \cdots, i\}$ and $\{i+1, \cdots, n\}$ stable, the series will converge when $\sigma_{\pi(i)} - \sigma_{\pi(i+1)} \ge b$ for all $i < n$.

We need the following combinatorial lemma to express certain s's as convex sums of points in the previous region, which is a union of certain reflections of the original convergence region.

Lemma 1. (Langlands) If $\gamma = (\gamma_1, \cdots, \gamma_r)$ is an r-tuple of real numbers and $b > 0$, there are r-tuples γ' and γ'' such that

- $\gamma = \frac{1}{2}(\gamma' + \gamma'')$
- $|\gamma_j - \gamma'_j| \le c$ and $|\gamma_j - \gamma''_j| \le c$ for all j
- there are permutations π' and π'' such that

$$\gamma'_{\pi'(j)} - \gamma'_{\pi'(j+1)} \ge b,\ \gamma''_{\pi''(j)} - \gamma''_{\pi''(j+1)} \ge b,\ j < r$$

where $c > 0$ is a constant depending on r and b only.

Proof: We proceed by induction. The case $r = 1$ is trivial. Suppose next that the lemma is proven for $1, \cdots, r-1$. Without loss of generality, assume $c_1(b) \le c_2(b) \le \cdots \le c_{r-1}(b)$ and $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_r$. If for some j, $\gamma_j - \gamma_{j+1} \ge 2c_{r-1}(b) + b$, then apply the induction hyptothesis to $(\gamma_1, \cdots, \gamma_j)$ and $(\gamma_{j+1}, \cdots, \gamma_r)$ and combine the two subsequences. If there is no such j, let $a = 2(2c_{r-1}(b) + b)$ and

$$\gamma'_1 = \gamma_1 + (r-1)a,\ \gamma'_2 = \gamma_2 + (r-2)a, \cdots \gamma'_r = \gamma_r$$

$$\gamma''_1 = \gamma_1 - (r-1)a,\ \cdots,\ \gamma''_r = \gamma_r$$

These γ' and γ'' satisfy the requirements of the lemma. $\square$

Fix $i < n$. We will show that the iterated sum expressing E^* converges in the region

$$\sigma_j - \sigma_k \ge c_1 + c_2 + b \ \text{ for all } j \le i, k > i \qquad (*)$$

where c_1 and c_2 are constants coming from the previous lemma in case of an i-tuple and an $(n-i)$-tuple of real numbers. Let $s = (\sigma_1 + i\tau_1, \cdots, \sigma_n + i\tau_n)$ be in this region. Apply the lemma to the tuples $(\sigma_1, \cdots, \sigma_i)$ and $(\sigma_{i+1}, \cdots, \sigma_n)$ to get $(\sigma'_1, \cdots, \sigma'_i)$ and $(\sigma'_1, \cdots, \sigma''_i)$ for the first tuple and $(\sigma'_{i+1}, \cdots, \sigma'_n)$ and $(\sigma''_{i+1}, \cdots, \sigma''_n)$ for the second. Combining the corresponding primed tuples

$$\begin{aligned} s' &= (\sigma'_1 + i\tau_1, \cdots, \sigma'_n + i\tau_n) \\ s'' &= (\sigma''_1 + i\tau_1, \cdots, \sigma''_n + i\tau_n) \end{aligned}$$

By the properties of s and the primed tuples, s' and s'' both lie in the region where the iterated series expression for E^* converges and $s = \frac{1}{2}(s' + s'')$. Moreover, from the entireness of the normalized Eisenstein series for smaller dimensions, this expression equals

$$
\begin{aligned}
&|\det g|^{-\frac{i}{2}} \sum_{\gamma\in P_{i,n-i}(\mathbf{Z})\backslash G(\mathbf{Z})} \frac{e^{-\frac{1}{4}}}{2\pi i} \\
&\qquad \int_{1-i\infty}^{1+i\infty} e^{t^2}(t-\frac{1}{2})^{-1} Z_i(s(t)) E^*(s_1(t)-\frac{n}{4},\cdots,m_1(\gamma g))\, E^*(s_{i+1}(t),\cdots,m_2(\gamma g))\, dt \\
&\qquad + \int_{i\infty}^{-i\infty} e^{t^2}(t-\frac{1}{2})^{-1} Z_i(s(t)) E^*(s_1(t)-\frac{n}{4},\cdots,m_1(\gamma g))\, E^*(s_{i+1}(t),\cdots,m_2(\gamma g))\, dt
\end{aligned}
$$

Let us consider the integral on $\mathrm{Re}(s)=1$. For $t=1+ix$, the integrand is bounded by

$$
\begin{aligned}
e^{1-x^2} \prod_{k>j} \left| \frac{(s_j-s_k+1)(s_j-s_k-1)}{4} \right| \prod_{\substack{i\geq k>j \\ \text{or} \\ k>j>i}} \pi^{-\frac{\sigma_j-\sigma_k+1}{2}} \Gamma(\frac{\sigma_j-\sigma_k+1}{2})\zeta(\sigma_j-\sigma_k+1) \\
\times\ |E(s_1(t)-\frac{n}{4},\cdots,s_i(t)-\frac{n}{4},m_1(\gamma g))|\,|E(s_{i+1}(t),\cdots,s_n(t),m_2(\gamma g))|
\end{aligned}
$$

Note that $\mathrm{Re}(s_1(t),\cdots,s_i(t)) = (\sigma'_1,\cdots,\sigma'_i)$ and $(\sigma'_1,\cdots,\sigma'_i)$ was chosen so that for some permutation π_1 of $\{1,\cdots,i\}$, $\sigma'_{\pi_1(j)}-\sigma'_{\pi_1(j+1)} \geq b$ for all $j<i$. Then using the symmetry of the $E(s_1(t)-\frac{n}{4},\cdots,s_i(t)-\frac{n}{4})$ we have

$$
\begin{aligned}
|E(s_1(t)-\frac{n}{4},\cdots,s_i(t)-\frac{n}{4},m_1(\gamma g))| &= |E(s_{\pi_1(1)}(t)-\frac{n}{4},\cdots,s_{\pi_1(i)}(t)-\frac{n}{4},m_1(\gamma g))| \\
&\leq E(\sigma'_{\pi_1(1)}-\frac{n}{4},\cdots,\sigma'_{\pi_1(i)}-\frac{n}{4},m_1(\gamma g))
\end{aligned}
$$

The inequality holds because $(s_{\pi_1(1)}(t)-\frac{n}{4},\cdots,s_{\pi_1(i)}(t)-\frac{n}{4})$ is in the region of convergence of the original series definition. Similar considerations show that

$$
|E(s_{i+1}(t),\cdots,s_n(t),m_2(\gamma g))| \leq E(\sigma'_{\pi_2(i)},\cdots,\sigma'_{\pi_2(n)},m_2(\gamma g))
$$

where π_2 is a permutation of $\{i+1,\cdots,n\}$ so that $(\sigma'_{\pi_2(i)},\cdots,\sigma'_{\pi_2(n)})$ lies in the region of convergence of the original series definition. Therefore, we can bound the integral on $\mathrm{Re}(s)=1$ by

$$
C\cdot E(\sigma'_{\pi_1(1)}-\frac{n}{4},\cdots,\sigma'_{\pi_1(i)}-\frac{n}{4},m_1(\gamma g))E(\sigma'_{\pi_2(i)},\cdots,\sigma'_{\pi_2(n)},m_2(\gamma g))
$$

where the constant C depends only on $\mathrm{Re}(s)$. Similarly, the integral on $\mathrm{Re}(s)=0$ is bounded by a constant multiple of

$$
E(\sigma''_{\pi'_1(1)}-\frac{n}{4},\cdots,\sigma''_{\pi'_1(i)}-\frac{n}{4},m_1(\gamma g))E(\sigma''_{\pi'_2(i)},\cdots,\sigma''_{\pi'_2(n)},m_2(\gamma g))
$$

When we sum these bounds on $P_i(\mathbf{Z})\backslash G(\mathbf{Z})$, we get the iterated series expressions for

$$
E(\sigma'_{\pi_1(1)}-\frac{n}{4},\cdots,\sigma'_{\pi_1(i)}-\frac{n}{4},\sigma'_{\pi_2(i)},\cdots,\sigma'_{\pi_2(n)},g)
$$

and

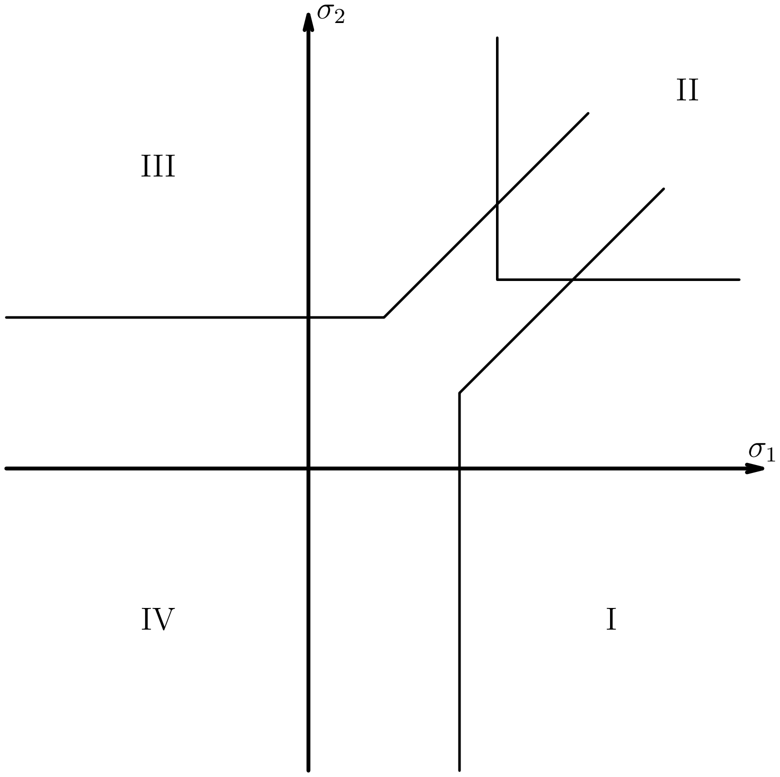
$$
E(\sigma''_{\pi'_1(1)}-\frac{n}{4},\cdots,\sigma''_{\pi'_1(i)}-\frac{n}{4},\sigma''_{\pi'_2(i)},\cdots,\sigma''_{\pi'_2(n)},g)
$$

respectively. This proves that the expression with sum and integral is absolutely convergent, and hence defines a continuation of E^* to the region given by (*). Moreover, when we interchange summation and integration we obtain

$$
\begin{aligned}
E^*(s,g) &= \frac{e^{-\frac{1}{4}}}{2\pi i}\int_{1-i\infty}^{1+i\infty} e^{t^2}(t-\frac{1}{2})^{-1}E^*(s(t),g)\,dt \\
&\quad + \frac{e^{-\frac{1}{4}}}{2\pi i}\int_{i\infty}^{-i\infty} e^{t^2}(t-\frac{1}{2})^{-1}E^*(s(t),g)\,dt
\end{aligned}
$$

Since the $E^*(s(t),g)$ inside the integrals is of polynomial growth, we obtain the same property for $E^*(s,g)$.

Now we prove the continuation to the whole plane. Let region I be the above defined region for $i=1$ and region II the one for $i=2$. From the above considerations the series has been continued to regions I and II. The symmetric image of this series in s_1 and s_2 will be defined in region III and will agree with the previous one on the intersection of regions II and III. Therefore, it gives a meromorphic continuation to the first three regions. So the picture, when projected to the σ_1,σ_2-plane is



The meromorphic continuation to the region IV will be achieved by a Cauchy integral. For c large enough, consider

$$
\begin{aligned}
&\frac{e^{-(s_1-s_2)^2}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\zeta^2}(\zeta-(s_1-s_2))^{-1}E^*(\frac{s_1+s_2+\zeta}{2},\frac{s_1+s_2-\zeta}{2},s_3,\cdots,s_n,g)\,d\zeta \\
&\quad + \frac{e^{-(s_1-s_2)^2}}{2\pi i}\int_{-c+i\infty}^{-c-i\infty} e^{\zeta^2}(\zeta-(s_1-s_2))^{-1}E^*(\frac{s_1+s_2+\zeta}{2},\frac{s_1+s_2-\zeta}{2},s_3,\cdots,s_n,g)\,d\zeta
\end{aligned}
$$

First fix c. For $s=(s_1,\cdots,s_n)$ in regions I, II or III, this expression is equal to $E^*(s_1,\cdots,s_n,g)$. Therefore, for fixed c, this expression is the meromorphic continuation of E^* to the region

$$
\left\{ s : (\frac{s_1+s_2\pm c}{2},\frac{s_1+s_2\mp c}{2},s_3,\cdots,s_n) \text{ lies in the regions I, II or III} \right\}
$$

For different c's, the corresponding expressions agree on a connected intersection, hence the continuations are the same. $\square$

3. Meromorphic continuation for Sp_n

In this section let $G = Sp_n$ and $P = P_{\min}$ the minimal parabolic in Sp_n. As before we define φ_s corresponding to a tuple $s = (s_1, .., s_n)$ of complex numbers by

$$\varphi_s(g) = \varphi_s(nak) = a_1^{s_1} \cdots a_n^{s_n}$$

where $g = nak$ is the Iwasawa decomposition $G = NAK$, a_i are the first n of the diagonal entries of a. The Eisenstein series with this data is

$$E(s, g) = E(\varphi_s, g) = \sum_{\gamma \in P(\mathbf{Z}) \backslash G(\mathbf{Z})} \varphi_s(\gamma g) \delta_P^{1/2}(a)$$

where $\delta_P = \prod_i a_i^{2n+2-2i}$ is the modulus function of P.

Theorem. *The series defining E is absolutely convergent for $s = (s_1, .., s_n)$ with* $\operatorname{Re}(s_i - s_{i+1})$ *(for $i < n$) and* $\operatorname{Re}(s_n)$ *all sufficiently large.*

Proof: Let $\eta_i(g)$ be defined by

$$\eta_i(g) = a_1(g)^{-2} \cdots a_i(g)^{-2}$$

Then

$$\varphi_s(g) \delta_P^{1/2}(g) = \prod_{i=1}^{n} \eta_i(g)^{-\frac{s_i - s_{i+1} + 1}{2}}$$

with $s_{n+1} = 0$ by convention. Hence as before

$$|E(g)| = \left| \sum_{\gamma \in P(\mathbf{Z}) \backslash G(\mathbf{Z})} \prod_{i=1}^{n} \eta_i(\gamma g)^{-\frac{s_i - s_{i+1} + 1}{2}} \right| \leq \prod_{i=1}^{n} \sum_{\gamma \in P_i(\mathbf{Z}) \backslash G(\mathbf{Z})} \eta_i(\gamma g)^{-\frac{\sigma_i - \sigma_{i+1} + 1}{2}}$$

where P_i denotes the i-th maximal parabolic subgroup in Sp_n consisting of those matrices with 0's in the lower left $(2n - i) \times i$ block.

With the norm ρ_i on $\wedge^i \mathbf{R}^{2n}$ defined by

$$\rho_i(w_1 \wedge \cdots \wedge w_i) = \det(w^t (g g^t)^{-1} w)^{1/2}$$

for $w_1, \cdots, w_i \in \mathbf{R}^{2n}$, we can bound each term in the product by

$$\sum_{\substack{v \in \wedge^i \mathbf{Z}^{2n} \\ v \text{ primitive}}} \rho_i(v)^{-\sigma_i + \sigma_{i+1} - 1}$$

and this sum converges for $\sigma_i - \sigma_{i+1} + 1 > \binom{2n}{i}$. Note that when $i = n$, this gives the condition $\sigma_n + 1 > \binom{2n}{i}$. $\square$

We normalize the Eisenstein series for Sp_n as follows:

$$E^*(s, g) = \prod_{i > j} Z\left(\frac{1 + s_j - s_i}{2}\right) Z\left(\frac{1 + s_j + s_i}{2}\right) \prod_i Z\left(\frac{s_i + 1}{2}\right) E(s, g)$$

Theorem. *The normalized Eisenstein series for Sp_n can be analytically continued to an entire function of $s = (s_1, \cdots, s_n) \in \mathbf{C}^n$ invariant under signed permutations of the s_i.*

Proof: As before rewrite E as an iterated sum, factoring through $P_i(\mathbf{Z})\backslash G(\mathbf{Z})$ for each $i \leq n$.

$$E(s,g) = \sum_{\gamma \in P_i(\mathbf{Z})\backslash G(\mathbf{Z})} \sum_{\delta \in P(\mathbf{Z})\backslash P_n(\mathbf{Z})} \varphi_s(\delta\gamma g)\delta_P^{1/2}(\delta\gamma g)$$

In the first case let $i = n$. Identify $P(\mathbf{Z})\backslash P_n(\mathbf{Z})$ with $P_{\min}(\mathbf{Z})\backslash GL_n(\mathbf{Z})$ by

$$\delta_1 \in P_{\min}(\mathbf{Z})\backslash GL_n(\mathbf{Z}) \quad \longleftrightarrow \quad \delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & (\delta_1^t)^{-1} \end{bmatrix} \in P(\mathbf{Z})\backslash P_n(\mathbf{Z})$$

For such $\delta_1 \in P_{\min}(\mathbf{Z})\backslash GL_n(\mathbf{Z})$

$$\varphi_s(\delta\gamma g)\delta_P^{1/2}(\delta\gamma g) = \prod_{i=1}^{n} a_i(\delta_1 m(\gamma g))^{s_i+n-i+1}$$

where $m(\gamma g)$ is obtained from the Iwasawa decomposition

$$\gamma g = n \begin{bmatrix} m(\gamma g) & 0 \\ 0 & (m(\gamma g)^t)^{-1} \end{bmatrix} k$$

with respect to P_n. The iterated sum becomes

$$E(s,g) = \sum_{\gamma \in P_n(\mathbf{Z})\backslash G(\mathbf{Z})} (GL)E(s + \frac{n+1}{2}, m(\gamma g))$$

The corresponding iterated expression for the normalized Eisenstein series is

$$E^*(s,g) = \prod_{i>j} Z(\frac{1+s_j-s_i}{2}) \prod_i Z(\frac{s_i+1}{2}) \sum_{\gamma \in P_n(\mathbf{Z})\backslash G(\mathbf{Z})} (GL)E^*(s + \frac{n+1}{2}, m(\gamma g))$$

This expression is valid for all s satisfying $\mathrm{Re}(s_i - s_{i+1}) > b$ for $i < n$ and $\mathrm{Re}(s_n) > b$, for a sufficiently large constant b. From the results on Eisenstein series for GL, each summand in the previous expression is entire and symmetric in the s_i. So the series is defined for s which, under a permutation of the indices, lies in the previous region. As in the GL_n case, by using the combinatorial lemma, we can express points in the region $\mathrm{Re}(s_i) > c_n(b) + b$ for all i, as a convex combination of points in the previous region, therefore using the same method as before we show that $E^*(s,g)$ converges in region $\mathrm{Re}(s_i) > c_n(b)+b$ for all i (where $c_n(b)$ is the constant occuring in the combinatorial lemma) .

Now let $i = n-1$ in the iterated sum expression. Identify $P(\mathbf{Z})\backslash P_{n-1}(\mathbf{Z})$ with $P_{\min}(\mathbf{Z})\backslash GL_{n-1}(\mathbf{Z}) \times P_{\min}(\mathbf{Z})\backslash Sp_1(\mathbf{Z})$ by

$$(\delta_1, \delta_2) = (\delta_1, \begin{bmatrix} a & b \\ c & d \end{bmatrix}) \quad \in \quad P_{\min}(\mathbf{Z})\backslash GL_{n-1}(\mathbf{Z}) \times P_{\min}(\mathbf{Z})\backslash Sp_1(\mathbf{Z})$$

$$\longleftrightarrow \quad \delta = \begin{bmatrix} \delta_1 & & & \\ & a & & b \\ & & (\delta_1^t)^{-1} & \\ & c & & d \end{bmatrix} \in P(\mathbf{Z})\backslash P_{n-1}(\mathbf{Z})$$

For such $\delta \in P(\mathbf{Z})\backslash P_{n-1}(\mathbf{Z})$,

$$\varphi_s(\delta\gamma g)\delta_P^{1/2}(\delta\gamma g) = \prod_{i=1}^{n-1} a_i(\delta_1 m_1(\gamma g))^{s_i+n+1-i} a_1(\delta_1 m_2(\gamma g))^{s_n+n+1-i}$$

where $m_1(\gamma g)$ and $m_2(\gamma g)$ occur in the Iwasawa decomposition

$$\gamma g = n \begin{bmatrix} m_1(\gamma g) & & & \\ & (m_2(\gamma g))_{11} & (m_2(\gamma g))_{12} & \\ & & (m_1(\gamma g)^t)^{-1} & \\ & (m_2(\gamma g))_{21} & & (m_2(\gamma g))_{22} \end{bmatrix} k$$

with respect to P_{n-1}. So the iterated sum becomes

$$\begin{aligned} E(s,g) &= \sum_{\gamma \in P_{n-1}(\mathbf{Z})\backslash G(\mathbf{Z})} \sum_{\delta_1 \in P_{\min}(\mathbf{Z})\backslash GL_{n-1}(\mathbf{Z})} \prod_1^{n-1} a_i(\delta_1 m_1(\gamma g)) \sum_{delta_2 \in P_{\min}(\mathbf{Z})\backslash SL_2(\mathbf{Z})} a_1(\delta_2 m_2(\gamma g)) \\ &= \sum_{\gamma \in P_{n-1}(\mathbf{Z})\backslash G(\mathbf{Z})} (GL)E(s_1 + \frac{n}{2} + 1, \cdots, s_{n-1} + \frac{n}{2} + 1, m_1(\gamma g)) \\ &\qquad \times (SL)E(s_n + \frac{1}{2}, \frac{1}{2}, m_2(\gamma g)) \end{aligned}$$

When we take the normalizations into account, the iterated expression is

$$\begin{aligned} E^*(s,g) &= \prod_{i>j} Z(\frac{1+s_j+s_i}{2}) \prod_{i<n} Z(\frac{s_i+1}{2}) \\ &\qquad \times (GL)E^*(s_1 + \frac{n}{2} + 1, \cdots, s_{n-1} + \frac{n}{2} + 1, m_1(\gamma g))(SL)E^*(s_n + \frac{1}{2}, \frac{1}{2}, m_2(\gamma g)) \end{aligned}$$

This expression is valid in the region $\mathrm{Re}(s_i) > b$ for all i. Using the fact that GL-Eisenstein series are symmetric in s_i's, we get that $(SL)E^*(s_n + \frac{1}{2}, \frac{1}{2}, g)$ is invariant under $s_n \to -s_n$. Therefore the iterated expression converges in the region $\mathrm{Re}(s_i) > b$ for all $i < n$ and $\mathrm{Re}(s_n) < -b$. If we apply the Phragmén-Lindelöf principle, we get continuation to the region $\mathrm{Re}(s_i) > b$ for all $i < n$ and $\mathrm{Re}(s_n)$ arbitrary.

The region where each $\mathrm{Re}(s_i) > b$ for all i is in the intersection of the regions where one of the real parts is arbitrary. Furthermore, the series is symmetric in the intersection. Hence, we can continue the series by using symmetry to those regions where one of the real parts is arbitrary. The convex hull of those regions is $\mathbf{C}^n$. However, we can apply the Cauchy integral formula as we did in the GL-Eisenstein series case only to linear combinations with two terms. So we proceed step by step starting with two regions where the first one is s_i, s_{i_1} arbitrary and $\mathrm{Re}(s_j) > b$ for all $j \neq i, i_1$ and the second one is s_i, s_{i_2} arbitrary and $\mathrm{Re}(s_j) > b$ for all $j \neq i, i_2$. The convex hull of these two regions will be s_i, s_{i_1} and s_{i_2} arbitrary and $\mathrm{Re}(s_j) > b$ for all $j \neq i, i_1, i_2$. At each step we increase the number of arbitrary coordinates by one, eventually reaching the whole $\mathbf{C}^n$. $\square$

References

[Langlands 1976] Robert P. Langlands, "On the Functional Equations Satisfied by Eisenstein Series", *Lecture Notes in Mathematics*, Vol. 544, Springer-Verlag, New York, 1976.