1. *Total energy of a closed two-body system.* In this problem we calculate the total energy of a system consisting of one body (mass $m$) orbiting another body (mass $M \gg m$) along a closed elliptical orbit. In so doing we will make more general the result we found in tutorial for a circular orbit of radius $R_o$:

$$E_{tot} = -\frac{GmM}{2R_o} \qquad \textit{(circular orbits only)}$$

Consider a comet (mass $m$) orbiting a star (mass $M$) in a highly eccentric orbit. Let the center of the star be located at the origin ($r = 0$) of a polar coordinate system.

Recall that the angular momentum per unit mass $(l)$ of the comet can be written $l = r^2\dot{\theta}$, and also that the latus rectum $\alpha$ of the orbit is related to $l$ according to: $l = (GM\alpha)^{1/2}$.

a. We first need to express the kinetic energy of the comet. To do this we note that the velocity of the comet can be written in polar coordinates as follows: $\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$.

   Starting with this expression for velocity, write <u>two</u> expressions for the **kinetic energy** of the comet: (i) one that is written in terms of $r$, $\dot{r}$, and $\dot{\theta}$, and then (ii) eliminate $\dot{\theta}$ from your expression so that it is written instead in terms of $r$, $\dot{r}$, and $l$. Clearly show all work.

b. Now express the **total energy** ($E_{tot}$) of the comet-star system, and do this in <u>two</u> ways: (i) write one expression in terms of $r$, $\dot{r}$, $l$, and given constant parameters, and then (ii) eliminate $l$ from your expression so that it is written instead in terms of $r$, $\dot{r}$, and $\alpha$. Clearly show all work.

c. You result in part b is valid for all points along the orbit of the comet, although in its present form it may not appear very useful. However, your expression can be simplified when considering the distances of closet approach (perihelion) and farthest approach (aphelion) of the comet.

   i. What simplification can be made for perihelion and aphelion? Explain.

   ii. Show that your expression for $E_{tot}$ simplifies to the <u>same</u> expression for <u>both</u> perihelion ($r_P = \alpha/(1 + \varepsilon)$) <u>and</u> aphelion ($r_A = \alpha/(1 - \varepsilon)$). Your final expression will be in terms of just the <u>semi-major axis *a*</u> of the comet's orbit, the masses $m$ and $M$ of the comet and star, and $G$.

   *Hint:* The latus rectum $\alpha$ can be expressed in terms of $a$ and $\varepsilon$ as follows: $\alpha = a(1 - \varepsilon^2)$.

2. Imagine a shuttlecraft in an elliptical orbit of semi-major axis $a$ around a planet of mass $M$. Show that the speed $v(r)$ of the shuttlecraft at any point along its orbit is expressed as the following function of $r$:

$$v(r) = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

3. Imagine that we lived in a world where Newton's law of gravitation was an inverse *cubic* law:

$$\vec{F} = -\frac{kMm}{r^3}\hat{e}_r$$

   a. Determine the potential energy function corresponding to this force. Show all work.

   b. In terms of the given quantities, express the escape velocity $v_e$ for a rocket launched from the earth's surface ($r = R_E$). Show all work. (Ignore the effect of the Earth's rotation as well as the change in mass of the rocket.).

 