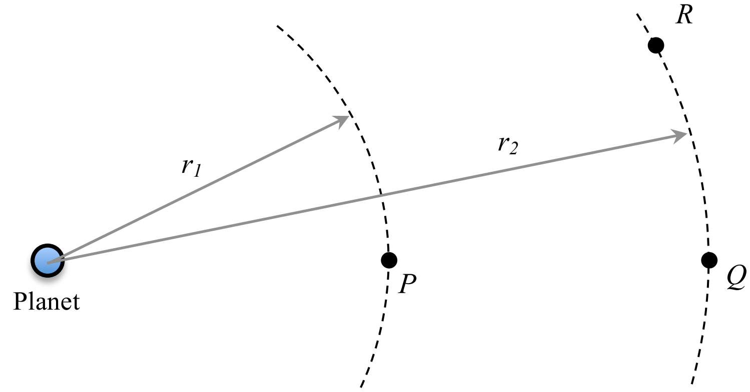
GRAVITATION AND CONSERVATION OF ENERGY

In this tutorial, we will use Newton's Law of Gravitation, $\vec{F}^G(\vec{r}) = -\dfrac{GMm}{r^2}\hat{e}_r$, to develop some problem-solving strategies in orbital mechanics. We focus on two-body problems for which one is much more massive than the other *($m \ll M$)*.

I. Work done by gravitational forces

Consider an unmanned space probe (mass m) undergoing maneuvers in the vicinity of a planet (mass M). In the figure below, three locations are labeled near the planet; point P is a distance r_1 away from the center of the planet, while points Q and R are a distance r_2 away.



A. Suppose the probe moved *directly away from* the planet from P to Q. Compute by <u>direct integration</u> the work done on the probe by the planet from P to Q. Check your work with your partners. We will call your result "Equation 1."

(Note: Keep careful track of signs [+/-]. Since $r_2 > r_1$, should the work be *positive* or *negative?)*

[Eq. 1]

B. Suppose that after arriving at point Q, the probe then maneuvered along the circular arc from Q to point R. What can be said about the work done by the planet for this part of the probe's motion? Explain.

C. Suppose now that the probe maneuvered from P to R but along a path that was different from the path $P \rightarrow Q \rightarrow R$. How, if at all, would that affect the amount of work done by the planet on the probe? (*Big hint:* At every point along the probe's path, whatever it may be, each next "step" along the path can be thought of as having components *parallel* and *perpendicular* to the direction of the force.)

Carefully discuss your reasoning with your partners—this is the 2nd most important question on the entire tutorial!

II. Gravitational potential energy

Your results thus far suggest that work done by gravitational forces depends not upon the path taken by an object but *only* upon the endpoints of that path. We can define gravitational potential energy U^G so that the change in gravitational energy is equal to the opposite of the work done, or: $W^G = -\Delta U^G$.

A. Your results from the preceding section suggest that gravitational potential energy can be expressed by a function like this (where r denotes the distance separating the centers of mass of the two bodies):

$$U^G(r) = \pm \frac{GMm}{r} \qquad (\text{"+" or "−"?})$$

Here's the question: Which sign, "+" or "–," belongs? Explain how your choice of sign will make the relationship $W^G = -\Delta U^G$ consistent with Equation 1 from the preceding page. (This is the most important question in the tutorial!) Check your thinking by answering the following questions:

- *Intuitively,* as the distance r increases, would you expect potential energy to *increase* or *decrease?*

- Now more *formally,* explain how your choice of sign will make the relationship $W^G = -\Delta U^G$ consistent with Equation 1.

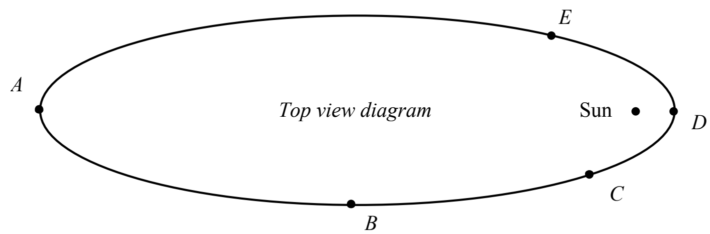
Please **STOP** here to briefly check your results here with an instructor.

Now we're in business! We will now see how we can use conservation of energy for systems of celestial objects in which one object orbits another as a result of just gravitational forces.

B. Consider a comet following a highly elliptical orbit around the Sun (see diagram below).

1. If the comet (mass m) has speed v when it is located a distance r from the Sun (mass M), write down an expression for the total energy (kinetic + potential) of the Sun-comet system in terms of these quantities and physical constants.

   (We will call your result "Equation 2.")



E_{tot} = [Eq. 2]

2. Use Equation 2 to rank the labeled points A – E according to the speed of the comet as it passes that point in its orbit. Discuss your reasoning with your partners.

III. Application: Circular orbits as examples of bound systems

Consider now the motion of a planet (mass m) that moves with constant speed v_o in orbit around a Sun-like star (mass M) along a circular orbit of radius R_o. (The Earth's orbit is almost circular.)

A. Write down an expression for the total energy of the planet-star system (kinetic + potential) in terms of the given parameters.

Because this situation involves the special case of a circular orbit, we can rewrite the above expression so that the total energy is written in terms in terms of just R_o, m, M, and G (and not v_o). Do so. (You will need Newton's Second Law in conjunction with Newton's Law of Gravitation.)

B. With your partners, interpret your (perhaps surprising!) result from part A:

1. Is the total energy of the planet-star system *positive* in value, *negative* in value, *or equal to zero*?

2. If the radius of the planet's orbit were larger than it actually was, would that change cause the total energy of the planet-star system to be *larger* or *smaller* than what it actually was? (Watch +/- signs!)

3. Suppose instead that the planet were located VERY far away from the star (as in, infinitely far away) and had ZERO velocity. Using Equation 2, what can be said about the total energy of the planet-star system for this extreme case?

4. Systems of objects like a star and an orbiting planet are often called *"bound systems."* Justify this term in light of what you now know about the total energy of such a system.

Please **STOP** here to briefly check your results here with an instructor.

IV. Application: "Escape velocity"

The term "escape velocity" refers to the minimum speed v_{esc} that a rocket or other spacebound vehicle must have at the time it leaves the Earth's surface (*i.e.,* at $r = r_E$) in order for it to avoid being recaptured in orbit around the Earth.

Calculate the escape velocity for a rocket leaving the Earth's surface, and in so doing show that it is independent of the mass of the rocket. Discuss your reasoning with your partners and show all work. (Ignore the effect of the Earth's rotation as well as the change in mass of the rocket.)

(*Big hint:* What is the minimum value for the total energy of the rocket-Earth system if this system is *not* to be a bound one?)

